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§6.2 Matrix representation

Q: 怎样的函数 f: (open) $A \subset \mathbb{R}^n \to \mathbb{R}^m$ 会在 x 处上可微呢?

一函数, x 不同就有不同 mapping.

A: 从必要条件想起. Assume f is diff. at x.
Then $\exists$ a linear function $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$
s.t. $f(x+h) - f(x) = Df(x)(h) + o(h)$ → 允许误差

把函数的形式写看[2]

有 ∞ 多 h, 故 $Df(x)$ 这次 $\Rightarrow$ 能成立

Let $f = (f_1, \dots, f_m)$, $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ (a 坐标用 i 表, b j 表). Then $f_j(x+h) - f_j(h)$
$= (Df(x)(h))_j + o(h)$, $\forall j = 1, \dots, m$.

↳ 6-1 证的 Prop.

$\because Df(x)$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}$
$\therefore [Df(x)(h)]_j$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}$.
$\Rightarrow f_j$ is diff at x, $\forall j = 1, \dots, m$
$\Rightarrow$ 只要 f 可微, 每一个坐标函数都可微
$\Rightarrow$ 定义 A open 是为了让此 linear function unique 存在

且 $[Df(x)(h)]_j = Df_j(x)(h) = \nabla f_j(x) \cdot h$

如果要展示 $Df(x)$, 则要找出基底, 取 $\mathbb{R}^n$, $\mathbb{R}^m$ 中的标准基底 $\{e_1, \dots, e_n\}$, $\{e_1, \dots, e_m\}$
$\Rightarrow [Df(x)(e_i)]_j = \nabla f_j(x) \cdot e_i = \frac{\partial f_j}{\partial x_i}(x)$

$\Rightarrow$ The matrix representation of the linear function $Df(x)$ w.r.t. the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ given by

is

Double A

各坐標對 x_1 微分

f_1 之 gradient $\rightarrow$ $\dfrac{\partial f_1}{\partial x_1}(x) \quad \cdots \quad \dfrac{\partial f_1}{\partial x_n}(x)$

f_m 之 gradient $\rightarrow$ $\dfrac{\partial f_m}{\partial x_1}(x) \quad \cdots \quad \dfrac{\partial f_m}{\partial x_1}(x)$

$\in \mathbb{R}^{m\times n}$ a matrix-valued function

(跟線代不大一樣)

分子表列

$\triangleq \left(\dfrac{\partial f_j}{\partial x_i}\right)(x) \triangleq \dfrac{\partial f_j}{\partial x_i}(x)$

→ 分母表行

$* \quad Df: A \to L(\mathbb{R}^n, \mathbb{R}^m)$

Thm. Let $A \subset \mathbb{R}^n$ be open. If $f=(f_1,\dots,f_m): A \to \mathbb{R}^m$ is diff at x, then $f_j: A \to \mathbb{R}$ is diff at x, $\forall j=1,\dots,m$ and the matrix of linear functin $Df(x)$ w.r.t. the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ is given by

$$\left(\frac{\partial f_j}{\partial x_i}(x)\right) \text{ or } Df(x) = \begin{pmatrix} \nabla f_1(x) \\ \vdots \\ \nabla f_m(x) \end{pmatrix}$$

Moreover, $Df(x)(e_i) = \begin{pmatrix} \frac{\partial f_1}{\partial x_i}(x) \\ \vdots \\ \frac{\partial f_m}{\partial x_i}(x) \end{pmatrix}$

哪一行LT在 哪一個向量上的取值的 第幾個坐標

即: $\dfrac{\partial f_j}{\partial x_i}(x)$ 為 $Df(x)(e_i)$ (m 維向量) 的第 j 個坐標

ex.

$$Df(x)(e_3) = \begin{pmatrix} \frac{\partial f_1}{\partial x_3}(x) \\ \vdots \\ \frac{\partial f_m}{\partial x_3}(x) \end{pmatrix}$$

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f本身没有偏微分，是它的坐标函数有偏微分

Cor. If $f=(f_1,\dots,f_m): \text{open } A\subset\mathbb{R}^n\to\mathbb{R}^m$ is diff at x, then the partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist at x, $\forall i=1,\dots,n$, $\forall j=1,\dots,m$

and $Df(x)=\left(\frac{\partial f_j}{\partial x_i}(x)\right)$, and the directional derivatives of each f_j exists. in every direction at x.

一个每一元都是函数的matrix，称matrix-valued

Def. The matrix of functions $\left(\frac{\partial f_j}{\partial x_i}\right)$ (function不大一样)

is called the Jacobian matrix of the derivative matrix of f.

现在已知道f可微 ⇒ 只分量偏微分都存在，因为偏微分所成的矩阵就是$Df(x)$了，且在A中每一点x都存在

⇏ 反过来不对，在$m=1$时就不对了.

⇏ 有没有容易检验可微的方法？要再加多少条件？

先考虑最简单的函数，猜常数函数之微分是0 L.T.

Prop. (1) If $f: A \overset{\text{open}}{\subset} \mathbb{R}^n\to\mathbb{R}^m$ is a constant map, then f is diff on A and $Df(x)=0$, $\forall x\in A$.

一阶多项式

(2). If $f: A \overset{\text{open}}{\subset} \mathbb{R}^n\to\mathbb{R}^m$ is a linear map, then f is diff. on A and $Df(x)=f$. $\forall x\in A$

⇒ Df is a constant function. $Df: A\to L(\mathbb{R}^n,\mathbb{R}^m)$，是指$Df(x)$为一个定函数f.

Double A

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pf. ∵ A is open
∴ $Df(x)$ is uniquely determined by f.

(1). ∵ $f(x+h)-f(x) = O = O\cdot h + O = O\cdot h + o(h)$.
(m维向量) (剩余项定义 $\frac{o(h)}{\|h\|}\to 0$)
∴ f is diff. at x. and $Df(x)=O$, $\forall x\in A$.

(2). Let $x\in A$, ∵ f is linear
∴ $f(x+h)-f(x) = f(x+h-x) = f(h) = f(h)+O$
$= f(h)+o(h)$
$\Rightarrow f$ is diff at x and $Df(x)=f$.

Q.E.D.

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§6.4 Conditions for differentiability

$= (f_1, \dots, f_n)$

Q: Let $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ s.t. all partial derivatives $\frac{\partial f_i}{\partial x_i}$ exists at x. 什麼時候 f 在 x 处可微?

A: Let $J = \left[\frac{\partial f_i}{\partial x_i}(x)\right]$ 怎樣的條件会使得可微, 即: $f(x+h) - f(x) = J(h) + o(h)$ 成立.

$\iff f_j(x+h) - f_j(x) = (J(h))_j + o(h)$, $\forall j = 1, \dots, m$

檢驗一个成立, 其他就成立了, 故可簡化為 $m = 1$.

Assume $m = 1$ (W.L.O.G.) that is, $f: A \subset \mathbb{R}^n \to \mathbb{R}$ and $J = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$

$\Rightarrow$ 找條件使剩餘項為 $o(h)$, 即:

$f(t) - f(x) = J(t-x) + o(t-x)$ 会成立.

f, J 都是給定的, 但顯然不可能, 除非是特別的 f.

$$\iff \lim_{t \to x} \frac{|f(t) - f(x) - J(t-x)|}{\|t-x\|} = 0 \text{ 成立}$$

得討論怎樣的 f 使此極限性 $= 0$?

從「極限」之所求想起

Let $\varepsilon > 0$, 考虑不等式 (必需加條件使此不等式成立)

$|f(t) - f(x) - J(t-x)| < \varepsilon \|t-x\|$. 要先可以化簡

↓

此時 J 是 gradient, 有多變数

$\Rightarrow$ 可把多变数看成单变数

$\Rightarrow$ 將 $n-1$ 个变数 固定.

$\Rightarrow$ 一次只动一个变数 (加一项减一项)

(思考)

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Let $t=(t_1,\dots,t_n)$ and $x=(x_1,\dots,x_n)$

$\Rightarrow f(t)-f(x)-J(t-x)=f(t_1,\dots,t_n)-f(x_1,\dots,x_n)-J(t-x)$

化成n項差 $\begin{cases} =f(t_1,\dots,t_n)-f(x_1,t_2,\dots,t_n)+f(x_1,t_2,t_3,\dots,t_n) \\ -f(x_1,x_2,t_3,\dots,t_n)+\cdots+f(x_1,\dots,x_{n-1},t_n)-f(x_1,\dots,x_n) \\ -J(t-x) \end{cases}$

要變數差值換成微分 $\Rightarrow$ MVT!

$\Rightarrow$ 要加的條件即為MVT成立之條件，即：f 在 $(x_1,t_2,\dots,t_n)$ 附近可微，其他次對正是顯然

Assume $\frac{\partial f}{\partial t_i}$ exists in a nbh of x, $\forall i=1,\dots,m$

Then by MVT, $\exists u_i\in(t_i,x_i)$ or (x_i,t_i), $\forall i=1,\dots,n$ s.t.

$$f(t)-f(x)-J(x)(t-x)=\frac{\partial f}{\partial t_1}(u_1,t_2,\dots,t_n)(t_1-x_1)$$
$$+\frac{\partial f}{\partial t_2}(x_1,u_2,\dots,t_n)(t_2-x_2)$$
$$+\dots.$$
$$+\frac{\partial f}{\partial t_n}(x_1,x_2,\dots,u_n)(t_n-x_n)-J(t-x)$$

$\sum_{i=1}^{n}\frac{\partial f}{\partial t_i}(x_1,\dots,x_n)\cdot(t_i-x_i)$ $=$ $J(t-x)$

$$=\sum_{i=1}^{n}\left[\frac{\partial f}{\partial t_i}(x_1,\dots,x_{i-1},u_i,t_{i+1},\dots,t_n)-\frac{\partial f}{\partial t_i}(x_1,\dots,x_n)\right](t_i-x_i)$$

$$\Rightarrow |f(t)-f(x)-J(x)(t-x)|\le\left[\sum_{i=1}^{n}\left|\frac{\partial f}{\partial t_i}(x_1,\dots,x_{i-1},u_i,t_{i+1},\dots,t_n)-\frac{\partial f}{\partial t_i}(x_1,\dots,x_n)\right|\right]\|t-x\|$$

令此項$<\frac{\varepsilon}{n}$時，可解$\|t-x\|$嗎？

(有n項)

Assume $\frac{\partial f}{\partial t_i}$ is conti. at x, $\forall i=1,\dots,n$.

For $\frac{\varepsilon}{n}$, $\exists \delta>0$ s.t. $\forall t\in A$ with $\|t-x\|<\delta$,

$$\left|\frac{\partial f}{\partial t_i}(t)-\frac{\partial f}{\partial t_i}(x)\right|<\frac{\varepsilon}{n},\ \forall i=1,\dots,n$$

$\Rightarrow \forall t\in A$ with $\|t-x\|<\delta$, $\|(x_1,\dots,x_{i-1},u_i,t_{i+1},\dots,t_n)-(x_1,\dots,x_n)\|<\delta$

$$\Rightarrow \left|\frac{\partial f}{\partial t_i}(t)-\frac{\partial f}{\partial t_i}(x)\right|<\frac{\varepsilon}{n},\ \forall t\in A$$

$$\Rightarrow \|f(t)-f(x)-J(x)(t-x)\|\le\left(\frac{\varepsilon}{n}\cdot n\right)\|t-x\|$$

$$=\varepsilon\|t-x\|$$

Therefore f is diff at x.

(f1,..,fm) → 取Jacobian，用到微一性 ⇒ open不可省

Thm. If $f: A\overset{\text{open}}{\subset}\mathbb{R}^n\to\mathbb{R}^m$ has partial derivatives $\frac{\partial f_j}{\partial x_i}$ in a nbh of x, and $\frac{\partial f}{\partial x_i}$ is conti. at x, $\forall i=1,\dots,n$, $j=1,\dots,m$, then f is diff at x.

§6-3 Continuity of Differentiable mappings: Differentiable Paths.

Q: If $f: A \overset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$ is a diff at x, is f cont. at x?

A: 去證該点极限值=函数值, let $\varepsilon>0$, 解函数差值$<\varepsilon$的不等式, 解变数差值.

Let $\varepsilon>0$, 由不等式 $\|f(t)-f(x)\|<\varepsilon$ 解 $\|t-x\|$?

$\because f$ is diff at x

$\therefore f(t)-f(x) = Df(x)(t-x) + \vartheta(t-x)$

$\Rightarrow \|f(t)-f(x)\| \le \underbrace{\|Df(x)(t-x)\|}_{\le \frac{\varepsilon}{2}} + \underbrace{\|\vartheta(t-x)\|}_{\le \frac{\varepsilon}{2}}$

但 $\|Df(x)(t-x)\| \le \frac{\varepsilon}{2}$ 可解嗎?

lemma. If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear function, then $\exists M>0$ s.t. $\|L(x)\| \le M\|x\|$, $\forall x \in \mathbb{R}^n$
In particular, $\|L(x)-L(y)\| \le M\|x-y\|$, i.e., L is Lipschitz.

pf. 看到 linear functin ⇒ 取值可以基底來表.
Let $x=(x_1,\dots,x_n)$ and $\{e_1,\dots,e_n\}$ a standard basis of $\mathbb{R}^n$.
Then $L(x) = L\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i L(e_i)$

$$\Rightarrow \|L(x)\| \le \sum_{i=1}^n \left(|x_i|\,\|L(e_i)\|\right)$$

$$\le \|x\| \underbrace{\sum_{i=1}^n \|L(e_i)\|}_{\text{定数}}$$

Q.E.D.

$\because \lim_{t\to x} \frac{\|\theta(t-x)\|}{\|t-x\|} = 0$ $\quad \forall t\in A$ with $\|t-x\|<\delta$,

$\therefore$ For $\varepsilon=1$, $\exists \delta_1>0$ s.t. $\|\theta(t-x)\| < 1\cdot\|t-x$

$\because Df(x)$ is a linear function

$\therefore \exists M>0$ s.t. $\forall t\in\mathbb{R}^n$, $\|Df(x)(t)\| \le M\|t\|$

$\Rightarrow \forall t\in A$ with $\|t-x\|<\delta$, $\|f(t)-f(x)\| \le$

$(M+1)\|t-x\|$

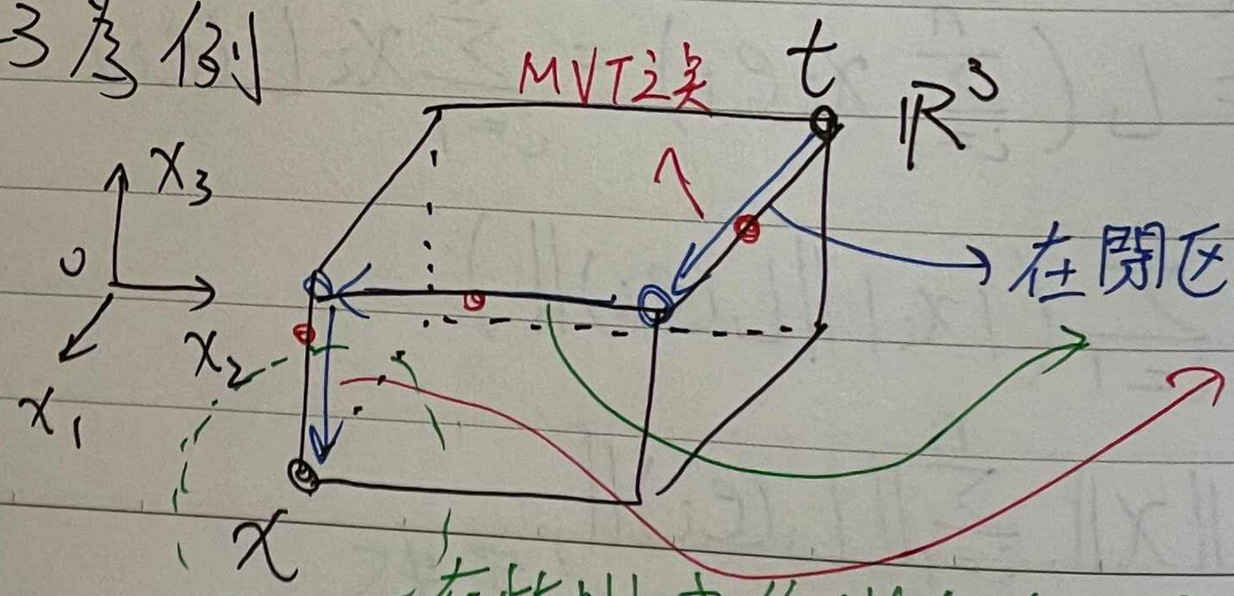
↳ 證了f連續更強，是Lipchitz (local)

Therefore f is conti. at x.

Prop. If a function $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ is diff at x, then f is conti. at x_0. In fact, there exists a constant M and a $\delta>0$ s.t. $\forall x\in A$ with $\|x-x_0\|<\delta$, $\|f(x)-f(x_0)\| < M\|x-x_0\|$ (That is called the local Lipschitz property).

Def. Let $f:(a,b)\subset\mathbb{R}\to\mathbb{R}^m$ be a diff vector-valued function. Then f is usually called a diff. curve or a path in $\mathbb{R}^m$

舉n=3為例



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但其实造的出一个函数，它在该点偏微分存在，
在该点偏微分不连续，可是函数依旧可微

§6.5 Chain Rule

單變數的 chain rule: $(g\circ f)'(x) = (g(f(x)))' = g'(f(x))\cdot f'(x$
我們希望這對多變數也成立: $D(g\circ f)(x)$
$= Dg(f(x))\circ Df(x)$

Thm. Let $f: A \overset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$ be diff at x_0
and $g: B \overset{\text{open}}{\subseteq} \mathbb{R}^m \to \mathbb{R}^p$ be diff. at $f(x_0)$.
Then $g\circ f$ is diff. at x_0, and
$D(g\circ f)(x) = Dg(f(x))\circ Df(x)$

pf. To show $\lim_{x\to x_0} \frac{\|g\circ f(x) - g\circ f(x_0) - Dg(f(x_0))\circ Df(x_0)(x-x_0)\|}{\|x-x_0\|}$

搭配可化簡的條件 $\Rightarrow g$ 可微

Let $\varepsilon > 0$ $\|g\circ f(x) - g\circ f(x_0) - Dg(f(x_0))\circ Df(x_0)(x-x_0)\|$
$\le \|g(f(x)) - g(f(x_0)) - Dg(f(x_0))(f(x)-f(x_0))\|$
$+ \|Dg(f(x_0))(f(x)-f(x_0)) - Df(x_0)(x-x_0)\|$

$\because$ g is diff. at $f(x_0)$
$\therefore$ For $\varepsilon/2M$, $\exists \delta_1 > 0$ s.t. $\forall y \in B$ with $\|y - f(x_0)\|$
$< \delta_1$, $\|g(y) - g(f(x_0)) - Dg(f(x_0))(y - f(x_0))\|$
$< \frac{\varepsilon}{2M}\|y - f(x_0)\|$ ($y = f(x)$)
$\because$ f is diff at x_0
$\therefore$ $\exists M > 0$ and $\delta_2 > 0$ s.t. $\forall x \in A$ with $\|x - x_0\| < \delta_2$
$\|f(x) - f(x_0)\| \le M\|x - x_0\| \le M\delta_2$
Let $0 < \delta_3 \le \delta_2$ s.t. $M\delta_3 \le \delta_1$
$\Rightarrow \forall x \in A$ with $\|x - x_0\| < \delta_3$,
$\|g(f(x)) - g(f(x_0)) - Dg(f(x_0))(f(x) - f(x_0))\| \le \frac{\varepsilon}{2M} M\|x - x_0$
$= \frac{\varepsilon}{2}\|x - x_0\|$

$\therefore Dg(f(x_0))$ is a linear function

$\therefore \exists L > 0$ s.t.

$\|Dg(f(x_0))(f(x) - f(x_0) - Df(x_0)(x - x_0)\|$

$\leq L\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|$

$\therefore f$ is diff at x_0

$\therefore$ For $\frac{\varepsilon}{2L}$, $\exists \delta_4 > 0$ s.t. $\forall x \in A$, with

$\|x - x_0\| < \delta_4$,

$\|f(x) - f(x_0) - Df(x_0)(x - x_0)\| < \frac{\varepsilon}{2L}\|x - x_0\|$

$\Rightarrow \forall x \in A$ with $\|x - x_0\| < \delta_4$,

$\frac{(x - x_0)\|}{} = 0$

$\|Dg(f(x_0))(f(x) - f(x_0) - Df(x_0)(x - x_0))\| < L \cdot \frac{\varepsilon}{2L}\|x - x_0\|$

Let $\delta_5 = \min\{\delta_3, \delta_4\}$. Then $\forall x \in A$ with

$\|x - x_0\| < \delta_5$,

$\|g(f(x)) - g(f(x_0)) - Dg(f(x_0))(Df(x_0)(x - x_0))\| < \varepsilon\|x - x_0\|$

$\Rightarrow g \circ f$ is diff. at x and $D(g \circ f)(x_0)$

$= Dg(f(x_0)) \circ Df(x_0)$.

Q.E.D.

Cor. The Jacobian matrix of $g \circ f$ at x is the matrix product of the Jacobian matrix of $g = (g_1, \ldots, g_p)$, evaluated at $f(x)$ with the Jacobian matrix of $f = (f_1, \ldots, f_m)$ evaluated at x. (in that order). i.e.,

$$D(g\circ f)(x) = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial y_1} & \cdots & \frac{\partial g_p}{\partial y_m} \end{pmatrix}\Bigg|_{y=f(x)} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix}(x)$$

Cor. Let $h = g\circ f$ and $h=(h_1,\dots,h_p)$. Then

$$\frac{\partial h_j}{\partial x_i} = \sum_{\ell=1}^{m} \frac{\partial g_j}{\partial y_\ell}\cdot\frac{\partial f_\ell}{\partial x_i}, \quad \forall j=1,\dots,p;\ i=1,\dots,n$$

ex. $h(r,\theta) = f(r\cos\theta, r\sin\theta)$. Then

$$\frac{\partial h}{\partial r}(r,\theta) = \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\cdot\cos\theta + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\cdot\sin\theta$$

$$\frac{\partial h}{\partial \theta}(r,\theta) = \text{"}\quad (-r\sin\theta) + \text{"}\quad r\cos\theta \ \#$$

ex. Verify the chain rule for $f(u,v,w) = u^2v+wv^2$ and $u=xy$, $v=\sin x$, $w=e^x$

-sol-. Let $h(x,y,z) = f(xy,\sin x, e^x) = x^2y^2\sin x + e^x\sin^2 x$. Then

$$\frac{\partial h}{\partial x}(x,y,z) = 2xy^2\sin x + x^2y^2\cos x + e^x 2\sin x\cos x + e^x\sin^2 x$$

Double A

By chain rule,

$$\frac{\partial h}{\partial x}(x,y,z) = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}$$

(x)
$$= 2xy\sin x \cdot y + (x^2y^2 + 2e^x\sin x)\cos x + \sin^2 x e^x$$
$$= 2xy^2\sin x + x^2y^2\cos x + e^x 2\sin x\cos x + e^x\sin^2 x$$

$\frac{\partial h}{\partial y}$, $\frac{\partial h}{\partial z}$ 同理 #

ex. Let $f:\mathbb{R}\to\mathbb{R}$ and $F:\mathbb{R}^2\to\mathbb{R}$ be given by $F(x,y) = f(xy)$. Verify that $x\frac{\partial F}{\partial x} = y\frac{\partial F}{\partial y}$.

pf. by chain rule,

$$\frac{\partial F}{\partial x} = f'(xy)\cdot y \Rightarrow x\frac{\partial F}{\partial x}(x,y) = xyf'(xy)$$

Similar for $y\frac{\partial F}{\partial y}$ #

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§6.6 Product Rule and Gradient

Thm. (微分的四則運算)
Let functions $f, g : A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ and $u : A \subset \mathbb{R}^n \to \mathbb{R}$ be diff at x_0. Then:
(1) $f+g$ is diff at x_0 and $D(f+g)(x_0)$
(2) αf is diff at x_0 and $D(\alpha f)(x_0) = \alpha Df(x_0)$
$\forall \alpha \in \mathbb{R}$
(3) uf is diff at x_0 and $D(uf)(x_0)$
$= Du(x_0)f(x_0) + u(x_0)Df(x_0)$. precisely for any $e \in \mathbb{R}^n$, $D(uf)(x_0)\cdot e = (Du(x_0)\cdot e)f(x_0) + u(x_0)Df(x_0)\cdot e$
(4) $f\cdot g$ is diff at x_0 and $D(f\cdot g)(x_0)$
$= D(f)(x_0)\cdot g(x_0) + f(x_0)\cdot Dg(x_0)$. More precisely, for any $e \in \mathbb{R}^n$ $D(f\cdot g)(x_0)(e) = [Df(x_0)(e)]\cdot g(x_0) + f(x_0)\cdot[Dg(x_0)(e)]$

渗出能用u可微的部份和能用f可微的部份

pf. (3) Let $\varepsilon > 0$
$\|u(x)f(x) - u(x_0)f(x_0) - u(x_0)Df(x_0)(x-x_0) - [Du(x_0)(x-x_0)]f(x$
$\le \|u(x)f(x) - u(x)f(x_0) - u(x)Df(x_0)(x-x_0)\|$
$+ |u(x)-u(x_0)|\cdot\|Df(x_0)(x-x_0)\|$
$+ \|f(x_0)\|\,|u(x)-u(x_0)-Du(x_0)(x-x_0)|$

$u(x)$連续，故附近之函数值可用极限值来估

$\because u$ is diff at x_0
$\therefore u$ is cont. at x_0
$\Rightarrow \exists\, \delta_1 > 0$ s.t. $\forall x \in A$ with $\|x-x_0\| < \delta_1$, $|u(x)-u(x_0)\| <$
$\Rightarrow |u(x)| \le |u(x_0)| + 1$, $\forall x \in A$ with $\|x-x_0\| < \delta_1$.

$\because f$ is diff at x_0

Double A

$\therefore$ For $\frac{\varepsilon}{3(|u(x_0)|+1)}$, $\|f(x)-f(x_0)-Df(x_0)(x-x_0)\|$

$< \frac{\varepsilon}{3(|u(x_0)|+1)}\|x-x_0\|$, $\forall x\in A$ with $\|x-x_0\|<\delta_1$

$\because Df(x_0)$ is a linear function

$\therefore \exists M>0$ s.t. $\|Df(x_0)(x-x_0)\|\le M\|x-x_0\|$

$\because u$ is cont. at x_0

$\therefore$ For $\frac{\varepsilon}{3M}$, $\exists \delta_2>0$ s.t. $\forall x\in A$ with

$\|x-x_0\|<\delta_2$, $|u(x)-u(x_0)|<\frac{\varepsilon}{3M}$.

$\because u$ is diff at x_0

$\therefore$ For $\frac{\varepsilon}{3(\|f(x_0)+1)}$, $\exists \delta_3>0$ s.t. $\forall x\in A$ with

$\|x-x_0\|<\delta_3$, $\|u(x)-u(x_0)-Df(x_0)(x-x_0)\|<\frac{\varepsilon}{3(\|f(x_0)+1)}$

Let $\delta=\min\{\delta_1,\delta_2,\delta_3\}$, then $\forall x\in A$ with

$\|x-x_0\|<\delta$, $\|u(x)f(x)-u(x_0)f(x_0)-u(x_0)Df(x_0)(x-x_0)$

$-[Du(x_0)(x-x_0)]f(x_0)\|<(|u(x_0)|+1)\frac{\varepsilon}{3(|u(x_0)|+1)}$

$+\frac{\varepsilon}{3M}\cdot M\|x-x_0\|+\|f(x_0)\|\cdot\frac{\varepsilon}{3(\|f(x_0)\|+1)}$

$<\left(\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\right)\|x-x_0\| = \varepsilon\|x-x_0\|$

$\Rightarrow uf$ is diff at x_0 and $D(uf)(x_0)$

$= u(x_0)Df(x_0)+Du(x_0)f(x_0)$.

(4) Let $f=(f_1,\dots,f_m)$ and $g=(g_1,\dots,g_m)$

$\because f$ and g are diff at x_0

$\therefore f_j$ and g_j are diff at x_0, $\forall j=1,\dots,m$

$\because f\cdot g=\sum_{j=1}^{m} f_j g_j$

$\therefore$ By ③ and ①, $f_j g_j$ and $f\cdot g$ are diff at x_0, and

$$\begin{aligned} D(f\cdot g)(x_0) &= \sum_{j=1}^{n} D(f_j g_j)(x_0) \\ &= \sum_{j=1}^{m}\left[g_j(x_0)Df_j(x_0)+f_j(x_0)Dg_j(x_0)\right] \\ &= g(x_0)\cdot Df(x_0)+f(x_0)\cdot Dg(x_0) \end{aligned}$$

Q.E.D.

Cor. Same assumptions as in prop. If $u(x_0)\neq 0$, then $\frac{f}{u}$ is diff at x_0 and

$$D\left(\frac{f}{u}\right)(x_0)=\frac{u(x_0)Df(x_0)-Du(x_0)f(x_0)}{u^2(x_0)}$$

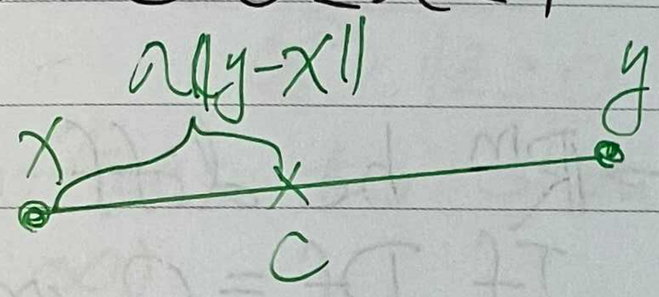
$$\left(\text{cf. } D\left(\frac{f}{u}\right)=\frac{uDf-(Du)f}{u^2}\right)$$

pf. Apply $\frac{1}{u}$ to ③.

§6.7 Mean-Value Thm.

Q: $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ 若 f 可微，则 MVT 会成立吗？
A: 由微积分(一)已知不会！

Def. ① Let $x, y, c \in \mathbb{R}^n$. We say that c is on the line segment joining x and y, or is between x and y iff $c = x + \lambda(y-x)$, for some $0 \le \lambda \le 1$



② A set $A \subset \mathbb{R}^n$ is said to be convex if for each x and y in A, the line segment joining x and y lies in A.

Thm. (MVT).

① Suppose that $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}$ is diff on A. If the line segment joining x and y lies in A, then $\exists\, c$ on the line segment s.t. $f(y) - f(x) = Df(c)(y-x)$.

② Suppose that $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ is diff on A. If the line segment joining x and y lies in A, then $\exists\, c_1, \dots, c_m$ s.t.
(这就有m个等式!)
$$f_j(y) - f_j(x) = Df_j(c_j)(y-x), \quad \forall j = 1, \dots, m$$

pf. (1) Let $h: [0,1] \to \mathbb{R}$ be $h(t) = f(x + t(y-x))$.
Then h is diff on $[0,1]$.
$\Rightarrow$ By MVT, $\exists\, t_0 \in (0,1)$ s.t.

$$h(1)-h(0) = h'(t_0)(1-0)$$
$$= h'(t_0)$$
$$= Df(x+t_0(y-x))\ (y-x)$$

$$\Rightarrow f(y)-f(x) = Df(x+t_0(y-x))(y-x)$$

(2) follows from (1)

Q.E.D.

Thm. Let $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ be diff. Suppose that A is convex. If $Df \equiv 0$ on A, then f is a constant function

pf. Let $x \in A$. Let $y \in A$.
$\because$ A is convex
$\therefore$ the line segment joining x and y lies in A.
By MVT (2), Let $f=(f_1,\dots,f_m)$, $\exists\ c_1,\dots,c_m$ on line segment joining x and y s.t.
$f_j(y)-f_j(x) = Df_j(c_j)(y-x)$, $\forall j=1,\dots,m$

$$\because Df = \begin{pmatrix} Df_1 \\ \vdots \\ Df_m \end{pmatrix} = 0$$

$\therefore Df_j = 0$ on A, $\forall j=1,\dots,m$
$\Rightarrow f_j(y)-f_j(x)=0$, $\forall j=1,\dots,m$
$\Rightarrow f(y)=f(x)$
$\Rightarrow f$ is a constant function

Q.E.D.

ex. Suppose that $f:[0,\infty)\to\mathbb{R}$ is cont, $f(0)=0$ and f is diff. on $(0,\infty)$ and f' is nondecreasing. Prove that $g(x)=\frac{f(x)}{x}$ is nondecreasing on $(0,\infty)$.

-sol-. To show that $g'\geq 0$ on $(0,\infty)$.

$$g'(x)=\frac{xf'(x)-f(x)}{x^2}$$ 此式 ≥ 0.

It suffices to show that $xf'(x)-f(x)\geq 0$ on $(0,\infty)$.

$\because f(x)=f(x)-f(0)$ ∴ By MVT, $\exists\, c\in(0,x)$ s.t. $f(x)=f'(c)x$

$\because f'$ is nondecreasing on $(0,\infty)$

$\therefore f'(c)\leq f'(x)$

$\Rightarrow f(x)\leq xf'(x)$

$\Rightarrow xf'(x)-f(x)\geq 0$

$\Rightarrow g'\geq 0$ on $(0,\infty)$

$\Rightarrow g$ is non-decreasing on $(0,\infty)$ ※

MON TUE WED THU FRI SAT SUN

§6.8 Taylor theorem and Higher order

Q: Let $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then $Df: A' \subset A \subset \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$. 可考虑对 Df 作微分吗?

$L(\mathbb{R}^n, \mathbb{R}^m)$

A: 第一个问题是对应域到底是不是欧氏空间呢? 若在 $\mathbb{R}^n$ 与 $\mathbb{R}^m$ 上各取一个基底, say standard basis, $\{e_1, \dots, e_n\}$, 则 $L(\mathbb{R}^n, \mathbb{R}^m) \simeq M_{m\times n}$ (= the set of $m\times n$ matrices) $\simeq \mathbb{R}^{mn}$.

Then Df can be viewed as a matrix of functions $\left(\frac{\partial f_j}{\partial x_i}\right): A' \subset \mathbb{R}^n \to M_{mn}$ defined by $\left(\frac{\partial f_j}{\partial x_i}\right)(x) = \left(\frac{\partial f_j}{\partial x_i}(x)\right)_{m\times n}$ where $f = (f_1, \dots, f_m)$ and $x = (x_1, \dots, x_n)$

因此我们可以考虑 the derivative of Df

Def. ① $D(Df) \triangleq D^2 f$. Then $D^2 f: A'' \subset A' \subset \mathbb{R}^n \to L(\mathbb{R}^n, L(\mathbb{R}^n; \mathbb{R}^m))$

② If Df is diff at x, then f is said to be twice diff at x.

③ In general, $D^k f = D(D^{k-1} f)$

Q: 上面的討論是邏輯延伸出來的, 如何具体認知 $D^k f$, 能想像出是一个怎样的函数?

A: ① 为了方便起見, assume $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ is diff on A. 回頭看 f 可微時如何得到 Df 的.

Let $f = (f_1, \dots, f_m)$ and $x = (x_1, \dots, x_n)$

derivatives

今用行向量表示$\mathbb{R}^n$布$\mathbb{R}^m$中的点，故将f寫成一个一维的行向量.

$$f=\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \xrightarrow{D} ? \quad \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

第i个column

如何認知這个過程呢？我們可以想成f這个行向量中每一个位置作 gradient 往右生成一个横的n维向量，而得到一个二維的矩陣且$Df(x_0)(e_i)$

=此矩值在x_0後的第i个column　矩陣取值在x_0　取第i行

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_i}(x_0) \\ \vdots \\ \frac{\partial f_m}{\partial x_i}(x_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_i} \\ \vdots \\ \frac{\partial f_m}{\partial x_i} \end{pmatrix} (x_0) \in \mathbb{R}^m = 对$$

$\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$作$\frac{\partial}{\partial x_i}$後再取值在$x_0$. 故$\frac{\partial f}{\partial x_i}(x_0) = Df(x_0)(e_i)$

此m維向量的第j个作標

Note. $\because Df(x_0)$ is linear in e_i

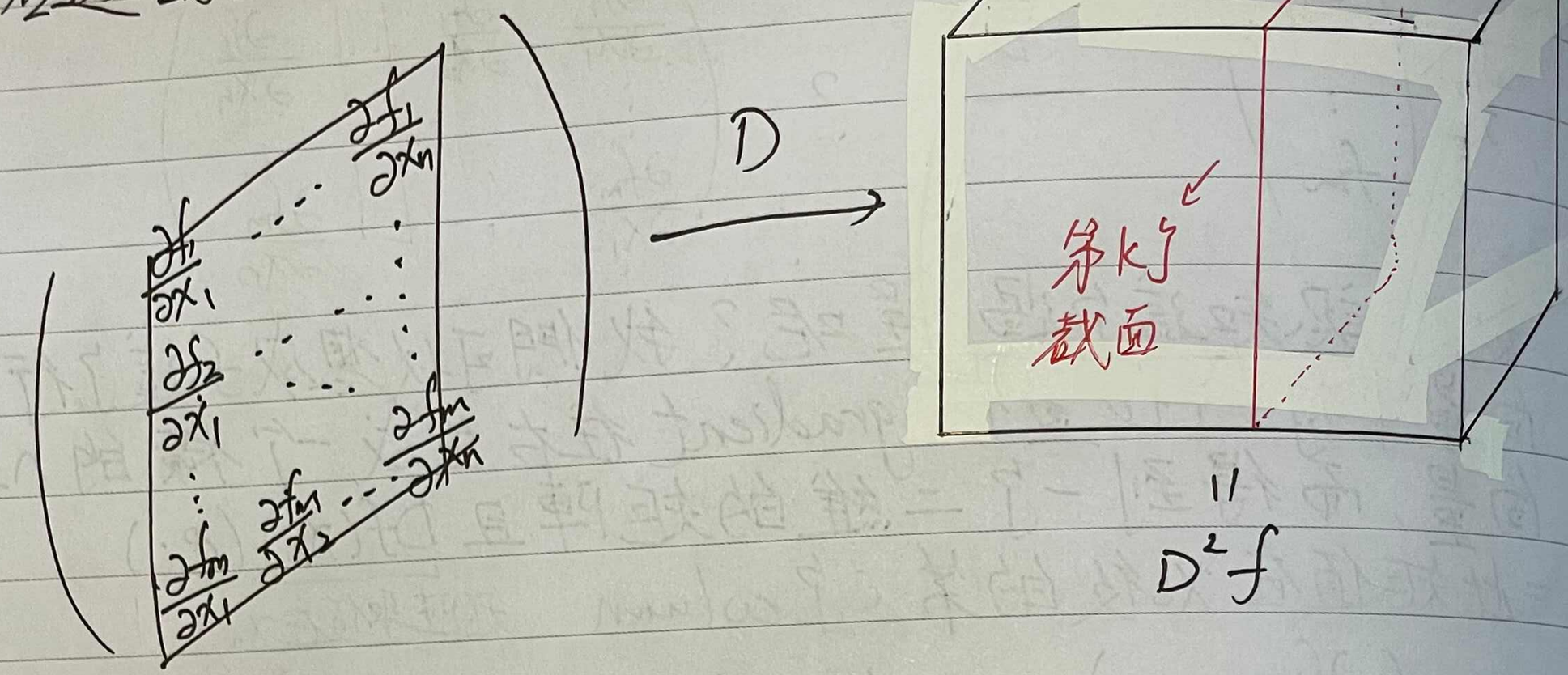
$$\therefore Df(x_0)\left(\sum_{i=1}^{n} a_i e_i\right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} a_i$$

② 現在將上述的想法再用在Df起Q $D(Df)=D^2f$上.

Assume $Df: A$ 些$\mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$ is diff on A. 將$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$此二维矩陣以第一行

軸心

向量為軸心向後旋轉90°使矩陣面朝右，空出右邊的空間，見下圖



今將此矩陣中的每个位置作 gradient 向右生成一个核的 n 维向量而得到一个三维矩陣，此時 $D^2f(x_0)(e_k)$ 為此三维矩陣取值 x_0 後的第 k 个前-後截面，以此截面最前面的 column 為軸心（見圖）往前旋轉 90° 使截面面朝前，即可得見

$$D^2f(x_0)(e_k) \in \mathbb{R}^n = \begin{pmatrix} \frac{\partial^2 f_1}{\partial x_k \partial x_1}(x_0) & \cdots & \frac{\partial^2 f_1}{\partial x_k \partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f_m}{\partial x_k \partial x_1}(x_0) & \cdots & \frac{\partial^2 f_m}{\partial x_k \partial x_n}(x_0) \end{pmatrix} \in L(\mathbb{R}^n, \mathbb{R}^m)$$

$$\left(D^2f(x_0)(e_k)\right)(e_i) = \begin{pmatrix} \frac{\partial^2 f_1}{\partial x_k \partial x_i}(x_0) \\ \vdots \\ \frac{\partial^2 f_m}{\partial x_k \partial x_i}(x_0) \end{pmatrix} \in \mathbb{R}^m = \text{對 } f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

作 $\frac{\partial^2}{\partial x_k \partial x_i}$ 再取值在 x_0 上，且 $\frac{\partial^2 f_j}{\partial x_k \partial x_i}(x_0) =$ $(D^2 f(x_0)(e_k))(e_i)$ 的 m 維向量第 j 分座標

Note: $\because D^2 f(x_0)$ 对 e_k 与 e_i 都 linear

$$\therefore \left[D^2 f(x_0)\left(\sum_{k=1}^{n} a_k e_k\right)\right]\left(\sum_{i=1}^{n} b_i e_i\right) = \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k \partial x_i}(x_0) a_k b_i \in \mathbb{R}^m$$

$D^2 f(x_0)$ 对 e_k, e_i 作用 $\Rightarrow$ 对 f 做 2 次偏微

Def. Denote $\left[D^2 f(x_0)(a)\right](b) = D^2 f(x_0)(a, b)$, $\forall a, b \in \mathbb{R}^n$. Then $D^2 f(x_0) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a bilinear function.

第二次偏微 第一次偏微

Thm. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be twice diff. on A. Then all second order partial derivatives of f exist on A s.t. $D^2 f(x)(e_k, e_i) = \frac{\partial^2 f}{\partial x_k \partial x_i}(x)$, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$

Thm. Let $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ be a functim. Suppose that all the seaund order partial derivatives are cont. on A, then f is twice diff. on A.

pf. Let $f = (f_1, \dots, f_m)$. $\because$ All seaund order partial derivatives of f are cont. on A

∴ The map $\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} : A \to L(\mathbb{R}^n, \mathbb{R}^m)$

is diff.

∵ $\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$ is cont. on A

∴ $\frac{\partial f_j}{\partial x_i}$ is cont. on A, $\forall i=1,\dots,n,\ j=1,\dots,m$

$\Rightarrow f$ is diff. on A and $Df = \left(\frac{\partial f_j}{\partial x_i}\right)$

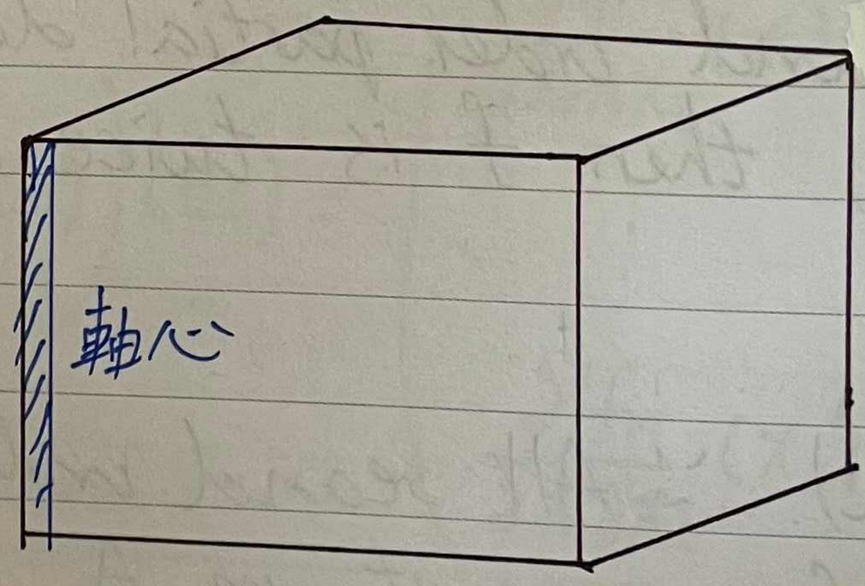
∵ Df is diff on A

∴ f is twice diff on A.

Q.E.D.

Q：如何認知 D^3f 呢？

A: Assume $D^2f: A \overset{\text{open}}{\subseteq} \mathbb{R}^n \to L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))$ is diff on A.

$D^2f =$ 

$\xrightarrow{D}$?

想像 4 維空間中，將此 3 維矩陣 D^3f 以藍色的那一 column 為軸心向第 4 維的方向往後旋轉 90° 使得此矩陣的面朝右，空出右邊的空間，而後對此 3 維矩陣的每一個位置作 gradient 向右生成一個核的 n 維向量而得到一個 4 維的矩陣. 則 $(D^3f(x_0))(e_l)$ = 此 4 維矩陣取值在 x_0 後的第 l 個前後截面（它是 3 維的矩陣），以此截面的最前面的行向量為軸心，將此截面往前旋轉 90° 使面朝前，即得見

$D^3f(x_0)(e_l) \in L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))$ s.t.

$[[D^3f(x_0)(e_l)](e_k)](e_i)$ = 對 $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$ 作 $\dfrac{\partial^3}{\partial x_l \partial x_k \partial x_i}$

後取值在 $x_0 \in \mathbb{R}^m$.

$$\frac{\partial^3 f_j}{\partial x_l \partial x_k \partial x_i}(x_0) = \left([D^3f(x_0)(e_l)](e_k)\right)(e_i)$$ 的第

j 個座標.

Note : $D^3f(x_0)$ 對 e_l, e_k, e_i 皆 linear

$$\therefore \left([D^3f(x_0)\left(\sum_{l=1}^{n} a_l e_l\right)]\left(\sum_{k=1}^{n} b_k e_k\right)\right)\left(\sum_{i=1}^{n} c_i e_i\right)$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial^3 f}{\partial x_l \partial x_k \partial x_i}(x_0)\, a_l b_k c_i \in \mathbb{R}^m$$

Def. Denote $\left([D^3f(x_0)(a)](b)\right)(c) = D^3f(x_0)(a,b,c)$
$\forall a, b, c \in \mathbb{R}^n$. then $D^3f(x_0) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$
is triple linear.

In general, $D^k f$ is a $k+1$ 維的矩陣且

$D^k f(x_0)(e_{i_1}, \dots, e_{i_k}) = \dfrac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}(x_0) \in \mathbb{R}^m$.

Thm. Let $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ be r times diff. on A. Then all r^{th} partial derivatives exist on A and $([D^r f(x_0)(e_{i_1})](e_{i_2}) \cdots (e_{i_r}))$

$\triangleq D^r f(x_0)(e_{i_1}, e_{i_2}, \dots, e_{i_r}) = \dfrac{\partial^r f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_r}}(x_0)$

$\in \mathbb{R}^m$

Thm. Let $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ be a function. If all r^{th} partial derivatives are cont. at x_0, then f is r-times diff. at x_0.

Thm. (Symmetry of mixed partials)
Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be twice diff. on the open set A. If all the second partial derivatives of f are cont. at x_0, then $\dfrac{\partial^2 f_j}{\partial x_k \partial x_i}(x_0)$ where $i, k = 1, \dots, n$ and $f = (f_1, \dots, f_m)$, $\forall j = 1, \dots, m$. That is $D^2 f(x_0)(e_k, e_i) = D^2 f(x_0)(e_i, e_k)$

$\forall i, k = 1, \dots, n$.

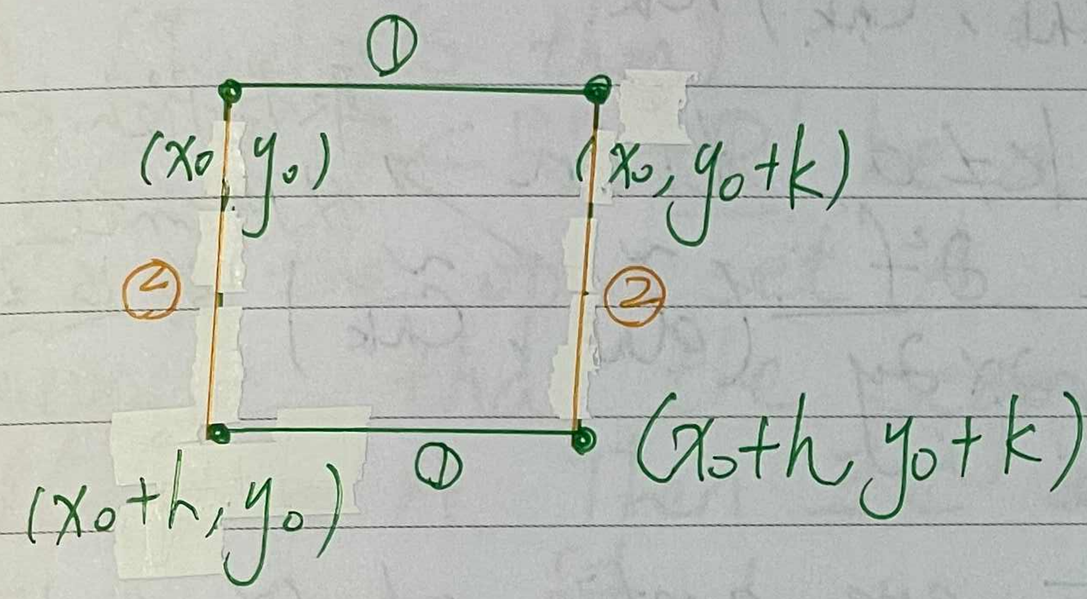
$\iff D^2 f(x_0)(a, b) = D^2 f(x_0)(b, a), \ \forall a, b \in \mathbb{R}^n$

pf: By holding all other variables besides x_i and x_k fixed we may assume $R^n \cong \mathbb{R}^2$. And for j fixed, we may assume that

$\mathbb{R}^m \cong \mathbb{R}$. Let $x_0 = (x_0, y_0)$

從這觀點來看 $\frac{\partial^2 f}{\partial y \partial x}(x_0) = \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right](x_0, y_0)$

考慮 $\frac{\partial f}{\partial x}(x_0, y_0+k) - \frac{\partial f}{\partial x}(x_0, y_0)$



對 $\frac{\partial f}{\partial x}(x_0, y_0+k)$ 需要考慮 $f(x_0+h, y_0+k) - f(x_0, y_0+k)$

對 $\frac{\partial f}{\partial x}(x_0, y_0)$ ∴ $f(x_0+h, y_0) - f(x_0, y_0)$

Let $S_{h,k} = [f(x_0+h, y_0+k) - f(x_0, y_0+k)] - [f(x_0+h, y_0) - f(x_0, y_0)]$ — ①

一次把/
二次MVT
做掉

對 $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$ 來說：$[f(x_0+h, y_0+k) - f(x_0+h, y_0)]$

$-[f(x_0, y_0+k) - f(x_0, y_0)] = S_{h,k}$ — ②.

Let $g(u) = f(x_0+h, u) - f(x_0, u)$. Then $S_{h,k} = g(y_0+k) - g(y_0)$

∵ f is diff on A

∴ g is diff in a nbh of y_0

By MVT, $\exists\ c_{h,k} \in (y_0, y_0+k)$ s.t. $S_{hk} = g'(c_{hk}) \cdot k$

$= \left[\frac{\partial f}{\partial y}(x_0+h, c_{hk}) - \frac{\partial f}{\partial y}(x_0, c_{hk})\right] k$.

∵ f is twice diff on A

∴ By MVT,

$\exists\, d_{hk} \in (x_0, x_0+h)$ s.t.

$$S_{hk} = \frac{\partial^2 f}{\partial x \partial y}(d_{hk}, c_{hk}) \cdot hk$$

Similarly, $\exists\, \tilde{c}_{hk} \in (y_0, y_0+k)$ and $\tilde{d}_{hk} \in (x_0, x_0+h)$

s.t. $S_{hk} = \frac{\partial^2 f}{\partial y \partial x}(\tilde{d}_{hk}, \tilde{c}_{hk})\, hk$

Choose $h \neq 0$ and $k \neq 0$, then

$$\frac{\partial^2 f}{\partial y \partial x}(d_{hk}, c_{hk}) = \frac{\partial^2 f}{\partial x \partial y}(\tilde{d}_{hk}, \tilde{c}_{hk})$$

取極限 $(h, k \to 0)$

使用 conti 之條件

強迫交於 (x_0, y_0)

$\because \frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are conti. at (x_0, y_0)

$\therefore$ Letting $h \to 0$, $k \to 0$, $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$

Q.E.D.

Thm. It follows that the continuity of all partial derivatives of f at x_0 will imply the symmetry of mixed partial derivatives at x_0 up to the order r.

(or $D^r f(x_0)$ as r linear is symmetry)

$D^{r-1} f(x_0)$ 對稱, ..., $D^2 f(x_0)$ 對稱

Def. ① A function is said to be class C^r if the first r derivatives of f exists and conti. (i.e. $Df, D^2 f, \ldots, D^r f$ exist and conti.

② A function is said to be smooth or of class C^∞ if it is of class C^r for

all $r \in \mathbb{N}$.

Remark. By chain rule, the composite of C^r functions is also C^r.

Thm. (Taylor's thm)
Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function of class C^r on the open set A. Let $x, y \in A$ and suppose that the segment joining x and y lies in A. Then $\exists$ C^r on that segment s.t.

$$f(y) = f(x) + \sum_{k=1}^{r-1} \frac{1}{k!} D^k(x)(\underbrace{y-x, \cdots, y-x}_{k\text{次}}) + \frac{1}{r!} D^r f(c)$$

Setting $y = x+h$, we can write Taylar formula as $f(x+h) = f(x) + \sum_{k=1}^{r-1} \frac{1}{k!} D^k(x)(h, \ldots, h) + R_{r-1}(x,h)$

where $R_{r-1}(x,h)$ is the Remainder with $\frac{R_{r-1}(x,h)}{\|h\|^{r-1}} \to 0$ as $h \to 0$

Pf. $\because f$ is C^1 $\xrightarrow{2^{nd}\text{ FTC}}$ $\therefore f(x+h) - f(x) = \int_0^1 \frac{d(f(x+th))}{dt}\, dt$

$\xrightarrow{\text{Chain rule}}$ $= \int_0^1 \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(x+th) h_i \right) dt$, where $h = (h_1, \ldots, h_n)$

$\xrightarrow{1 = \frac{-d(1-t)}{dt}}$ $= -(1-t) \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x+th) h_i \Big|_{t=0}^{t=1}$

$+ \int_0^1 (1-t) \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i}(x+th) h_i h_j \, dt$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) h_i - \frac{(1-t)^2}{2} \sum_{j,i=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(x+th) h_j h_i \Big|_0^1$$

$$+ \int_0^1 \frac{(1-t)^2}{2!} \sum_{k,j,i=1}^{n} \frac{\partial^3 f}{\partial x_k \partial x_j \partial x_i}(x+ih) h_k h_j h_i \, dt$$

$$= Df(x)(h) + \frac{1}{2!} D^2 f(x)(h,h) + \cdots + \frac{1}{(r-1)!} D^{r-1} f(x)(\underbrace{h, h, \ldots, h}_{r-1 次})$$

$$+ \underbrace{\frac{1}{(r-1)!} \int_0^1 (1-t)^{r-1} \sum_{i_1, \ldots, i_r=1}^{n} \frac{\partial^r f}{\partial x_{i_r} \cdots, \partial x_{i_1}}(x+th) h_{i_r} \cdots h_{i_1} \, dt}_{R_{r-1}(x,h)}$$

$\because D^r f$ is conti. at x

$\therefore \dfrac{\partial^r f}{\partial x_{i_r} \cdots \partial x_{i_1}}$ is bounded in a nbh of x.

(i.e., for $\|h\|$ is small enough) $\forall i_1, \ldots, i_r = 1, \ldots, n$

$$\Rightarrow \exists M > 0 \text{ s.t. } \sum_{i_1, \ldots, i_r=1}^{n} \left| \frac{\partial^r f}{\partial x_{i_r} \cdots \partial x_{i_r}}(x+th) \right| \leq M.$$

for $\|h\|$ small

$$\Rightarrow |R_{r-1}(x,h)| \leq \int_0^1 M \|h\|^r \cdot 1 \, dt = M \|h\|^r$$

$$\Rightarrow \frac{|R_{r-1}(x,h)|}{\|h\|^{r-1}} \to 0 \text{ as } h \to 0$$

$$\Rightarrow \frac{R_{r-1}(x,h)}{\|h\|^{r-1}} \to 0 \text{ as } h \to 0.$$

Moreover, $\because R_{r-1}(x,h) = \int_0^1 \boxed{\frac{(1-t)^{r-1}}{(r-1)!}} D^r f(x+th)(h, \ldots, h) dt$

≥ 0

Apply 2nd MVT for integral

and $\frac{(1-t)^{r-1}}{(r-1)!} \geq 0$ on $[0,1]$

$\therefore$ By 2nd MVT for integral, $\exists\, t_0 \in [0,1]$
s.t. $R_{r-1}(x,h) = D^r f(x+t_0 h)(h,\dots,h_r) \int_0^1 \frac{(1-t)^{r-1}}{(r-1)!}\,dt$

$= \frac{1}{r!} D^r f(c)(h,\dots,h)$, where $c = x + t_0 h$

Q.E.D.

留下好的证法 做积分

Remark. With a bit more effort, one prove a stronger thm: If f is of class C^r, then

$$f(x+h) = f(x) + \sum_{k=1}^{r} D^k f(x)(h\dots,h) + R_r(x,h)$$

where $\frac{R_r(x,h)}{\|h\|^r} \to 0$ as $h \to 0$

Def. ① If f is of C^∞, then

$$\sum_{k=0}^{\infty} \frac{1}{k!} D^k f(x_0)(x-x_0, x-x_0, \dots, x-x_0)$$

is called the Taylar series of f about x_0 (which is a power series which need not converge to $f(x)$!, 但在 x_0 這点收敛)

② If the Taylar series of f about x_0 收敛半径 >0, converges to f in a nbh of x_0, then we say that f is real analytic at x_0.

③ We say that f is analytic on A if f is analytic at every point of A.

Remark. f is analytic at $x_0 \Leftrightarrow$ the remainder term $\frac{1}{r!} D^r f(c)(x-x_0, \dots, x-x_0) \to 0$ as $r \to \infty$ in some nbd of x.

ex. If f is C^∞ on $\mathbb{R}$ and for each interval $[a,b]$ there is a constant M s.t. $|f^{(n)}(x)| \le M^n, \forall n \in \mathbb{N}, \forall x \in [a,b]$. Show that f is analytic.

pf. Let $x_0 \in \mathbb{R}$. Choose $b > 0$ s.t. $x_0 \in (-b, b)$.

The remainder term $|R_{r-1}(x_0, x-x_0)| = \frac{f^{(r)}(c)}{r!}(x-x_0)^r$

$\le \boxed{\frac{M^r |x-x_0|^r}{r!}}$ 要證 $\to 0$

⇒用級數判斂

$\because$ By ratio test, $\sum_{r=0}^{\infty} \frac{M^r |x-x_0|^r}{r!}$ conv.

$\therefore \frac{M^r |x-x_0|^r}{r!} \to 0$ as $r \to \infty$

$\Rightarrow \forall x \in (-b, b)$, f is analytic of x_0

$\Rightarrow f$ is analytic in $\mathbb{R}$. #

Q: 是否有無限可微的函數不 analytic?

A: 有!

ex. Give an example of C^∞ function that is not analytic.

-sol-. Let $f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0. \end{cases}$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{x}=\lim_{x\to 0}\frac{\frac{1}{x}}{e^{\frac{1}{x^2}}}$$

$t=\frac{1}{x}$

$$\overset{\downarrow}{=}\lim_{t\to\infty}\frac{t}{e^{t^2}}=\left(\frac{\infty}{\infty}\right)\overset{\uparrow}{=}\lim_{t\to\infty}\frac{1}{2te^{t^2}}=0 \quad (\to -\infty \text{ 亦可})$$

分成趨到$+\infty,-\infty$討論

L'Hospital rule

$-\infty$.

$$\Rightarrow f'(x)=\begin{cases} \frac{2}{x^3}e^{-\frac{1}{x^2}}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

Similarly, $f^{(k)}(0)=0.\ \forall k\in\mathbb{N}$

Hence the Taylar series of f about $x=0$ is the zero function #

ex. Compute the second order Taylar formula for $f(x,y)=\cos(x+2y)$ at $(0,0)$.

pf. (思考)

1. $\sum_{k=0}^{r}\frac{1}{k!}D^kf(x_0)(x-x_0,\dots,x-x_0)+R_r(x_0,x-x)$

2. $D^kf(x_0)(v,v,\dots,v)=\sum_{i_1,\dots,i_k=1}^{n}\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(x_0)v_{i_1}\cdots v_{i_k}$

$v=\sum v_i e_{i_1}\ \ e_{i_2}\ \ e_{i_k}$

$f(0,0)=1,\ \frac{\partial f}{\partial x}(x,y)=-\sin(x+2y),\ \frac{\partial f}{\partial y}(x,y)=-2\sin(x+2y)$

$\Rightarrow \frac{\partial f}{\partial x}(0,0)=0,\ \frac{\partial f}{\partial y}(0,0)=0$

Double A

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\cos(x+2y), \quad \frac{\partial^2 f}{\partial y \partial x}(x,y) = -2\cos(x+2y),$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -4\cos(x+2y)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(0,0) = -1, \quad \frac{\partial^2 f}{\partial y \partial x}(0,0) = -2, \quad \frac{\partial^2 f}{\partial y^2}(0,0) = -4.$$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

$$\Rightarrow f(h,k) = 1 + 0\cdot h + 0\cdot k + \frac{1}{2!}\left((-1)h^2 + (-2)hk\cdot 2 + (-4)k^2\right) + R_2((0,0),(h,k))$$

$$= 1 - h^2 - 2hk - 2k^2 + R_2((0,0),(h,k)) \quad \#$$

§6.9 Maxima and Minima

Thm. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be diff on the open set A. If $x_0 \in A$ is a local extreme point, then $Df(x_0) = 0$.

pf. Assume $Df(x_0) \neq 0$. Then $v \in \mathbb{R}^n$ s.t.
$Df(x_0)(v) = c \neq 0$, say $c > 0$.
$\because f$ is diff at x_0
$\therefore$ For $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall \|\lambda v\| < \delta$,
$|f(x_0+\lambda v) - f(x_0) - Df(x_0)(\lambda v)| < \varepsilon \|\lambda v\|$
$\Rightarrow \forall |\lambda| \|v\| < \delta$, $|f(x_0+\lambda v) - f(x_0) - \lambda c| < \varepsilon |\lambda| \|v\|$
$\Rightarrow \lambda c - \varepsilon |\lambda| \|v\| < f(x_0+\lambda v) - f(x_0) < \lambda c + \varepsilon |\lambda| \|v\|$

既然ε是任意取，何不取一个可幫我們化簡的$\varepsilon \Rightarrow \varepsilon = \frac{c}{2\|v\|}$

Choose $\varepsilon = \frac{c}{2\|v\|}$, then $\forall |\lambda| < \frac{\delta}{\|v\|}$,
$c(\lambda - \frac{|\lambda|}{2}) < f(x_0+\lambda v) - f(x_0) < (\lambda + \frac{|\lambda|}{2})c$

→不知c正負，直回去補條件

If $\lambda > 0$, then $f(x_0+\lambda v) - f(x_0) > 0$.
If $\lambda < 0$, then $f(x_0+\lambda v) - f(x_0) < 0$.

Hence $f(x_0)$ is neither a local max. nor local min of $\rightarrow\leftarrow$
Therefore $Df(x_0) = 0$

Q.E.D.

Remark. Critical point $\not\Rightarrow$ local extreme points.

ex. $n=1$, $f(x)=x^3$, $x=0$.
$n=2$, $f(x,y)=y^2-x^2$ at $(0,0)$

Def. A critical point that is not a local extreme point is called a saddle point.

Def. If $f: A\subset\mathbb{R}^n\to\mathbb{R}$ is of class C^2, then Hessian of f at x_0 is defined to be the bilinear function $Hx_0(f)$: $\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is given by $Hx_0(f)(v,w)$
$=D^2f(x_0)(v,w)$, $\forall v,w\in\mathbb{R}^n$

Remark $\because D^2f$ w.r.t. the standard basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ is just the matrix of function
$\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)$
$\therefore \left\|\frac{\partial^2 f}{\partial x_i\partial x_j}(x_0)\right\|$, the norm of $D^2f(x_0)$ in the normed space $L(\mathbb{R}^n, L(\mathbb{R}^n,\mathbb{R}))\simeq\mathbb{R}^{n^2}$
is $\|D^2f(x_0)\|=\sqrt{\sum_{i=1}^{n}\sum_{j=1}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}(x_0)\right)^2}$

Moreover, $\because D^2f(x_0)(v,w)=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}(x_0)v_iw_j$

$\therefore D^2f(x_0)$ as a bilinear function defined on $\mathbb{R}^n\times\mathbb{R}^n$ is a cont. function in $\mathbb{R}^n\times\mathbb{R}^n$
(注e linear function 是 cont. 同理)

⇒ 我們將 $D^2f(x_0)$ 看成非常高維度的歐氏空間中的一矣，也看成 $\mathbb{R}^n \times \mathbb{R}^n$ 上的 bilinear functin

⇒ $D^2f(x_0)(v,w)$ 可用 v, w 的 num 來估嗎

(的 num)

$$\left|D^2f(x_0)(v,w)\right| \le \sum_{i=1}^n \sum_{j=1}^n \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)\right| |v_i||w_j|$$

$$= \sum_{j=1}^n \left[|w_j| \left(\sum_{i=1}^n \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right| |v_i| \right) \right]$$

By Schwarz ineq,

$$\le \sum_{j=1}^n |w_j| \left(\sum_{i=1}^n \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n v_i^2 \right)^{\frac{1}{2}}$$

$$= \|v\| \sum_{j=1}^n \left[|w_i| \left(\sum_{i=1}^n \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right|^2 \right)^{\frac{1}{2}} \right]$$

$$= \|v\| \left(\sum_{j=1}^n w_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \sum_{i=1}^n \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right|^2 \right)^{\frac{1}{2}}$$

$$= \|v\| \|w\| \|Df(x_0)\|$$

Therefore $|D^2f(x_0)(v,w)| \le \|Df(x_0)\| \|v\| \|w\|$, $\forall$ $v, w \in \mathbb{R}^n$

Def. A bilinear form $B: V \times V \to \mathbb{R}$ is called positive definite (negative) (<), if $B(x,x) > 0$, $\forall x \neq 0$ in V, and is called positive semidefinite, if $B(x,x) \ge 0$, $\forall x \in V$

Thm. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a C^2 functim on an open set A. Suppose $x_0 \in A$ is a critical point of f.

① If $Hx_0 f$ is negative definite, then f has a local maximum at x_0

② If f has a local maximum at x_0, then $Hx_0 f$ is negative semidefinite

pf. $\because x_0$ is a critical pt of f and f is diff at x_0

$\therefore Df(x_0) = 0$

$\because f$ is of C^2

$\therefore$ By Taylor thm, $f(y) - f(x_0) = D^2 f(c)(y-x_0, y-x_0)$ where c lies on the segment of y and x_0. We want to show that $\exists \delta > 0$ s.t. $\forall y \in D(x_0, \delta) \subset A$ s.t. $f(y) - f(x_0) < 0$. Therefore x_0 is a local maximum point of f.

$$D^2 f(c)(y-x_0, y-x_0) = D^2 f(x_0)(y-x_0, y-x_0) + D^2 f(c)(y-x_0, y-x_0) - D^2 f(x_0)(y-x_0, y-x_0)$$

↓ 为2"||"必大

$$\leq D^2 f(x_0)(y-x_0, y-x_0) + |(D^2 f(c) - D^2 f(x_0))(y-x_0, y-x_0)|$$

↓ bilinear 先相减

$$\leq \|y-x_0\|^2 Df(x_0)\left(\frac{y-x_0}{\|y-x_0\|}, \frac{y-x_0}{\|y-x_0\|}\right) + \|D^2 f(c) - D^2 f(x_0)\| \|y-x_0\|^2$$

动大 固定点

$$= \left(D^2 f(x_0)\left(\frac{y-x_0}{\|y-x_0\|}, \frac{y-x_0}{\|y-x_0\|}\right) + \|D^2 f(c) - D^2 f(x_0)\|\right)\|y-x_0\|^2$$

要的是固定x_0得到的 bilinear fum，对任意的 y, x_0 取2次時恒<0

⇒ 最簡單的方式就是設$Df(x_0)$負定

這項不能正太多，要多小有多小

⇒ 只要$D^2 f$在x_0連續

⇒ 已假設C^2

$\because f$ is of C^2

$\therefore D^2f$ is cont. on A

$\Rightarrow D^2f$ is cont at x_0.

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall y \in \underline{D(x_0, \delta)} \subset A$,

$\|D^2f(y) - D^2f(x_0)\| < \varepsilon$

↳ C在 x_0, y 线段上, 故 C 必在此 ball 中.

$\because C$ lies in the line segment of x_0 and y_0,

$\therefore C \in D(x_0, \delta)$

$\Rightarrow \|D^2f(C) - D^2f(x_0)\| < \varepsilon$. → ε 选多少, 使前一步加上此 ε 会比负

Notice $\frac{y - x_0}{\|y - x_0\|} \in S^1 = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$

that

$\because D^2f(x_0)$ is a bilinear form on $\mathbb{R}^n \times \mathbb{R}^n$

$\therefore D^2f(x_0)(y, y)$ is cont. on $\mathbb{R}^n$

$\because S^1$ is compact

$\therefore D^2f(x_0)$ has a maximum point at $\bar{x}$.

$\because D^2f(x_0)$ is negative definite

$\therefore D^2f(x_0)(\bar{x}, \bar{x}) < 0$. Choose $\varepsilon = \frac{-Df(x_0)(\bar{x}, \bar{x})}{2}$

Then $\forall y \in D(x_0, \delta)$, $D^2f(C)(y - x_0, y - x_0)$

$< D^2f(x_0)(\bar{x}, \bar{x}) + \left[\frac{-Df(x_0)(\bar{x}, \bar{x})}{2}\right] < 0$.

Therefore $f(x_0)$ is a local maximum.

② Suppose that $H_{x_0}(f)$ is not negative semidefinite. Then $\exists\, 0 \neq \bar{x} \in \mathbb{R}^n$ s.t.

$H_{x_0}f(\bar{x}, \bar{x}) > 0$.

Let $g(t) = f(x_0 + t\bar{x})$

Double A

$\because$ A is open
$\therefore$ f is defined in a nbd of x_0
$\Rightarrow$ g is defined a nbd of $0 \in \mathbb{R}$
$\because$ f is C^2
$\therefore$ g is twice diff. in a nbd of $0 \in \mathbb{R}$
$\because$ f has a local max. at x_0
$\therefore$ g has a local maximum at 0
$\Rightarrow$ $g'(0) = 0$.

$$\Rightarrow g''(0) = \sum_{i,k=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_k}(x_0)\, \bar{x}_i \bar{x}_k$$

$$= H_{x_0} f(\bar{x}, \bar{x})$$

$$> 0$$

$\Rightarrow$ By second derivative test, g has a local minimum at x_0 $\rightarrow\leftarrow$
$\Rightarrow$ $H_{x_0}(f)$ is negative semidefinite.

Q.E.D.

Remark. For the case of local minima, the negative in ① and ② in the above Thm is changed to positive in ① and ②.

Note. Let $A = (a_{ij})$ be $n \times n$ matrix. Let $\Delta_k =$ the determinant of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{k1} & \cdots & \cdots & a_{kk} \end{pmatrix}$$

(i.e. A中左上角的 $k \times k$ matrix)

Thm. ① A symmetric matrix is positive definite
$\iff \Delta_k > 0,\ \forall k = 1, \dots, n$
② A symmetric matrix is positive semidefinite
$\Rightarrow \Delta_k \geq 0,\ \forall k = 1, \dots, n$. ($\Leftarrow$ 有反例)

Cor. ① A symmetrix matrix A is negative definite $\iff \Delta_k < 0$, for k odd and $\Delta_k > 0$ for k even
② A symmetrix matrix A is negative semidefinite $\Rightarrow \Delta_k \leq 0$, $\forall k$ odd and $\Delta_k \geq 0$ for k even

Remark. Assume x_0 is a critical pt of f and f is of C^2.
$$\Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}$$
$\Rightarrow H_{x_0}(f)$ is symmetric.
$$\Rightarrow \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right) \text{ is symmetric.}$$
If $\Delta_k > 0$ for some odd k, then $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right)$ is not negative (semi) definite
$\Rightarrow f$ cannot have a local maximum at x_0.

If $\Delta_k < 0$ for some even k, then $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right)$ is not positive (semi) definite nor negative (semi) definite.

Therefore x_0 must be a saddle point.

⇒ 当 $n=2$ 時就是微積分二的二次導数檢驗法！此即為何成立.

ex. Show that the matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ is positive definite $\Leftrightarrow a>0$, $ad-bc>0$.

-sol-. $\Delta_1 = a$, $\Delta_2 = ad-bc$

ex. Investigate the nature of the critical point $(0,0)$ of $f(x,y)=x^2-xy+y^2$

-sol-.
$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 2x-y \\ \frac{\partial f}{\partial y}(x,y) = -x+2y \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 f}{\partial x^2}(x,y)=2, \ \frac{\partial^2 f}{\partial y^2}(x,y)=2 \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) = -1 \end{cases}$$

$$\Rightarrow H_{(0,0)}(f) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$\because \Delta_1 = 2 > 0$ and $\Delta_2 = 3 > 0$

$\therefore H_{(0,0)}(f)$ is positive-definite

$\Rightarrow$ By Thm, f has a local min at $(0,0)$ #

Ch7 The inverse and Implicit function Theorems and related topics.

§7.1 Inverse Function Thm.

System of n linear eq. of n variables.

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x^n = y_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x^n = y_n \end{cases}$$

can be solved uniquely for $x_1, \ldots, x_n$ when $y_1, \ldots, y_n$ are given if the matrix (a_{ij}) is nonsingular. that is, $\det(a_{ij}) \neq 0$.

n變數 + n relation, 何時有解?

When can we solve a system of the form

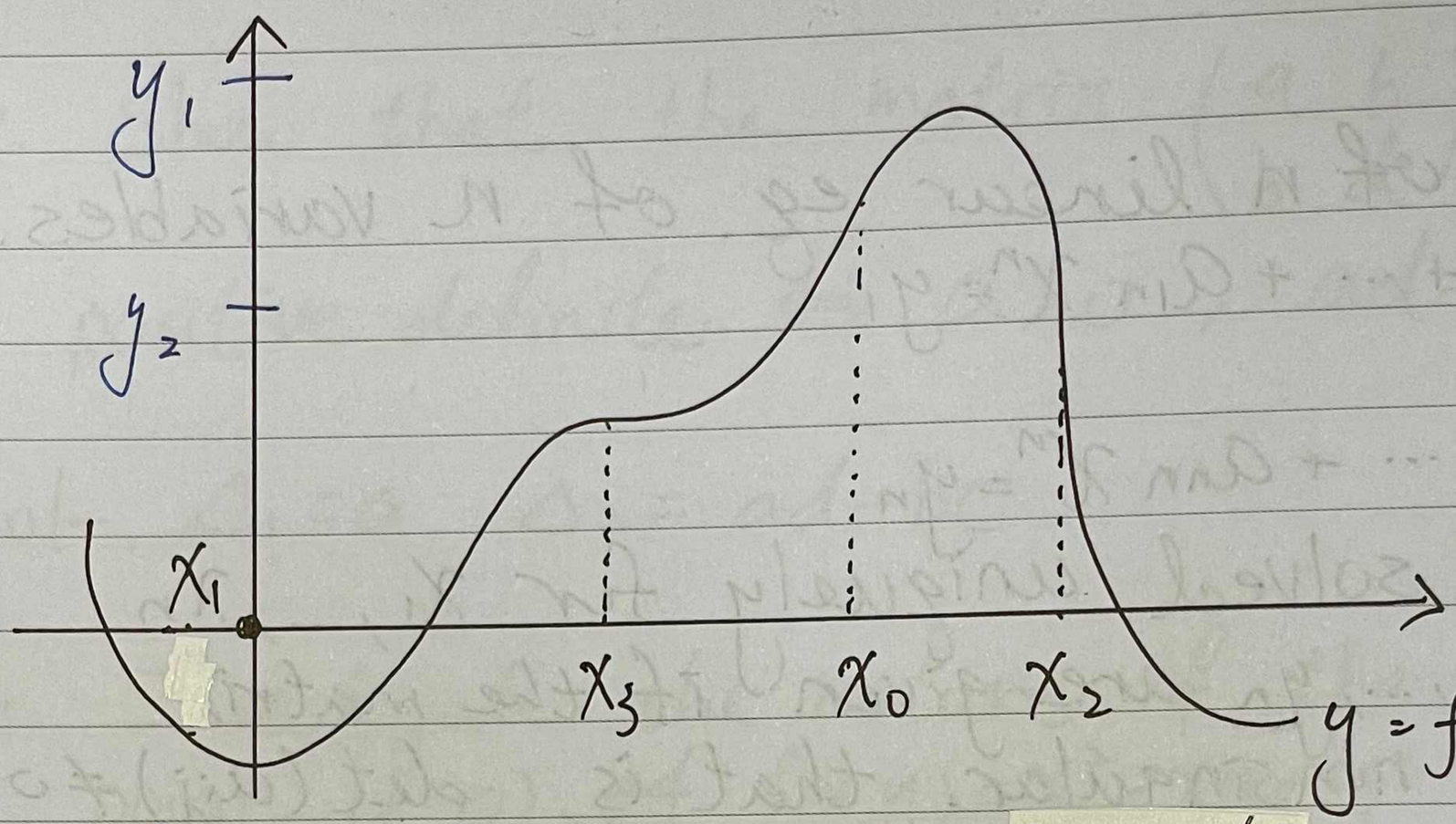
$$\begin{cases} f_1(x_1, \ldots, x_n) = y_1 \\ \vdots \\ f_n(x_1, \ldots, x_n) = y_n \end{cases} \Rightarrow (a_{ji}) \to \left(\frac{\partial f_j}{\partial x_i}\right)$$

Let $f = (f_1, \ldots, f_n)$ and $x = (x_1, \ldots, x_n)$. Then $\left(\frac{\partial f_j}{\partial x_i}\right) = Df$. ↳ $\mathbb{R}^n \to \mathbb{R}^n$

Def. The Jacobian determinant is defined by $Jf(x) = \det(Df(x))$. Sometimes write Jf as $\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$

猜：$Jf(x) \neq 0$ 時有解.

Let $f: \mathbb{R} \to \mathbb{R}$



$Jf(x_1) = Jf(x_3) = 0$ ⇒ 但 x_1 附近無一对一，x_3 卻可，故未必可判斷

$Jf(x_0) \neq 0$ ⇒ 給 y 也未必能找到 x，如：y_1.

只能解值域裡的 y

⇒ 此問題下的太大，無法 solve. 頂多 solve image 裡的 y

⇒ 就算只解值域裡的 y，解也未必唯一存在 如：y_2

⇒ 不要看整个图形，只看一个点（如：x_0）的附近，則解会存在且唯一.

Thm. (Inverse function Thm)

Let $A \subset \mathbb{R}^n$ be open and let $f: A \subset \mathbb{R}^n \to \mathbb{R}^n$ be of class C^1. Let $x_0 \in A$. If $Jf(x_0) \neq 0$ then $\exists$ a nbd. U of x_0 in A and a nbd W of $f(x_0)$ s.t. $f(U) = W$ and $f|_U$

has a C^1 inverse $f^{-1}: W \to V$. Moreover, for $y \in W$ and $x = f^{-1}(y)$, $Df^{-1}(y) = [Df(x)]^{-1}$, the inverse of $Df(x)$ (as a linear map). If f is of C^p, $p \geq 1$, then so is f^{-1}.

思考.

$$\|Df(x_0)(x)\| \leq \sum_{i=1}^{n} \|x_i \, Df(x_0)(e_i)\|$$

(倍数 → 向量)

by Schwarz ineq. $\leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \|Df(x_0)(e_i)\|^2\right)^{\frac{1}{2}}$

$$= \|x\| \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial f_j}{\partial x_i}(x_0)\right)^2\right)$$

($\frac{\partial f_j}{\partial x_i}(x_0)$)

$= \|x\| \, \|Df(x_0)\|$ → 在欧式空间的长度

Prop. $\|Df(x_0)(x)\| = \|x\| \|Df(x_0)\| \Rightarrow$ 可微 imply local Lipchiz

最重要之定理为 Contractim mapping principle (只要 domain 夠好), 当 Lipchiz constant < 1 時就是 Contraction

但不一定能控制 $\|Df(x_0)\|$ 的長度是否 < 1, 因此可加一輔助函数使這件事能滿足, 即: 設 g 使

$\|D(g+f)(x_0)\| = \|Dg(x_0) + Df(x_0)\|$ (不能用乘的, 会被弱化)

$\leq \|Dg(x_0)\| + \|Df(x_0)\|$ 乘系数也不可, 会失有 local

$= 0$ (能控制到 < 1, 不如直接 $= 0$) Lipchiz)

只是 g 難找!! 除非 $Df(x_0) = I_n$, 即 $f(x) = x$

令 $\tilde{f} = f + g$, 若能做到 $Df(x_0) = I_n$, $D\tilde{f}(x_0) = 0$

則 只要 $D\tilde{f}$ 連续, 則 x 夠靠近 x_0 時, $\|D\tilde{f}(x_0)\|$

要多小有多小，f 為 contraction mapping.
→ 可以來證反函數定理了.

pf. Consider $L(\mathbb{R}^n, \mathbb{R}^n) \simeq M_{n\times n} \simeq \mathbb{R}^{n^2}$ as a normed space. Let $GL(\mathbb{R}^n, \mathbb{R}^n)$ = the set of all $n\times n$ invertible matrices (or the invertible linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$).
Thus $GL(\mathbb{R}^n, \mathbb{R}^n) = \{A \in M_{n\times n} \mid \det(A) \neq 0\}$.
Let $\mathcal{L}^{-1}: GL(\mathbb{R}^n, \mathbb{R}^n) \to GL(\mathbb{R}^n, \mathbb{R}^n)$ denote the map that takes on an invertible matrix A to its inverse.

Lemma. ① $GL(\mathbb{R}^n, \mathbb{R}^n)$ is an open set in $M_{n\times n}$.
② $\mathcal{L}^{-1}$ is a C^∞-mapping.

思考
① open: ∵ $GL(\mathbb{R}^n, \mathbb{R}^n)$ 可刻劃成一individual det在 $M_{n\times n}$ 上取值不為0的地方
⇒ 證取值為0的地方closed就好，因此希望 det 是 conti. 可把 closed set $\{0\}$ 拉回 closed set

pf (of lemma): ① ∵ det: $M_{n\times n} \to \mathbb{R}$ is a function and is a polynomial of degree n of n^2 variables
∴ All order of partial derivatives of det is also a polynomial
⇒ det is C^∞
∵ $\mathbb{R}\setminus\{0\}$ is open in $\mathbb{R}$ and det is cont.
∴ $GL(\mathbb{R}^n, \mathbb{R}^n)$ is open in $M_{n\times n}$

② $\because \mathcal{L}^{-1}(A) = A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$, where

$\operatorname{adj}(A)$ is an $n \times n$ matrix s.t. $(\operatorname{adj} A)_{ij}$
$= (-1)^{i+j} \det A(j|i)$, where $\det A(j|i)$
↳ A拿掉j列i行

= the det. of the matrix obtained from A by deleting the j-th row and i-th column

$\therefore (\operatorname{adj} A)_{ij}$ is C^∞.
$\Rightarrow \mathcal{L}$ is C^∞

Q.E.D.

pf. (續): Claim 1. We may assume that $x_0 = 0$ and $f(x_0) = 0$.

pf of claim: Let $h(x) = f(x + x_0) - f(x_0)$. Then
$h(0) = 0$, and $Dh(x) = Df(x + x_0) \circ I_n$.
$= Df(x + x_0)$, and $Dh(0) = Df(x_0)$

Note: $\because h(x) = f(x + x_0) - f(x_0)$
附近

$\therefore$ 若 h 在 0 某有反函數，則 f 在 x_0 附近有反函數且
$\because h(x - x_0) = f(x) - f(x_0)$, and let $y = f(x)$
$\therefore x - x_0 = h^{-1}(y - f(x_0))$
$\Rightarrow f^{-1}(y) = x = h^{-1}(y - f(x_0)) + x_0$

Claim 2: We may assume that $Df(0) = I_n$

pf of claim 2:

思考

$D(\square f)(x_0) = D\square(x_0) \circ Df(x_0) = I_n$

$\Rightarrow D\square(x_0) = (Df(x_0))^{-1}$

$\Rightarrow$ 取口为 $(Df(x_0))^{-1}$，因为它是 linear map，微分後还是自己.

Let $T = (Df(0))^{-1}$. Then $T \circ f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function s.t. $(T \circ f)(0) = 0$ and $D(T \circ f)(0) = DT(f(0)) \circ Df(0) = I_n$.

Note: If $T \circ f$ has a inverse, then $(T \circ f)^{-1} = f^{-1} \circ T^{-1} \Rightarrow f^{-1} = (T \circ f)^{-1} \circ T$

Define $g : A \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function by $g(x) = x - f(x)$. Then g is of class C^p if f is of class C^p. Moreover $g(x)$ and $Dg(0) = I_n - I_n = 0$.

$\because g$ is C^1

$\therefore Dg$ is cont. at 0

$\Rightarrow$ For $\frac{1}{2n} > 0$, $\exists r > 0$ s.t. $\forall x \in A$ with

要控制 Lipchitz constant 与 domain

$\|x\| \leq r$, $\|Dg(x) - Dg(0)\| = \|Dg(x)\| < \frac{1}{2n}$

要 complete 才能用 contraction principle

Let $g=(g_1,\dots,g_n)$. Then $Dg=\begin{pmatrix} Dg_1 \\ \vdots \\ Dg_n \end{pmatrix}$.

$\forall x\in B(0,r)$, $\|Dg_i(x)\|\le\|Dg(x)\|<\frac{1}{2n}$, $\forall i=1,\dots,n$.

$\because B(0,r)$ is a convex set. $\forall x\in B(0,r)$, by Mean-value Thm, $\exists c_1,\dots,c_n\in B(0,r)$ s.t.

$g_i(x)-g_i(0)\overset{=0}{=}Dg_i(c_i)(x-0)=\nabla g_i(c)\cdot x$, $\forall i=1,\dots,n$

$\Rightarrow |g_i(x)|\le\|Dg_i(c)\|\,\|x\|\le\frac{1}{2n}\|x\|$, $\forall i=1,\dots,n$

$\Rightarrow \|g(x)\|\le\sum_{i=1}^{n}|g_i(x)|\le\frac{\|x\|}{2}\le\frac{r}{2}$

$\Rightarrow g: B(0,r)\to B(0,r)$. Similarly, for $\forall x_1,x_2\in B(0,r)$, $\|g(x_1)-g(x_2)\|\le\frac{1}{2}\|x_1-x_2\|$

$\Rightarrow g$ is a contraction map

$\because B(0,r)$ is complete

$\therefore$ By contraction mapping principle, $\exists!\, x_0\in B(0,r)$ s.t. $g(x_0)=x_0$

$\Rightarrow x_0-f(x_0)=x_0$

$\Rightarrow f(x_0)=0$.

(思考 | 对其它的 y, $F(x)=f(x)-y$, 其 g 为 $x-f(x)+y$)

Define $g_y: B(0,r)\subset\mathbb{R}^n\to\mathbb{R}^n$ by $g_y(x)+y$ for y fixed. $\|g_y(x)\|\le\|y\|+\|g(x)\|\le\|y\|+\frac{\|x\|}{2}$

$\le\|y\|+\frac{r}{2}$

$\Rightarrow\forall y\in B(0,\frac{r}{2})$, $g_y: B(0,r)\to B(0,r)$

$\because g_y$ is a contraction map and $B(0,r)$ is complete

$\therefore$ By contraction mapping principle, $\exists!\,\bar{x}\in B(0,r)$ s.t. $g_y(\bar{x})=\bar{x}$

$\Rightarrow \exists!\ \bar{x} \in B(0,r)$ s.t. $f(\bar{x}) = y$.

$\Rightarrow$ f has an inverse $f^{-1}: B(0, \frac{r}{2}) \subset \mathbb{R}^n \to B(0,r)$ which is defined by $f^{-1}(y)$ = the unique fixed point of g_y.

Notice that from the proof. We have $f^{-1}|_{D(0,\frac{r}{2})} : D(0, \frac{r}{2}) \to D(0,r)$. Hence we write $f^{-1}|_{D(0,\frac{r}{2})}$ by f^{-1}.

Claim 4: The inverse f^{-1} is cont. on $D(0, \frac{r}{2})$.

pf of claim: Let $y_1, y_2 \in D(0, \frac{r}{2})$ and let $f^{-1}(y_1) = x_1$, $f^{-1}(y_2) = x_2$. Then $g_{y_1}(x_1) = x_1$ and $g_{y_2}(x_2) = x_2$

$$\begin{aligned}
\Rightarrow \|x_1 - x_2\| &= g_{y_1}(x_1) - g_{y_2}(x_2) \\
&= \|(y_1 + g(x_1)) - (y_2 + g(x_2))\| \\
&\le \|y_1 - y_2\| + \|g(x_1) - g(x_2)\| \\
&\le \|y_1 - y_2\| + \tfrac{1}{2}\|x_1 - x_2\|
\end{aligned}$$

$\Rightarrow \|x_1 - x_2\| \le 2\|y_1 - y_2\|$

$\Rightarrow$ f^{-1} is Lipschitz with Lipschitz constant ≤ 2

$\Rightarrow$ f^{-1} is cont. on $D(0, \frac{r}{2})$.

思考

1. 如果 f^{-1} is diff at y, then $f^{-1} \circ f$ is diff at x, when $f^{-1}(x) = y$.

$\Rightarrow I_n = Df^{-1}(y) \circ Df(x)$

$\Rightarrow Df^{-1}(y) = (Df(x))^{-1}$

∵ f is C^1 on $D(0,r)$

∴ $Df : D(0,r) \to M_{n\times n}$ is cont.

∵ $Df(0) = I_n \in GL(\mathbb{R}^n, \mathbb{R}^n)$ and $GL(\mathbb{R}^n, \mathbb{R}^n)$ is open

∴ $\exists\ \bar{r} > 0$ s.t. $D(I_n, \bar{r}) \subset GL(\mathbb{R}^n, \mathbb{R}^n)$

∵ Df is cont. at 0

∴ $\exists\ \tilde{r} < r$ s.t. $Df(D(0,\tilde{r})) \subset D(I_n, \bar{r})$.

即：当 $x \in D(0,\tilde{r})$, $Df(x)$ is invertible!

∵ $f^{-1}(0) = 0$ and f^{-1} is cont. at 0.

∴ $\exists\ \tilde{r}_0 > 0$ s.t. $f^{-1}(D(0,\tilde{r}_0)) \subset D(0,\tilde{r})$

Write $f^{-1}|_{D(0,\tilde{r}_0)}$ as f^{-1}. Then $f^{-1} : D(0,\tilde{r}_0) \to D(0,\tilde{r})$ is Lipchitz. with Lipschitz constant ≤ 2. s.t. $[Df(f^{-1}(y))]^{-1}$ exists

Claim 5. The inverse f^{-1} is diff. from $D(0,\tilde{r}_0)$ to $D(0,\tilde{r})$

Pf of claim 5: Let $\bar{y} \in D(0,\tilde{r}_0)$. Let $\varepsilon > 0$,

Let $y \in D(0,\tilde{r}_0)$ with $f^{-1}(y) = x$ and $f^{-1}(\bar{y}) = \bar{x}$

$$\|f^{-1}(y) - f^{-1}(\bar{y}) - (Df(\bar{x}))^{-1}(y-\bar{y})\|$$

$$= \|x - \bar{x} - (Df(\bar{x}))^{-1}(f(x) - f(\bar{x}))\|$$

可估, $\because Df(\bar{x})(x-\bar{x})$

$$= \|[Df(\bar{x})]^{-1}(Df(\bar{x})(x-\bar{x}) - (f(x)-f(\bar{x})))\|$$

$$\le \|(Df(\bar{x}))^{-1}\|\ \|f(x) - f(\bar{x}) - Df(\bar{x})(x-\bar{x})\|$$

∵ f is diff at $\bar{x}$

∴ For $\dfrac{\varepsilon}{\|[Df(\bar{x})]^{-1}\|}$, $\exists\ \delta > 0$ s.t. $\forall x \in D(\bar{x},\delta)$

$\subset D(0,\tilde{r})$ s.t.

$\|f(x)-f(\bar{x})-Df(\bar{x})(x-\bar{x})\| < \frac{\varepsilon}{\|[Df(\bar{x})]^{-1}\|}\|x-\bar{x}\|$

$\Rightarrow \forall x \in D(\bar{x},\delta) \subset D(0,\tilde{r}),\ \|y-\bar{y}-(Df(\bar{x}))^{-1}(y-\bar{y})\|$

$< \varepsilon\|x-\bar{x}\|$

$< 2\varepsilon\|y-\bar{y}\|$

$\Rightarrow \exists \tilde{\delta}$ s.t. $\tilde{\delta} < \frac{\delta}{2}$ s.t. $D(\bar{y},\tilde{\delta}) \subset D(0,\tilde{r}_0)$,

$\|y-\bar{y}-[Df(\bar{x})]^{-1}(y-\bar{y})\| < 2\varepsilon\|y-\bar{y}\|$

$\Rightarrow f^{-1}$ is diff at $\bar{y}$

$\Rightarrow f^{-1}$ is diff on $D(0,\tilde{r}_0)$

Let $W = D(0,\tilde{r}_0)$, and $U = Im(f^{-1}) =$ the preimage, $(f|_{D(0,\tilde{r})})^{-1}(W)$.

$\because f$ is cont and $D(0,\tilde{r})$ is open

$\therefore U$ is open in $\mathbb{R}^n$

$\Rightarrow f^{-1}: W \to U$ is a diff and bijection function.

Claim b: f^{-1} is of class C^p

pf of claim: $\because Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$, $\forall y \in W$

Notice that

$\because f^{-1}$ is cont. and Df is cont.

$\therefore Df^{-1}$ is cont. ← 合成函數 and $\mathcal{L}_r^{-1}$ is cont.

$\Rightarrow f^{-1}$ is C^1

Let $f \in C^p$, $p > 1$, Then $Df \in C^{p-1}$ with $p-1 \geq 1$

$\because f^{-1} \in C^1$ and $Df \in C^{p-1}$ and $\mathcal{L}_r$ is C^∞

$\therefore Df$ is C^1

$\Rightarrow f$ is C^2

Double A

$\Rightarrow$ by this process, we have $f^{-1} \in C^3, C^4, \dots, C^p$

This ends the proof of inverse function Thm.

Q.E.D.

Remark. Inverse function Thm is useful because it tells us when there are solutions $f^{-1}(y)$ and how to differentiate the solutions without solving the equation $f^{-1}(y)$ explicitly.

e.g. Consider the equations $\frac{x^4+y^4}{x} = u(x,y)$ and $\sin x + \cos y = v(x,y)$. Near which points (x,y) can we solve for x, y in terms of u, v?

-sol-. That is, near which points, the function $f(x,y) = (u(x,y), v(x,y))$ has an (local) inverse.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{3x^4-y^4}{x^2} & \frac{4y^3}{x} \\ \cos x & -\sin y \end{vmatrix}$$

$$= \frac{\sin y}{x^2}(y^4 - 3x^4) - \frac{4y^3}{x}\cos x \quad -(*)$$

Ans: 在使 $(*)$ 不为 0 之点 $\#$

e.g. Same assumption as above. Find

$$\frac{\partial x}{\partial u}\left(\frac{\pi^3}{4}, 1\right)$$

↳ 对应到上 x, y 为何？$(\ ,\ ) \xrightarrow{f} \left(\frac{\pi^3}{4}, 1\right)$

-sol-.

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{1}{Jf}\begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial u}{\partial y} \\ -\frac{\partial v}{\partial x} & \frac{\partial v}{\partial x} \end{pmatrix}$$

$\because f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \left(\frac{\pi^3}{4}, 1\right)$ and

$$Jf\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \begin{vmatrix} \frac{\pi^2}{2} & \pi^2 \\ 0 & -1 \end{vmatrix} = -\frac{\pi^2}{2}$$

$$\therefore \frac{\partial x}{\partial u}\left(\frac{\pi^3}{4}, 1\right) = \frac{-2}{\pi^2}\left(-\sin\frac{\pi}{2}\right) = \frac{2}{\pi^2} \quad \#$$

Note: If $f: \mathbb{R} \to \mathbb{R}$ is diff and $f'(x) \neq 0$ for all x.

$\therefore$ f' satisfies intermediate value thm

$\therefore$ Either $f' > 0$ or $f' < 0$ on $\mathbb{R}$

$\Rightarrow$ Either $f \nearrow$ on $\mathbb{R}$ or $f \searrow$ on $\mathbb{R}$

$\Rightarrow$ f is 1-1 on $\mathbb{R}$

Show that this is not true in the case $\mathbb{R}^2$ or $\mathbb{R}^1$ for $n \geq 2$.

Def. A function $f: A \subset \mathbb{R}^n \to B \subset \mathbb{R}^n$ is said to be (C^p) diffeomorphism if it is of class C^p and has inverse $f^{-1}: B \subset \mathbb{R}^n \to A \subset \mathbb{R}^n$ also of (class C^p).

Double A

§7.2 Implicit function Thm.

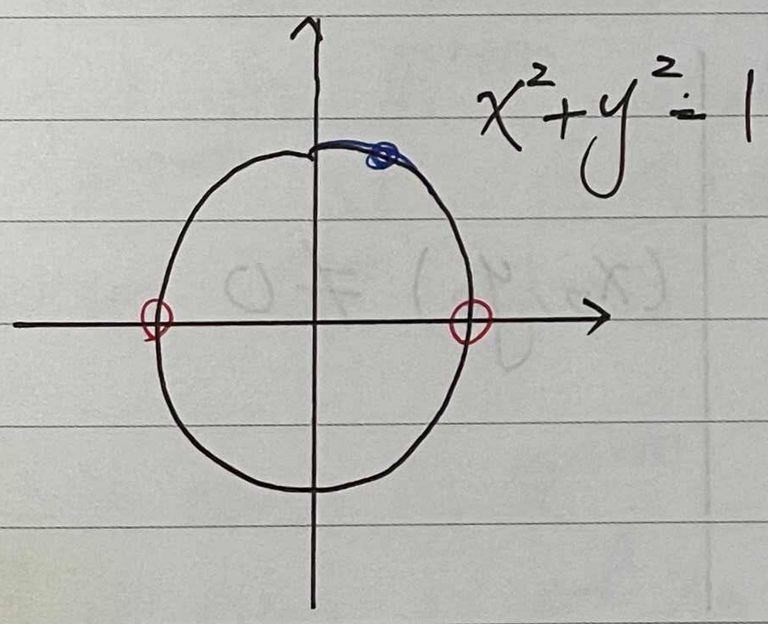
Q: For an under determined system of m equations in n+m variables

$$\begin{cases} F_1(x_1, \dots, x_{n+m}) = 0 \\ \vdots \\ F_m(x_1, \dots, x_{n+m}) = 0 \end{cases}$$

there is ∞多 solution in general.
那如何讓此system有finite solution呢？甚至於唯一解呢?

A: 從這n+m个变数挑出最合产的m个有解的变数，剩下n个当成常数

Let $F(x,y) = x^2+y^2-1=0$



在(1,0)和(-1,0)此二点 $F(x,y)=0$ 所得的y解不会是x的函数，故無法解y. 此時(1,0) and (-1,0)对F(x,y)滿足的條件為何？⇒避開之

$$\frac{\partial F}{\partial y}(1,0) = \frac{\partial F}{\partial y}(-1,0) = 0$$

For $y \neq 0$. 由 $F(x,y)=0$ 所得的y解是x的函数且唯一。此時对 $F(x,y)=0$ 滿足的條件為何呢?

Double A

$\frac{\partial F}{\partial y}(x,y) \neq 0$

$\Rightarrow$ 剛好在此時兩個條件把可解，不可解分開

可猜測應挑選怎樣的 m 個變數

$\Rightarrow$ 猜：要挑使 $\frac{\partial(F_1,\dots,F_m)}{\partial(x_{i_1},\dots,x_{i_m})}(\bar{x}) \neq 0$ 之 m 個變數

Let $F=(F_1,\dots,F_m)$ $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_m)$

Thm. (Implicit Function Thm)

Let $F: A \subset \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a function of class C^p on the open set A. Suppose that $(x_0,y_0) \in A$ and $F(x_0,y_0)=0$. Let $F=(F_1,\dots,F_m)$. If the determinant

$$\Delta = \begin{vmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{vmatrix} (x_0,y_0) \neq 0$$

open

then $\exists$ a nbh U of x_0 in $\mathbb{R}^n$ and a nbh V of y_0 in $\mathbb{R}^m$ and a unique function $f: U \to V$ s.t. $F(x,f(x))=0$, $\forall x \in U$. Furthermore, f is of class C^p.

pf. The Thm is proved by Inverse function Thm. Define $G: A \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by

$G(x,y) = (\underline{x}, F(x,y))$

↳ 希望 Jacobian 不會太多 $\Rightarrow$ 挑 x，因其 Jacobian 為 I_n

$\because$ F is C^p

$\therefore$ G is C^p and $G(x_0, y_0) = (x_0, 0)$.

$$\because DG(x,y) = \left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ & & I_n & \end{matrix}}^{n} & \overbrace{\begin{matrix} 0 & \cdots & 0 \\ & O & \end{matrix}}^{m} \\ \hline \begin{matrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_1} \end{matrix} & \begin{matrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{matrix} \end{array}\right) \begin{matrix} \\ \\ n \\ \\ m \end{matrix}$$

$\therefore$ $JG(x_0, y_0) = \Delta(x_0, y_0) \neq 0$

By inverse function Thm, $\exists$ a nbd S of (x_0, y_0) in $\mathbb{R}^{n+m}$ and a nbd W of $(x_0, 0)$ in $\mathbb{R}^{n+m}$ s.t. $G(S) = W$ and G has a C^p inverse $G^{-1}: W \to S$.

$\because$ S is open in $\mathbb{R}^{n+m}$

$\therefore$ $\exists$ a nbh U of x_0 in $\mathbb{R}^n$ and a nbd V of y_0 in $\mathbb{R}^m$ s.t. $U \times V \subset S$.

(插入一長方形)

Let $G(U \times V) = Y$. Then $Y \subset W$ is open in $\mathbb{R}^m$ and $(x_0, 0) \in Y$ s.t. $G: U \times V \to Y$ is a bijection C^p function.

$\because$ $G(x,y) = (x, F(x,y))$

$\therefore$ G^{-1} is of the form $G^{-1}(x,w) = (x, H(x,w))$

$\because$ G^{-1} is of C^p

$\therefore$ $H: Y \to V$ is of C^p with $H(x_0, 0) = y_0$
s.t. $F(x, H(x,w)) = w$
$\because (x_0, 0) \in Y$ and Y is open in $\mathbb{R}^{n+m}$
$\therefore \exists$ a nbd U of x_0 in $\mathbb{R}^n$ s.t. $U \times \{0\} \subset Y$
Define a function $f: U \to V \subset \mathbb{R}^m$ by $f(x) = H(x,0)$

$\Rightarrow$ Then $f(x_0) = y_0$ and $F(x, f(x)) = F(x, H(x,0))$
$= 0, \forall x \in U$.
$\because H$ is of C^p
$\therefore f$ is of C^p
By the inverse functin Thm, G^{-1} is uniquely determined. So H is uniquely determined
$\Rightarrow f$ is uniquely determined.

Q.E.D.

Cor. In the implicit function Thm, the $\frac{\partial f_i}{\partial x_i}$ are given by

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}$$

, where $f = (f_1, \ldots, f_m)$

pf. $\because F(x, f(x)) = 0$
$\therefore F_j(x, f(x)) = 0$, $\forall j = 1, \ldots, m$, where $F = (F_1, \ldots, F_m)$.

$$\Rightarrow 0 = \frac{\partial}{\partial x_i}\left(F_j(x, f(x))\right)$$

$$= \frac{\partial F}{\partial x_i} + \sum_{k=1}^{m} \frac{\partial F_j}{\partial y_k}(x, f(x)) \cdot \frac{\partial f_k}{\partial x_i}(x)$$

$(x, f(x))$

$$\Rightarrow -\frac{\partial F_j}{\partial x_i}(x, f(x)) = \sum_{k=1}^{m} \frac{\partial F_j}{\partial y_k}(x, f(x)) \frac{\partial f_k}{\partial x_i}(x)$$

矩阵乘法

$$\Rightarrow -\left(\frac{\partial F_j}{\partial x_i}\right)(x, f(x)) = \left[\left(\frac{\partial F_j}{\partial y_k}(x, f(x))\right)\left(\frac{\partial f_k}{\partial x_i}(x)\right)\right]$$

不要背！直接做 $F(x, f(x)) = 0$

解联立

Q.E.D.

Cor. If $m=1$, then $\frac{\partial f}{\partial x_i} = -\frac{\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial y}}$.

e.g. Consider the following system of equations:

$$\begin{cases} xu + yv^2 = 0 \\ xv^3 + y^2u^6 = 0 \end{cases}$$

Are they solvable for u and v in terms of $x=1, y=-1, u=1, v=-1$? Near $x=0, y=1, u=0, v=0$. Compute $\frac{\partial u}{\partial x}$ at $x=1, y=-1$ and at $x=0, y=1$, if it exists.

-sol-. Let $F(x, y, u, v) = (xu + yv^2, xv^3 + y^2u^6)$

$$\therefore \Delta = \begin{vmatrix} x & 2yv \\ 6y^2u^5 & 3xv^2 \end{vmatrix}$$

$$\therefore \Delta(1,-1,1,-1) = \begin{vmatrix} 1 & 2 \\ 6 & 3 \end{vmatrix} \neq 0$$

$\Rightarrow$ By Implicit functim Thm, $\exists!$ solution u and v in terms of x and y near $(1,-1,1,-1)$

$$\because \begin{cases} u + x\frac{\partial u}{\partial x} + 2yv\frac{\partial v}{\partial x} = 0 \\ v^3 + 3xv^2\frac{\partial v}{\partial x} + 6y^2u^5\frac{\partial u}{\partial x} = 0 \end{cases}$$

代入 $(1,-1,1,-1)$

$$\Rightarrow \begin{cases} 1 + \frac{\partial u}{\partial x}(1,-1) + 2\frac{\partial v}{\partial x}(1,-1) = 0 \\ -1 + 6\frac{\partial u}{\partial x}(1,-1) + 3\frac{\partial v}{\partial x}(1,-1) = 0 \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial x}(1,-1) = \frac{5}{9} \quad \#$$

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§7.3 Domain Straightening Thm.

Q: Let $f: A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}$ be a function. Let $c \in \text{Im} f$. 考虑 the set of the preimage $f^{-1}(c)$ 的维度

如果f不足条数，若考虑 c之 pre image，不大可能是整个A，所以在某一点附近维度不大可能是满的(n维)。最好就是n-1维。什麼時候 $f^{-1}(c)$ 维度是n-1维呢?

A: 若 $\exists i \in \{1, \dots, n\}$ s.t. $\frac{\partial f}{\partial x_i}(\bar{x}) \neq 0$, where $\bar{x} \in f^{-1}(c)$ and Df is cont. at $\bar{x}$

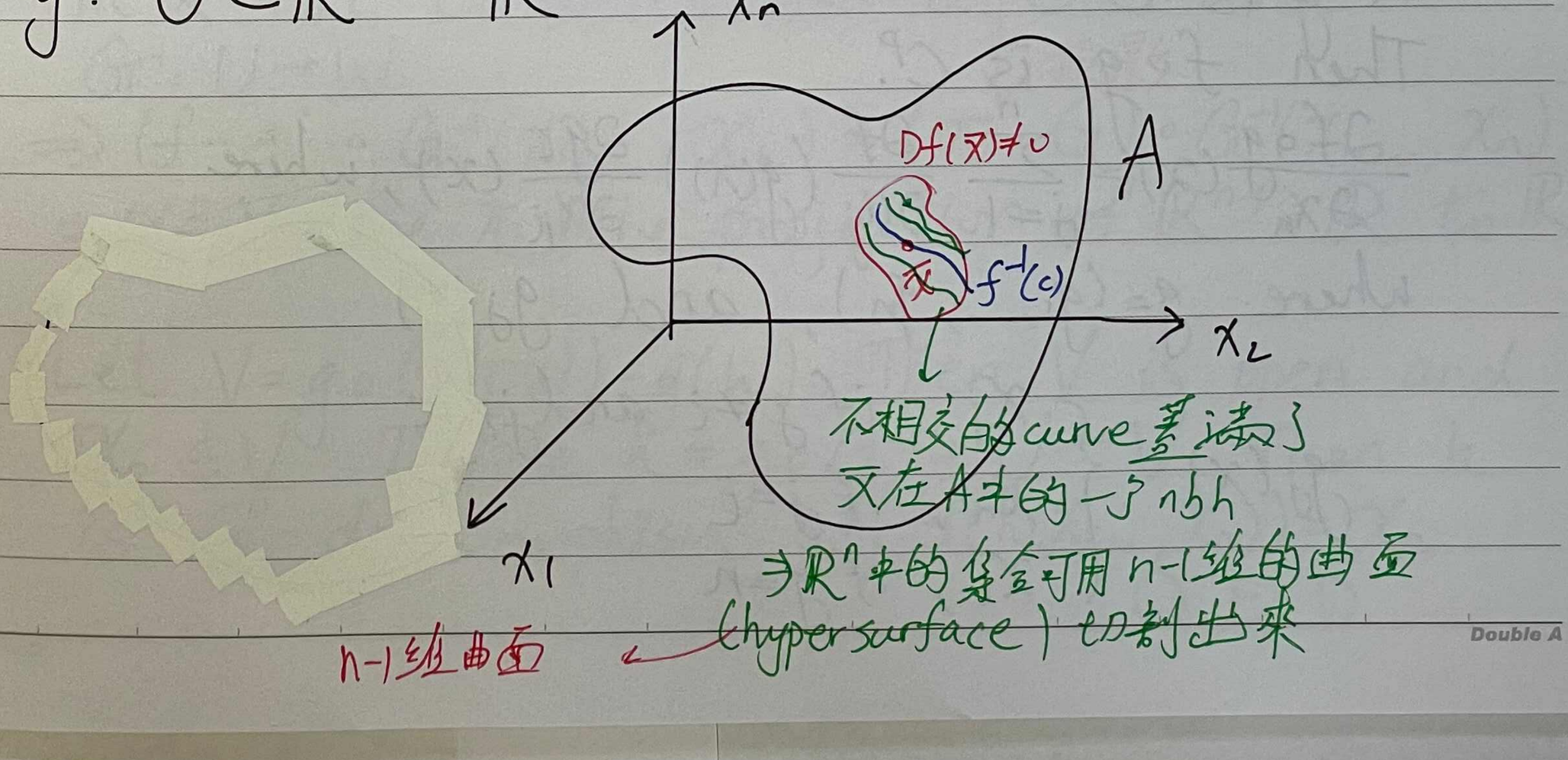
Implicit function thm tells us that

$\exists g: U \subset \mathbb{R}^n \to V \subset \mathbb{R}$ s.t. $x_i = g(x_1, \dots, \hat{x_i}, \dots, x_n)$ 被省略

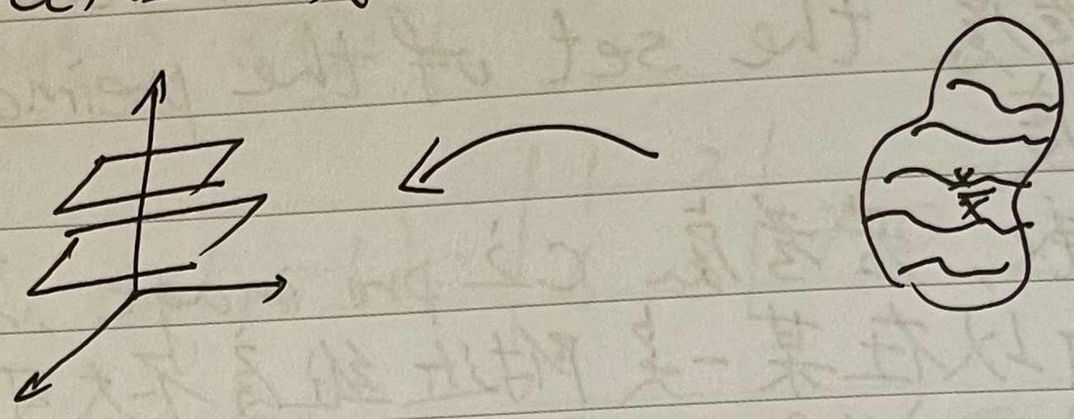
and $\bar{x}_i = g(\bar{x}_1, \dots, \hat{\bar{x}}_i, \dots, \bar{x}_n)$ where $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ s.t. $f(x_1, \dots, x_{i-1}, g(x_1, \dots \hat{x_i}, \dots x_n), x_{i+1}, \dots, x_n) = c$

$\Rightarrow (x_1, \dots, x_{i-1}, g(x_i \dots, \hat{x_i}, \dots, x_m), x_{i+1}, \dots, x_m) \in f^{-1}(c)$

Notice that $\{(x_1, \dots, x_{i-1}, g(x_1, \dots, \hat{x_i}, \dots, x_n), x_{i+1}, \dots, x_n) \mid x_1, \dots, \hat{x_i}, \dots, x_n \in U\}$ = the graph of $g: U \subset \mathbb{R}^{n-1} \to \mathbb{R}$



⇒存在一个(局部存在)坐标变换可以把这些 hypersurface (level set) 压平成平面



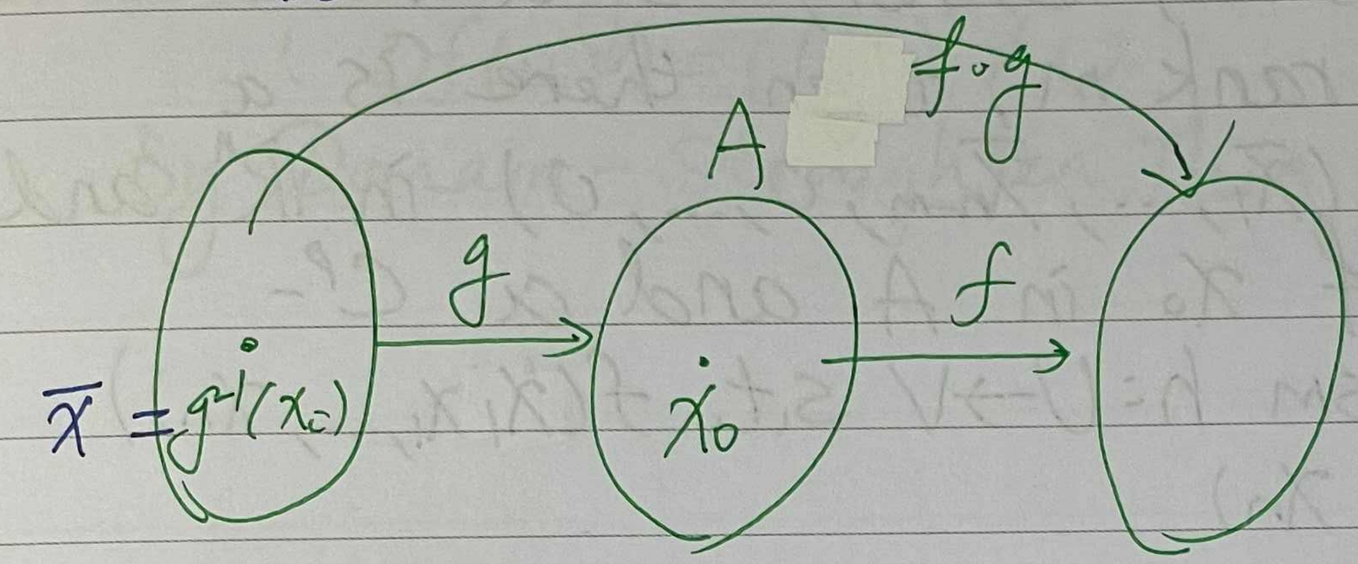
Thm. (Domain-Straightening Thm)
Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function C^p, $p \geq 1$, on the open set A. Let $x_0 \in A$ and $f(x_0) = 0$ and $Df(x_0) \neq 0$. Then $\exists$ a nbd U of $(\bar{x}, \ldots, \bar{x}_{n-1}, 0)$ in $\mathbb{R}^n$ and a nbd V of x_0 in A and a C^p-diffeomorphism $h: U \to V$ s.t. $f(h(x_1, \ldots, x_n)) = x_n$

pf. $\because Df(x_0) \neq 0$
$\therefore \exists\ i \in \{1, \ldots, n\}$ s.t. $\frac{\partial f}{\partial x_i}(x_0) \neq 0$.
Define $g: \mathbb{R}^n \to \mathbb{R}^n$ by $g(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$
$= (x_1, \ldots, x_{i-1}, x_n, x_{i+1}, \ldots, x_{n-1}, x_i)$
$\because g$ is 1-1 and linear
$\therefore g$ is C^∞.
Then $f \circ g$ is C^p.
$$\frac{\partial f \circ g}{\partial x_n}(x) = \sum_{j=1}^{n} \frac{\partial f}{\partial u_j}(g(x)) \frac{\partial g_i}{\partial x_n}(x),$$
where $g = (g_1, \ldots, g_n)$, and

$$g_j(x) = \begin{cases} x_j, & \text{if } j \neq i \text{ and } j \neq n \\ x_n, & \text{if } j = i \\ x_i, & \text{if } j = n \end{cases}$$

$$\Rightarrow \frac{\partial f\circ g}{\partial x_n} = \frac{\partial f}{\partial x_i}(g(x))$$

$$\Rightarrow \frac{\partial (f\circ g)}{\partial x_n}(g^{-1}(x_0)) = \frac{\partial f}{\partial x_i}(x_0) \neq 0$$



$\because$ g is a homeomorphism and A is open in $\mathbb{R}^n$

$\therefore$ $g^{-1}(A)$ = the image of $g^{-1} \triangleq B$ is open in $\mathbb{R}^n$ and $g^{-1}(x_0) \in B$

$B \subset$

Let $g^{-1}(x_0) = \bar{x}$. Define a function $G: \mathbb{R}^{n-1}\times\mathbb{R} \to \mathbb{R}^{n-1}\times\mathbb{R}$ by $G(x,y) = (x_i, f\circ g(x))$. Then G is C^p and $JG(\bar{x}) = \frac{\partial f}{\partial x_i}(x_0)$.

By inverse function Thm, $\exists$ an nbd W of $\bar{x}$ in B and a nbd U of $(\bar{x}_1, \dots, \bar{x}_{n-1}, 0)$ in $\mathbb{R}^n$ s.t. $G: W \to U$ has a C^p-inverse $G^{-1}: U \to W$

投影

$\Rightarrow (f\circ g)\circ G^{-1}(x_1, \dots, x_n) = (\pi\circ G)\circ G^{-1}(x_1, \dots, x_n)$

$= x_n$, where π = the projection of $\mathbb{R}^{n-1}\times\mathbb{R}$ to $\mathbb{R}$

Let $V = g\circ G^{-1}(U) = g(W)$. Then V is open and $x_0 \in V$. Define $h = g\circ G^{-1}: U \to V$. Then h is C^p-diffeomorphism s.t. $f(h(x_1, \dots, x_n)) = x_n$.

Q.E.D.

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Thm. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a C^p-function, $p \geq 1$, on the open set A, where $n \geq m$. Let $f(x_0) = 0$. If $Df(x_0)$, as a linear map, has rank m, then there is a nbd U of $(\bar{x}_1, \dots, \bar{x}_{n-m}, 0, \dots, 0)$ in $\mathbb{R}^n$ and a nbd V of x_0 in A and a C^p-diffeomorphism $h: U \to V$ s.t. $f(h(x_1, \dots, x_n)) = (x_{n-m+1}, \dots, x_n)$

pf. ex 18 on p441.

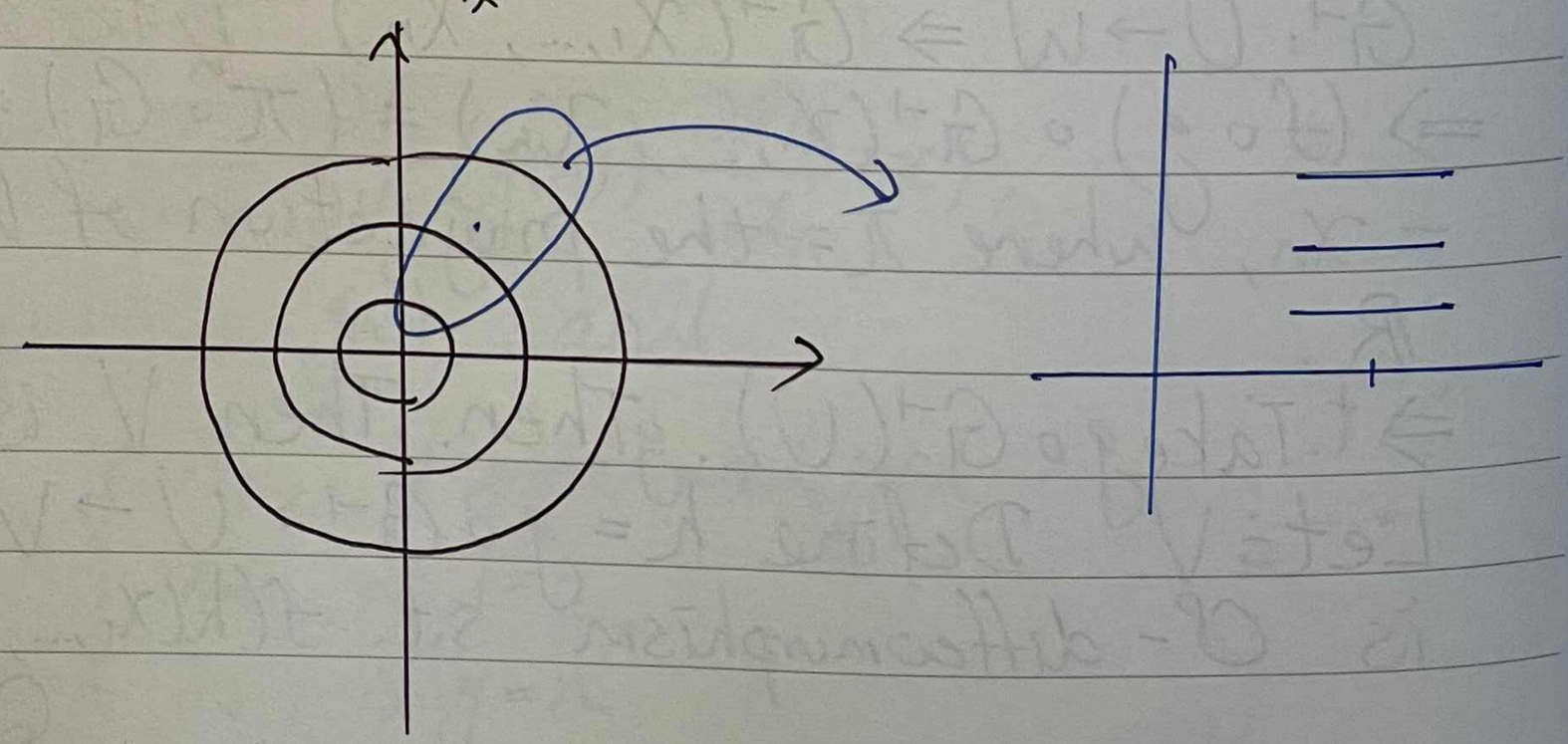
e.g. Let $f(x, y) = x^2 + y^2$. Can we "straighten out" f near $(0, 0)$? What about other points.

-sol-. $Df(x, y) = (2x, 2y)$. $\because Df(x, y) \neq 0$, if $(x, y) \neq (0, 0)$.

$\therefore$ It can be straighten out at $(x, y) \neq (0, 0)$

$\because f(0, 0) = 0$ and $f^{-1}(0) = \{(0, 0)\}$

$\therefore$ It can't be #



e.g. Let $f(x,y) = x^3 + x + y$. Can we straighten out f near $(0,0)$?

-sol-. $\because Df(x,y) = (3x^2+1, 1)$

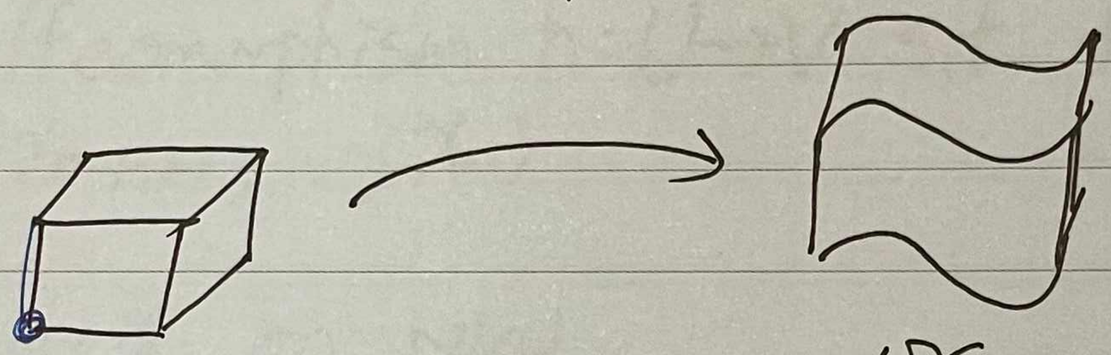
$\therefore Df(0,0) = (1,1) \neq 0$

By Domain-Straighten Thm, f can be. //

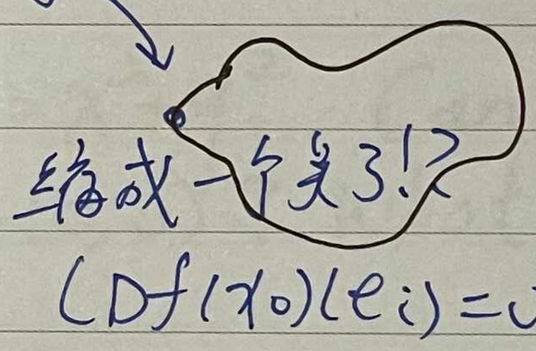
§7.4 Further Consequences of the Implicit Fu

Q: Let $f: A \overset{\text{open}}{\subset} \mathbb{R}^p \to \mathbb{R}^n$ be a function, where $p \le n$. 一般来說 Im(f) 不会超过定義域的维度 p. 那什麼時候 Im(f) 的維度 = p 呢?

Let $x \in A$, 以 x 為頂点的 p 维小立方体考虑此小立方体經 f 映射过去会是一个弯曲的 p- 体, 或是小於 p 维的曲体



某條边3这过去時



縮成一个点!?

$(Df(x_0)(e_i) = 0)$

Thm. (Range - Straightening Thm)

Let $f: A \subset \mathbb{R}^k \to \mathbb{R}^n$ be a function of class C^p, $p \ge 1$ and $k \le n$ on the open set A. Let $x_0 \in A$. If the rank of $Df(x_0)$ is k, then $\exists$ a nbd W of x_0 in A, a nbd U of $f(x_0)$ in $\mathbb{R}^n$ and an open set V in $\mathbb{R}^n$, and C^p-diffeomorphism $g: U \to V$ s.t. $g \circ f: W \to V$ has $g \circ f(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$

pf. $\because$ $Df(x_0)$ has rank k

$\therefore$ $\exists$ some k-k submatrix $Df(x_0)$ with a nonzero determinant.

By relabling, if necessary, we may

Double A

Function Theorem.

assume that

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_k}{\partial x_1} & & \frac{\partial f_n}{\partial x_n} \end{vmatrix} (x_0) \neq 0.$$

where $f = (f_1, \ldots, f_n)$. Define $\varphi : A \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by $\varphi(x, y) = f(x) + (0, y)$. Then $\varphi \in C^p$ and $\varphi(x_0, 0) = f(x_0)$.

Moreover, $\varphi = (\varphi_1, \ldots, \varphi_n)$ with $\varphi_j = f_j$, $\forall j \leq k$, and $\varphi_j = f_j + x_j$, $\forall j > k$, where $x_j = y_{j-k}$.

$$\Rightarrow D\varphi(x_0, 0) = \left(\begin{array}{ccc|c} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_k} & 0 \cdots 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_k} & 0 \cdots 0 \\ \hline \frac{\partial f_{k+1}}{\partial x_1} & \cdots & \frac{\partial f_{k+1}}{\partial x_k} & \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & & \frac{\partial f_n}{\partial x_k} & 0 \cdots 0 \end{array}\right.\left.\begin{array}{c} \\ \\ \\ \hline 1 \cdots 0 \\ \vdots \\ 0 \cdots 1 \end{array}\right) (x_0, 0)$$

(I_n)

$$\Rightarrow J\varphi(x_0, 0) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \vdots & & \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_k} \end{vmatrix} (x_0) \neq 0$$

By Inverse function Thm, $\exists$ a nbd V of $(x_0, 0)$ in $A \times \mathbb{R}^{n-k}$ and a nbd U of $f(x_0)$ in $\mathbb{R}^n$ s.t. $\varphi(V) = U$ and a C^p-function $g: U \to V$ s.t. $g = (\varphi|_V)^{-1}$

$\Rightarrow g \circ (f(x) + (0, y)) = (x, y)$, $\forall (x, y) \in V \subseteq A \times \mathbb{R}^{n-k}$

Take W be a nbd of x_0 in A s.t. $W \times \{0\} \subset V$. Then $g \circ f(x) = (x, 0)$, $\forall x \in W$.

Q.E.D.

Q: Let $f: A \overset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^N$ be a function.

由前面 $Df(x)$ 的 rank 的討論自然延伸出一个問題：当 $Df(x)$ 的 rank 皆是 m 時，此時 $\operatorname{Im} f$ 是一个 m 维度的集合 $\Rightarrow m \leq n$，則如何 n 个变数挑出 m 个变数来描述它的取值？

已经不可能去改变 f 本身的描述方式了（雖然它的值域是 m 维）那可不可以在定義域裡經过一个变数变换，在新的变数之下，用新的坐標来描述給的函数 f，使此子集的只用 m 个变数来表？

Thm. (Rank Thm)

Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^N$ be a C^p-function on the open set A. If $Df(x)$ has rank m for all x in a nbd of x_0 in A, then $\exists$ an open set V of $\mathbb{R}^n$ and a nbd U of x_0 in A and a C^p-diffeomorphism $h: V \to U$. s.t. $h \circ f(x_1, \ldots, x_n) = \tilde{f}(x_1, \ldots, x_m)$ for some C^p-function $\tilde{f}$

pf. WLOG, we may assume $x_0 = 0_n$ and $f(x_0) = 0 \in \mathbb{R}^N$ and $Df(x)$ has rank m, $\forall x \in A$.

Let $L = Df(x_0)$ and $\{c_1, \dots, c_n\}$ be a basis of $\mathbb{R}^n$ s.t. $\{c_{m+1}, \dots, c_n\}$ spans the null space of L.

Let $X_1 = \text{span}\{c_1, \dots, c_m\}$ and $X_2 = \text{span}\{c_{m+1}, \dots, c_n\}$. Let $Y_1 = \text{span}\{y_1 = L(c_1), \dots, y_m = L(c_m)\}$ and $\{y_{m+1}, \dots, y_N\}$ be chosen s.t. $\{y_1, \dots, y_m, y_{m+1}, \dots, y_N\}$ of $\mathbb{R}^N$. and let $Y_2 = \text{span}\{y_{m+1}, \dots, y_N\}$

Let P_1 and P_2 be linear transformatins in $\mathbb{R}^n$ defined by $P_1(x) = \sum_{i=1}^{n} \varphi_i(x) c_i$ and $P_2(x) = \sum_{i=m+1}^{n} \varphi_i(x) c_i$, where $x = \varphi_1(x) c_1 + \dots + \varphi_n(x) c_n$
Then $\text{Im} P_j = X_j$, $\forall j = 1, 2$.

Similarly, let Q_1, Q_2 be LTs in $\mathbb{R}^N$ s.t.
$Q_1(\bar{x}) = \sum_{i=1}^{m} \varphi_j(\bar{x}) y_i$, $Q_2(\bar{x}) = \sum_{i=M+1}^{N} \varphi_i(\bar{x}) y_i$
where $\bar{x} = \varphi_1(\bar{x}) y_1 + \dots + \varphi_N(\bar{x}) y_N$. Then $\text{Im} Q_j = Y_j$. $\forall j = 1, 2$. Clearly, $L_1 = L|_{X_1} : X_1 \to Y_1$ is a linear isomorphism. Let $T = L_1^{-1} : Y_1 \to X_1$.

Define $u : A \subset \mathbb{R}^n \to \mathbb{R}^n$ by $u(x) = T \circ Q_1 \circ f(x) + P_2(x)$

($f(x) \in \mathbb{R}^N$, $Q_1 \circ f(x) \in Y_1 \subset \mathbb{R}^N$, T: linear, $P_2(x) \in X \subset \mathbb{R}^n$, P_2: linear)

Then u is C^p and $u(0)=0$ and u maps $A\cap X_1$ into X_1

$\because Du(0) = T\circ Q_1\circ L + P_2$

$\therefore Du(0)(c_i) = c_i,\ \forall i=1,\dots,n.$

$\Rightarrow Du(0)$ is invertible

$\Rightarrow Ju(0)\neq 0$

By Inverse function Thm, $\exists$ a nbd U of 0 in A and a nbd V of 0 in $\mathbb{R}^n$ s.t. $V=u(U)$ and a C^p-inverse $w=(u|_U)^{-1}: V\to U$

We expect $f\circ w$ is the function we want.

$\Rightarrow$ 希望 $f\circ w$ 之与 m 个变数有关

$\Rightarrow$ 如何证 $f\circ w$ 之取值只跟 m 个变数有关？可用把剩下的 $n-m$ 个变数乱换後得到相同之值来证.

思考

即：$f\circ w(\underbrace{z_1,\dots,z_m}_{\text{有关}},\underbrace{z_{m+1},\dots,z_n}_{\text{無关緊要}})$

$-f\circ w(z_1,\dots,z_m,\overline{z_{m+1}},\dots,\overline{z_n}) = 0$ ← 用MVT去证!

要能用MVT $\Rightarrow$ 兩点连线要在 V 裡 $\Rightarrow$ V convex 嗎?

$\to$ 将 V 縮小成 r-ball 就好.

$f\circ w(z_1,\dots,z_m,z_{m+1},\dots,z_n) - f\circ w(z_1,\dots,z_m,\overline{z}_{m+1},\dots,\overline{z}_n)$

$\Rightarrow D(f\circ w)(c)\cdot(\underbrace{0,\dots,0}_{m},\underbrace{z_{m+1}-\overline{z}_{m+1},\dots,z_n-\overline{z}_n}_{\mathbb{R}^{n-m}})$

能証 $D(f\circ w)(z)(0\times\mathbb{R}^{n-m}) = 0,\ \forall z\in V$ 嗎?

Notice that $D(f\circ w)(z) = \underline{Df(w(z))}\circ w(z)$ has rank m, $\forall z\in V$.

rank m C^p-diffeomorphism

Re

$\because Z = U(\omega(z)) = T \circ Q_1 \circ (f \circ \omega)(z) + P_2(\omega(z))$,
$\forall z \in V$.

$\therefore L(z) = \boxed{L \circ T} = \boxed{Q_1 \circ (f \circ \omega)(z)}$, $\forall z \in V$.

$L|_{X_1} = L_1$ 在 X_1 上对消!

$\in X_1$ $\quad \in Y_1$

$= Q_1 \circ (f \circ \omega)(z)$, $\forall z \in V$.

$\Rightarrow L = Q_1 \circ (f \circ \omega)$, $\forall z \in V$.

$\Rightarrow L = Q_1 \circ D(f \circ \omega)(z)$, $\forall z \in V$.

$\because L$ and $D(f \circ \omega)(z)$ are rank m, $\forall z \in V$.

$\therefore Q$ is 1-1 on the Range of $D(f \circ \omega)(z)$.

$\Rightarrow$ If $\bar{z} \in \mathbb{R}^n$ s.t. $L(\bar{z}) = 0$, then $D(f \circ \omega)(z)(\bar{z}) = 0$.

$\Rightarrow D(f \circ \omega)(z)(z_2) = 0$, $\forall z_2 \in \underbrace{X_2}_{\ker L}$, $\forall z \in V$.

$\Rightarrow f \circ \omega(z + \bar{z}) = f \circ \omega(z)$, $\forall \bar{z} \in X_2$, $z \in V$.

Let α be the L.T. on $\mathbb{R}^n$ s.t. $\alpha(x_1, \dots, x_n)$ (坐标获向量)
$= \sum_{i=1}^{n} x_i c_i$. Let $\tilde{V} = \alpha^{-1}(V)$, then $\tilde{V}$ is open
in $\mathbb{R}^n$

$\Rightarrow$ for $\omega \circ \alpha : \tilde{V} \to U$ in C^r s.t.

$f \circ \omega \circ \alpha = \tilde{f}(x_1, \dots, x_m)$

Q.E.D.

Remark. Domain-straighten Thm is just a special case of rank Thm for $m = N$.

Cor. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^N$ be a function of C^p, $p \geq 1$, on the open set. Let $x_0 \in A$. If $Df(x)$ has rank m, then $\exists$ an open set U_1 of $\mathbb{R}^n$ and a nbd U_2 of x_0 in A, a nbd V_1 of $f(x_0)$ in $\mathbb{R}^N$ and open set V_2 in $\mathbb{R}^N$ and two C^p-diffeomorphisms $h: U_1 \to U_2$ and $g: V_1 \to V_2$ s.t. $g \circ f \circ h(x_1, \dots, x_n) = \tilde{f}(x_1, \dots, x_m, 0, \dots, 0)$

pf: 先把自变数的坐标找出来 $\Rightarrow$ rank Thm.

By Rank Thm. $\exists$ C^r-diffeomorphism $h: U \to V$ s.t. $f \circ h(x_1, \dots, x_n) = \tilde{f}(x_1, \dots, x_m)$

$= \tilde{f} \circ \pi(x_1, \dots, x_n)$, where $\pi: \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$, the projection

Then $D(\tilde{f} \circ \pi)(x) = D(f \circ h)(x) = Df(h(x)) \cdot Dh(x)$

$\Rightarrow D\tilde{f}(\pi(x)) \circ \pi = Df(h(x)) \circ Dh(x)$

$\Rightarrow D\tilde{f}(\pi(x))$ is rank m, $\forall x \in U$.

Let W be an open set in $\mathbb{R}^m$ s.t. $\pi^{-1}(W) \subset U$.

Then $\tilde{f}: W \to V$

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Note: 在 $f: A \subset \mathbb{R}^n \to \mathbb{R}^N$ in C^1. 若 $Df(x)$ has rank m, 根據 Df 是連續在 A 上. 因此 $Df(x)$ 在 x_0 附近的点 x 上, 其 rank 至少是 m. 因此在 rank thm 中的條件必須下「在 x_0 的附近的点, $Df(x)$ has rank m」.

§7.7 Constrained Extrema and Lagrange Mu

Q: Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function. Let $S \subset \mathbb{R}^n$, 如何找 f 在 S 上的 extrema?

A: 只能有系統地處理 S = level set 之交集

Thm. (Lagrange multiplier Thm)
Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ and $g: U \subset \mathbb{R}^n \to \mathbb{R}$ be given C^1 functions. Let $x_0 \in U$ s.t. $g(x_0) = C$. Let $S = g^{-1}(c_0)$ be the level set for g with value c_0. Assume that $\nabla g(x_0) \neq 0$
If $f|_S: S \to \mathbb{R}$ has a maximum or minimum at x_0, then $\exists \lambda \in \mathbb{R}$ st. $\nabla f(x_0) = \lambda \nabla g(x_0)$

Def. The tangent plane of the set S at $x_0 \triangleq \{\alpha'(0) \mid \alpha$ is a diff. curve in S s.t. $\alpha(0) = x_0\} \subset \mathbb{R}^n$ which is denoted $T_{x_0}(S)$

Fact: $T_{x_0}(S) = \{v \in \mathbb{R}^n \mid v \perp \nabla g(x_0)\}$ $\triangleq B$ = 在 $\mathbb{R}^n$ 中过 x_0 点以 $\nabla g(x_0)$ 为法向量的平面!(因此將 $T_{x_0}(S)$ 叫做 tangent plane)

pf of Fact: (i) "$\subseteq$": Let $v \in T_{x_0}(S)$. Then $\exists$ a diff. curve α in S s.t. $\alpha(0) = x_0$ and $\alpha'(0) = v$. Then $g(\alpha(t)) = c_0$

$$\Rightarrow 0 = \frac{d}{dt}\left(g(\alpha(t))\right)$$

$$\text{at } t=0 \quad = \nabla g(\alpha(t)) \cdot \alpha'(t)$$

$$\Rightarrow \nabla g(x_0) \cdot v = 0$$

$$\Rightarrow v \in B$$

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Multiplier.

Hence $T_{x_0}(S) \subseteq B$.

"$\supseteq$" Let $v \in B$. $\because g$ is C^1 and $\nabla g(x_0) \neq 0$

$\therefore$ By Domain-straightening thm $\exists$ a C^1-diffeomorphism $h: U \overset{\text{open}}{\subseteq} \mathbb{R}^n \to V \overset{\text{open}}{\subseteq} \mathbb{R}^n$ s.t. $x_0 \in V$

$g(h(x_1, \dots, x_n)) = x_n$.

$\because x_0 \in S$

$\therefore g(S) = c_0$

$\Rightarrow h^{-1}(S)$ lies in the coordinate plane $x_n = c_0$.

Let $w = Dh^{-1}(x_0)(v)$ and $h(y_0) = x_0$. Then $Dh(y_0)(w) = v$.

$\because g(h(x_1, \dots, x_n)) = x_n$

$\therefore \nabla g(h(y_0)) \circ Dh(y_0) = (0, 0, \dots, 0, 1) = e_n$

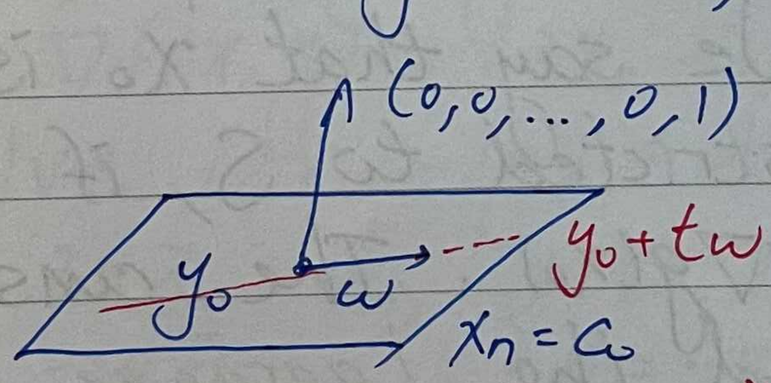
$\Rightarrow$ 兩邊取值在 w 向量上, $\langle \nabla g(\boxed{h(y_0)}), v \rangle = \langle e_n, w \rangle$ ($h(y_0) = x_0$)

$\because v \perp \nabla g(x_0)$

$\therefore \langle \nabla g(h(y_0)), v \rangle = 0 = \langle e_n, w \rangle$

$\Rightarrow w$ is perpendicular to the normal vector of the plane $x_n = c_0$

$\Rightarrow$ Let $\alpha(t) = h(y_0 + tw)$, where $y_0 + tw \in U$.



Then α is differentiable and $\alpha(t) \in S$. and

$\alpha(0) = x_0$ $\alpha'(0) = \boxed{Dh(y_0)}(w) = v$ (送回來)

$\Rightarrow v \in T_{x_0}(S)$

$\Rightarrow B \subseteq T_{x_0}(S)$

Therefore $B = T_{x_0}(S)$

Q.E.D.

Go back to the proof of Lagrange multiplier Thm.

pf of Thm: If f|s has an extremum at x_0, then $f(\alpha(t))$ has extreme at $t=0$ for every diff. curve α on S with $\alpha(0) = x_0$

$$\Rightarrow \frac{d}{dt} f(\alpha(t))\Big|_{t=0} = 0$$

$\Rightarrow \langle \nabla f(x_0), \alpha'(0) \rangle = 0$

$\Rightarrow \nabla f(x_0) \perp \alpha'(0), \forall \alpha'(0)$

$\Rightarrow \nabla f(x_0)$ is perpendicular to the tangent plane $T_{x_0}(S)$

$\Rightarrow \exists \lambda \in \mathbb{R}$ s.t. $\nabla f(x_0) = \lambda \cdot \nabla g(x_0)$

Q.E.D.

Def. Same assumptions as in previous Thm. Let $x_0 \in S$. We say that x_0 is a critical point of f restricted to S, if $\exists \lambda \in \mathbb{R}$ s.t. $Df(x_0) = \lambda \nabla g(x_0)$. The constant λ, then, is called the Lagrange multiplier.

Remark. 還需討論 critical pt 是 maximum or minimum, or neither.

Thm. If the surface S is defined by a number of constraints: $g_1(x_1,\dots,x_n)=c_1$, $g_2(x_1,\dots,x_n)=c_2,\dots, g_k(x_1,\dots,x_n)=c_k$.
If f has an extremum at x_0 in S and $\nabla_1 g(x_0),\dots,\nabla_k g(x_0)$ are linearly indep. then $\exists \lambda_1,\dots,\lambda_k\in\mathbb{R}$ s.t. $\nabla f(x_0)=\lambda_1\nabla g_1(x_0)+\cdots+\lambda_k\nabla g_k(x_0)$.

→ extremum在這些向量生成的切平面的交集上
⇒ 在 $\operatorname{span}\{\nabla_1 g(x_0),\dots,\nabla_k g(x_0)\}$ 上
⇒ 用線性組合描述之

e.g. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=x^2-y^2$ and let $S=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$. Find the critical point of f restricted to S.

pf. Let $g(x,y)=x^2+y^2$. Then g is C^1 and $S=g^{-1}(1)$. $\nabla g(x,y)=(2x,2y)\neq(0,0)$, $\forall (x,y)\in S$. $\nabla f(x,y)=(2x,-2y)$
To solve $(2x,-2y)=\lambda(2x,2y)$ for $\lambda\in\mathbb{R}$.
$$\Rightarrow\begin{cases}2x=2x\lambda\\ 2y=-2y\lambda\end{cases}\Rightarrow\begin{cases}x=x\lambda\\ y=-y\lambda\end{cases}$$
① If $x\neq 0$, then $\lambda=1\Rightarrow y=0$, 代 $x^2+y^2=1$
$\Rightarrow x=\pm 1$

$\Rightarrow (x,y) = (\pm 1, 0)$

② If $x=0$, then $\because x^2+y^2=1$

$\therefore y = \pm 1$, 代入 $-2y = -2y\lambda$

$\Rightarrow \lambda = -1$

$\Rightarrow (x,y) = (0, \pm 1)$

Therefore $(\pm 1, 0), (0, \pm 1)$ are critical point of f restricted to S //

e.g. Determine the nature of critical points of f restricted to S in the previous example.

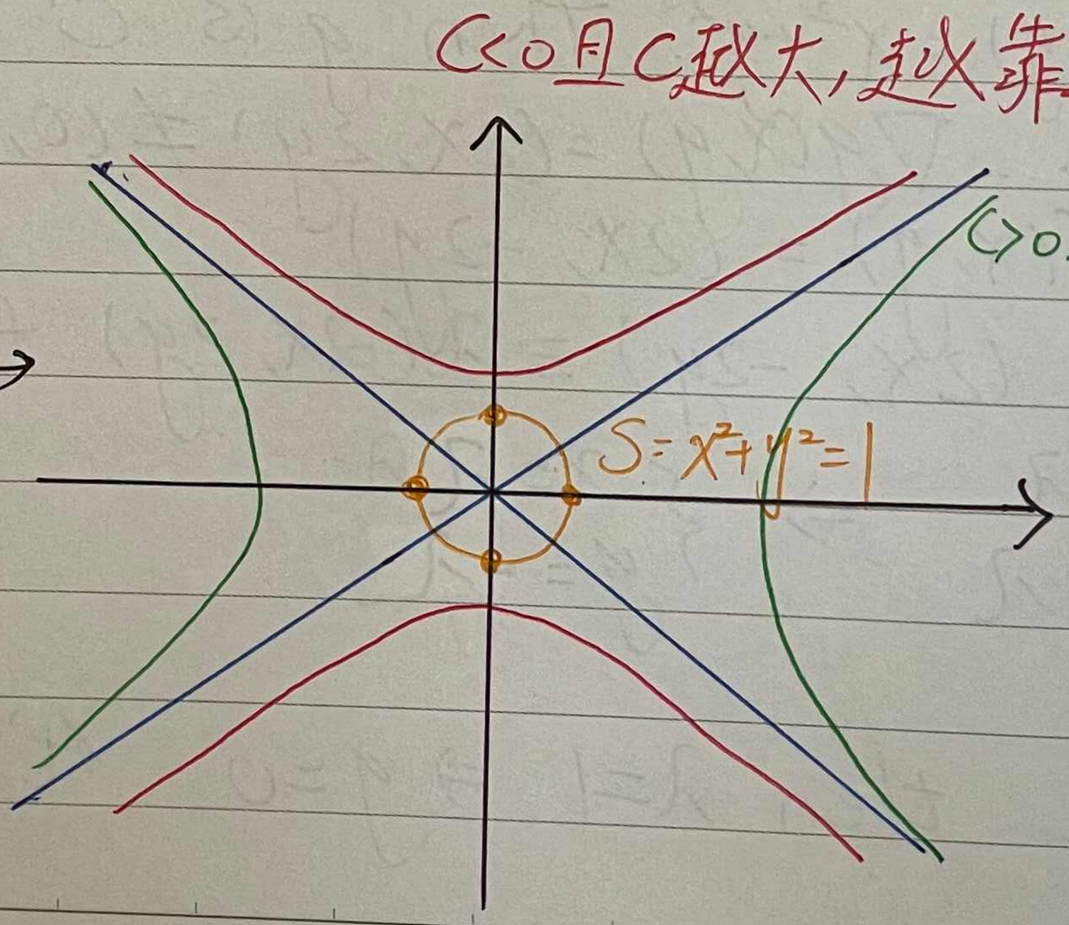
-sol- Let $f(x,y) = x^2 - y^2 = C$, C is constant

Then $f^{-1}(c) = \begin{cases} \text{双曲线}, C=0 \\ \text{两条相交于原点的直线}, C\neq 0 \end{cases}$

当 $C>0$ 时, $f^{-1}(c)$ 为开口朝左右且左右对称的双曲线

"<"

上下" 上下"



$\Rightarrow$ $\begin{cases} f(\pm 1, 0) \text{ 為 max. on } S \\ f(0, \pm 1) \text{ 為 min on } S \end{cases}$ #

Cor. If f, when constrainted to a surface, has an extremum at x_0, then $\nabla f(x_0)$ is perpendicular to S at x_0

e.g. Let $S \subset \mathbb{R}^2$ be a line through $(4,0)$ inclined at $45°$, and let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x,y) = x^2 + y^2$. Find the minimum of f on S.

-sol-. $\because S = \{(x,y) \in \mathbb{R}^2 \mid y - x - 1 = 0\}$

$\therefore$ Let $g(x,y) = y - x - 1$, then g is C^1 and $S = g^{-1}(0)$.

$\nabla g(x,y) = (-1, 1) \neq 0$ on S

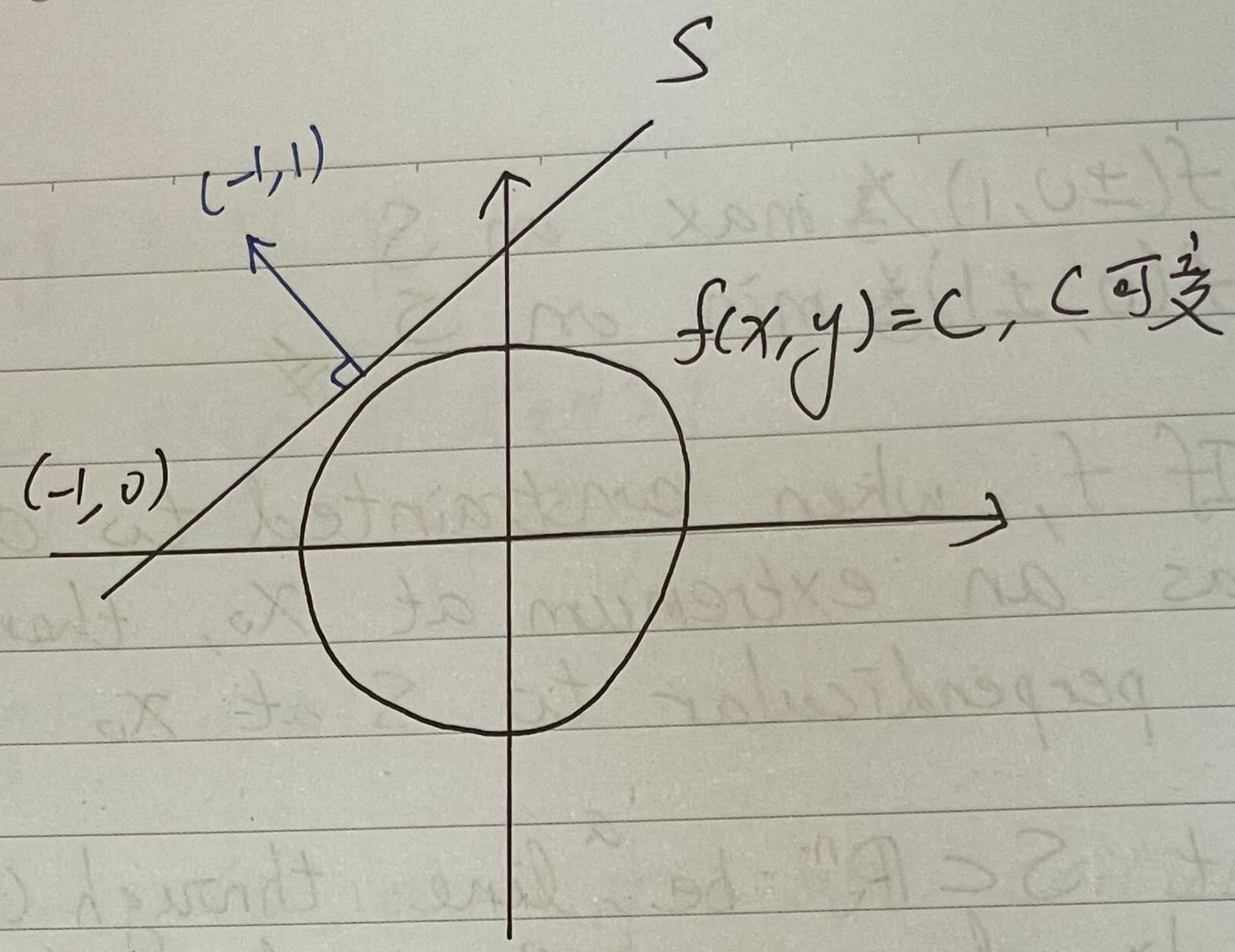
$\nabla f(x,y) = (2x, 2y)$. Solve $\begin{cases} 2x = -\lambda \\ 2y = \lambda \\ y - x - 1 = 0 \end{cases}$

$\Rightarrow \begin{cases} x = -\lambda/2 \\ y = \lambda/2 \end{cases}$ 代入 $y - x - 1 = 0$

$\Rightarrow \frac{\lambda}{2} + \frac{\lambda}{2} - 1 = 0$

$\Rightarrow \lambda = 1, \quad x = -1/2, \quad y = 1/2$

$\Rightarrow (-\frac{1}{2}, \frac{1}{2})$ is the only minimum.



$\nabla f(-\frac{1}{2}, \frac{1}{2}) = (-1, 1)$

∵ 此题几何意义为 S 上能圆心最近之点

∴ $f(-\frac{1}{2}, \frac{1}{2})$ 为 min ∎

e.g. Find the extreme point of $f(x,y,z) = x+y+z$ subject to the conditions $x^2+y^2=2$ and $x+z=1$

-sol-. Let $g_1(x,y,z) = x^2+y^2$ and $g_2(x,y) = x+z$.

Then g_1 and g_2 are C^1.

$\nabla g_1(x,y,z) = (2x, 2y, 0)$

$\nabla g_2(x,y) = (1, 0, 1)$.

Let $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=2 \text{ and } x+z=1\}$

Then $S = g_1^{-1}(2) \cap g_2^{-1}(1)$.

$\Rightarrow \nabla g_1(x,y,z) \neq 0$ on S

$\nabla g_2(x,y,z) \neq 0$ on S.

$\nabla f(x,y,z) = (1,1,1)$

To find $\lambda_1, \lambda_2 \in \mathbb{R}$ s.t. $\lambda_1(2x, 2y, 0) + \lambda_2(1, 0, 1) = (1, 1, 1)$, and $x^2+y^2=2$ and $x+z=1$, solve

Double A

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$$\begin{cases} 2\lambda_1 x + \lambda_2 = 1 \\ 2\lambda_1 y = 1 \quad (\Rightarrow \lambda_1 \neq 0) \\ \lambda_2 = 1 \\ x^2 + y^2 = 2 \\ x + z = 1 \end{cases} \Rightarrow 2x\lambda_1 = 0 \Rightarrow x = 0$$

$\Rightarrow (x, y, z, \lambda_1, \lambda_2) = (0, \pm\sqrt{2}, 1, \pm\frac{\sqrt{2}}{2}, 1)$

$\Rightarrow$ The critical points of $f|_S$ are $(0, \sqrt{2}, 1)$ and $(0, -\sqrt{2}, 1)$

$\because g_1^{-1}(2)$ is a cylinder and $g_2^{-1}(1)$ is a plane in $\mathbb{R}^3$

$\therefore S$ is an ellipse

$\Rightarrow S$ is compact in $\mathbb{R}^3$

$\because f$ is cont.

$\therefore f|_S$ has a maximum and minimum

$\therefore \begin{cases} f(0, \sqrt{2}, 1) = \sqrt{2} + 1 \\ f(0, -\sqrt{2}, 1) = -\sqrt{2} + 1 \end{cases}$

$\therefore f|_S$ has a maximum $\sqrt{2}+1$ and minimum $-\sqrt{2}+1$. #

剛好2個 critical pt 才能這樣證.

Ch8 Integration

§8.1 Integrable function

Def. A rectangle in $\mathbb{R}^n$ is a set of the forms: $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$

↳ 不在乎 open / closed

Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function on the bounded set A. Choose a rectangle $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$ that encloses A.

Extend f to be defined on the whole rectangle B by defining it to be zero outside A. Then we divide the rectangle B into smaller rectangles by partitioning each $[a_i, b_i]$ points $a_i = x_0^i < x_1^i < \cdots < x_{m_i}^i = b_i$, $\forall i = 1, \ldots, n$ where $x = (x^1, x^2, \ldots, x^n)$

Def. The set $P = \{[x_{i_1-1}^1, x_{i_1}^1] \times \cdots \times [x_{i_n-1}^n, x_{i_n}^n] \mid i_1 = 1, \ldots, m_1, \ldots, i_n = 1, \ldots, m_n\}$ formed by $m_1 m_2 \cdots m_n$ subrectangles is called a partition of the rectangle B.

Def. ① The volume of the rectangle

$$B = [a_1, b_1] \times \cdots \times [a_n, b_n] \triangleq V(B)$$
$$= (b_1 - a_1) \times \cdots \times (b_n - a_n)$$

② Let U be a set in $\mathbb{R}^n$ s.t. $U \subset A$.

Define $M_U = \sup_{x \in U} f(x)$ and $m_U = \inf_{x \in U} f(x)$

③ The lower sum of f for a partition P of the rectangle B is defined by $L(f,P) = \sum_{R \in P} m_R V(R)$, where the sum being over all subrectangles R of the partition P.

④ The upper sum of f for P is defined by $U(f,P) = \sum_{R \in P} M_R V(R)$

Note: $\because f$ is bounded $\therefore M_R$ and m_R exists, $\forall R \in P$

Prop. $U(f,P) \geq L(f,P)$, $\forall$ partition P of B

Def. Let P_1 and P_2 be two partitions of the rectangle B. We say that P_2 is a refinement of P_1 if each rectangle belonging to P_2 is contained in a rectangle belonging to P_1. Denote it by $P_2 < P_1$

Prop. ① If $P_2 < P_1$, then $L(f,P_1) \leq L(f,P_2)$ and $U(f,P_2) \leq U(f,P_1)$

② Let P_1 and P_2 be two partitions of the rectangle B, then $\exists$ a partition P_3 of the rectangle B s.t. $P_3 < P_2$ and $P_3 < P_1$.

pf. ① is obvious

② Let P_3 be partition of B arranged by using all subdivision points forming P_1 and P_2. Then $P_3 < P_1$ and $P_3 < P_2$

Q.E.D.

Def. P_3, thus, is called the refinement of P_1 and P_2.

Prop. Let P_1 and P_2 be two partitions of the rectangle B, then $L(f, P_1) \leq U(f, P_2)$ ☆

pf. Let P_3 be the refinement of P_1 and P_2. Then $L(f, P_1) \leq L(f, P_3) \leq U(f, P_3)$ $\leq U(f, P_2)$.

Q.E.D.

Def. ① The upper integral of f on A is defined by $\overline{\int_A} f = \inf \{U(f, P) \mid P$ 取 is a partition of the rectangle B containing $A.\}$

② The lower integral of f on A is defined by $\underline{\int_A} f = \sup \{L(f_i, P) \mid P$ " $\}$

Prop. $\underline{\int_A} f \leq \overline{\int_A} f$

pf. $\because L(f, P_1) \leq U(f, P_2)$, $\forall$ partitions P_1 and P_2 of the rectangle B.

Double A

$\therefore$ Let P_1 vary, Then $\underline{\int}_A f \le U(f, P_2)$, $\forall P_2$.
$\Rightarrow$ Let P_2 vary, $\underline{\int}_A f \le \overline{\int}_A f$.

Q.E.D.

Def. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function on the bounded set A. We say that f is (Riemann) integrable on A, if $\underline{\int}_A f = \overline{\int}_A f$. We write $\int_A f = \underline{\int}_A f = \overline{\int}_A f$.

Remark. ① Instead of $\int_A f$, the notations $\int_A f(x)\,dx$ or $\underbrace{\int \cdots \int}_{A} f(x_1, \ldots, x_n)\,dx_1 \cdots dx_n$ is frequently employed.

② The definition of $U(f,P)$, $L(f,P)$, $\underline{\int}_A f$, $\overline{\int}_A f$ or $\int_A f$ is independent of the choice of the rectangle B which contains A.

③ $\int_A f$ = the generalized volume between the graph of the function $y = f(x)$ and A

④ $L(f, P_1) \le \int_A f \le U(f, P_2)$, $\forall P_1$ and P_2 are partitions of the rectangle $B \supset A$.

Q: 怎樣的 bounded function 才會是 bounded set 上會是 integrable 呢?

Thm. (Rieman's condition)
Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function on a bounded set A. Let S be a rectangle in $\mathbb{R}^n$ s.t. $A \subset S$. Let f be extended to S by defining $f=0$ on $S \setminus A$ Then f is integrable on A
$\iff \forall \varepsilon > 0$, $\exists$ a partition P_ε of S s.t. $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$

pf. ($\Rightarrow$) $\because f$ is integrable
$\therefore \underline{\int}_A f = \overline{\int}_A f$, say I

Then $\forall \varepsilon > 0$, $\exists$ partitions P_1 and P_2 of S s.t. $L(f, P_1) > I - \varepsilon/2$ and $U(f, P_2) < I + \varepsilon$

Let P_3 be the refinement of P_1 and P_2.
Then $L(f, P_3) \geq L(f, P_1)$ and $U(f, P_3) < U(f, P_2)$
$\Rightarrow U(f, P_3) - L(f, P_3) \leq U(f, P_2) - L(f, P_1)$

$$< \left(I + \frac{\varepsilon}{2}\right) - \left(I - \frac{\varepsilon}{2}\right)$$

$$= \varepsilon.$$

($\Leftarrow$). Let $\varepsilon > 0$. Then $\exists$ a partition P of S s.t. $U(f, P) - L(f, P) < \varepsilon$.
$\because \underline{\int}_A f \geq L(f, P)$ and $\overline{\int}_A f \leq U(f, P)$

與ε無关，定数

$\therefore \overline{\int_A} f - \underline{\int_A} f \leq U(f,P) - L(f,P) = \varepsilon.$

$\because \lim_{\varepsilon \to 0^+} \varepsilon = 0$

$\therefore \overline{\int_A} f - \underline{\int_A} f \leq 0.$

$\because \overline{\int_A} f \geq \underline{\int_A} f$

$\therefore \overline{\int_A} f = \underline{\int_A} f$

$\Rightarrow$ f is integrable

Q.E.D.

Cor. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function on a bounded set A. Let S be a rectangle in Rieman's condition. Then f is integrable on A with integral $I \iff \forall \varepsilon > 0$, $\exists$ a partition P. (or partitions P_1, P_2) of S s.t. $U(f,P) < I + \varepsilon$ and $L(f,P) > I - \varepsilon$. (or $U(f,P_1) < I + \varepsilon$ and $L(f,P_2) > I - \varepsilon$)

Def. ① Let P be a partition of a rectangle B. Then the length of P denoted by $\|P\|$, is the maximum length of all sides of subrectangles in P.

② Let $P = \{S_1, \dots, S_N\}$ be a partition of the rectangle B. Let $x_i^* \in S_i$, $\forall i = 1, \dots, N$

$\sum_{i=1}^{N} f(x_i^*)V(S_i)$ is called a Riemann sum of f w.r.t. the partition P.

Prop. $L(f,P) \le \sum_{i=1}^{N} f(x_i^*)V(S_i) \le U(f,P)$, $\forall x_i^* \in S_i$, $i = 1, \dots, N$.

Q: Is $L(f,P)$ or $U(f,P)$ is a Riemann sum of f w.r.t. P?
⇒ S_i 上必可找到 x_i^* 取值到 sup 或 inf 嗎？

A: 不一定！S_i 是 bounded，但必需 closed 且 f 連續才取得到.

Thm. (Darboux's Thm.)
Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function on a bounded set A. Let S be a rectangle in $\mathbb{R}^n$ s.t. $A \subset S$. Then f is integrable with integral $I \iff \forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall$ partition $P = \{S_1, \dots, S_N\}$ of S with $\|P\| < \delta$ and for any $x_1 \in S_1, \dots, x_N \in S_N$, $\left\| \sum_{i=1}^{N} f(x_i)V(S_i) - I \right\| < \varepsilon$.
(i.e., $\lim_{\|P\| \to 0} \sum_{i=1}^{N} f(x_i)V(S_i) = I$)

pf. ⇐) 用 Riemann sum 去找 P_1, P_2 s.t. $U(f,P_1) < I + \varepsilon$, $L(f,P_1) > I - \varepsilon$.

解不等式找P.

Let $\varepsilon>0$. Let $P=\{S_1,\dots,S_N\}$ be a partition of S. To estimate $\sum_{i=1}^{N} M_{S_i}(f)V(S_i)$

$-I<\varepsilon$ in terms of $\left|\sum_{i=1}^{N} f(x_i)V(S_i)-I\right|<\varepsilon$

住<"|·|"对的时候去"||"会比较小.

估绝对值

可要多靠近有多靠近

$$|U(f,P)-I|\le\left|\sum_{i=1}^{N} M_{S_i}(f)V(S_i)-\sum_{i=1}^{N} f(x_i)V(S_i)\right|$$
$$+\left|\sum_{i=1}^{N} f(x_i)V(S_i)-I\right| \to \text{先选}P,\ \because x_i\in P$$

For $\frac{\varepsilon}{2}$, $\exists\delta>0$ s.t. $\forall$ partition $\tilde{P}=\{\tilde{S}_1,\dots,\tilde{S}_{\tilde{N}}\}$ with $\|\tilde{P}\|<\delta$, $\left|\sum_{i=1}^{\tilde{N}} f(x_i)V(\tilde{S}_i)-I\right|<\frac{\varepsilon}{2}$. $\forall x_i\in\tilde{S}_i$, $i=1,\dots,N$.

Choose P s.t. $\|P\|<\delta$. Then $\left|\sum_{i=1}^{N} f(x_i)V(S_i)-I\right|<\frac{\varepsilon}{2}$.

$\because M_{S_i}(f)=\sup\{f(x)\mid x\in S_i\}$

$\therefore$ For $\frac{\varepsilon}{2NV(S_i)}$, $\exists x_i\in S_i$ s.t.

$$f(x_i)>M_{S_i}(f)-\frac{\varepsilon}{2NV(S_i)}$$

比手法很常用

$$\Rightarrow |U(f,P)-I|<\sum_{i=1}^{N}\left(\frac{\varepsilon}{2NV(S_i)}\cdot V(S_i)\right)+\frac{\varepsilon}{2}=\varepsilon$$

$$\Rightarrow U(f,P)-I<\varepsilon$$
$$\Rightarrow U(f,P)<I+\varepsilon$$

Similarly, $\exists$ a partition $\tilde{P}$ of S s.t. $L(f,\tilde{P})>I-\varepsilon$

$\Rightarrow f$ is integrable with integral I.

⇒) Let $\varepsilon > 0$. Let $P = \{S_1, \dots, S_N\}$.
要由解 $\left| \sum_{i=1}^{N} f(x_i) V(S_i) - I \right| < \varepsilon$ 中去找 P.

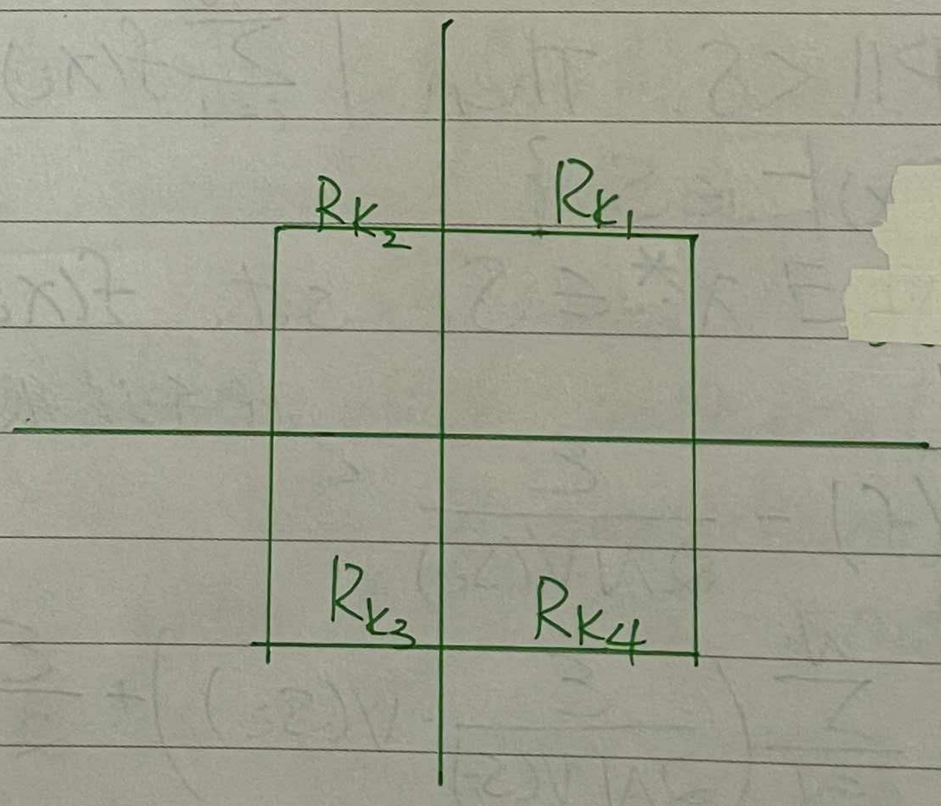
$\because f$ is integrable with $\int_A f = I$
$\therefore$ For this ε, $\exists$ a partition P_ε of S
s.t. $L(f, P_\varepsilon) > I - \varepsilon$ and $U(f, P_\varepsilon) < I + \varepsilon$.

Let $P_\varepsilon = \{R_1, \dots, R_m\}$,

$$\sum_{i=1}^{N} f(x_i) V(S_i) = \sum_{\substack{S_i \in P \\ S_i \subset R_{j(i)}}} f(x_i) V(S_i) + \sum_{\substack{S_i \in P \\ S_i \not\subset R_j, \forall j}} f(x_i) V(S_i)$$

$$\leq \sum_{S_i \in P, S_i \subset R_{j(i)}} M_{R_{j(i)}} V(S_i) + \sum_{S_i \in P, S_i \subset R_j, \forall j} f(x_i) V(S_i)$$

(思考:



$\because$ 在 $S_i \cap R_{kj}$, $j = 1, 2, 3, 4$,

$$f(x_i) = M_{R_j}(f) + f(x_i) - M_{R_{jk}}(f)$$
$$\leq M_{R_{jk}}(f) + 2M, \text{ where } |f(x)| \leq M, \forall x \in A.$$

$$f(x_i) V(S_i) = \sum_{j=1}^{4} f(x_i) V(S_i \cap R_{kj})$$

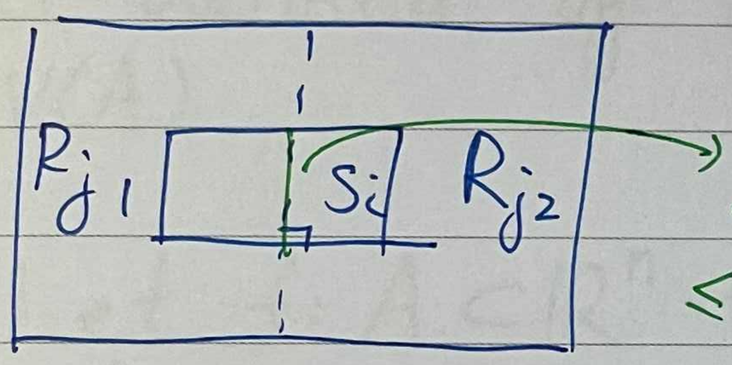
$$\leq \sum_{j=1}^{4} M_{R_{kj}} V(S_i \cap R_{kj}) + 2M V(S_i) \Big)$$

P变动时，

Hence $\sum_{i=1}^{N} f(x_i) V(S_i) \leq \underbrace{U(f, P_\varepsilon)}_{< I+\varepsilon} + \boxed{2M \sum_{\substack{S_i \subset P \\ S_i \not\subset R_j \\ \forall j=1,\dots,m}} V(S_i)}$

此项有 upper bound 吗？

Assume $\|P\| < \delta$



area of this face $= w_i$
$\leq$ area of the face of $R_{j1} \cap R_{j2}$

$V(S_i) \leq \delta \cdot w_i$

思考

⇒ Let T be the total area of the faces between any two rectangles of P_ε

Then $\sum_{\substack{S_i \subset P \\ S_i \not\subset R_j \\ \forall j=1,\dots,m}} V(S_i) \leq \sum \delta \cdot w_j = \delta \sum w_j \leq \delta T$

Let $\delta = \dfrac{\varepsilon}{2MT}$, then $\forall \|P\| < \delta$,

$$\sum_{i=1}^{N} f(x_i) V(S_i) < (I+\varepsilon) + \varepsilon = I + 2\varepsilon$$

Similarly, $\sum_{i=1}^{N} f(x_i) V(S_i) > L(f, P) - 2M \sum_{\substack{S_i \not\subset R_j \\ j=1,\dots,m}} V(S_i)$

$$> (I-\varepsilon)-\varepsilon = I-2\varepsilon.$$

$$\Rightarrow \left| \sum_{i=1}^{N} f(x_i) V(S_i) - I \right| < 2\varepsilon$$

Therefore f is integrable with integral I.

Q.E.D.

Remark. Hence we can use 多重 series 去估計 $\int_A f$.

MON TUE WED THU FRI SAT SUN | DATE: / / | NO.: | SUBJECT:

§8.2 Volume and sets of measure zero

Def. Let $A \subset \mathbb{R}^n$ be a bounded set.

① The characteristic function 1_A on A is a map defined by $1_A : \mathbb{R}^n \to \mathbb{R}$ by

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

② We say that A has volume (or content) if 1_A is integrable and the volume of A is defined by $\int_A 1_A$ and denoted by $V(A)$

Note: ① Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function Then $f \cdot 1_A$ is the extension required in the def. of the integration

② If $n=1$, then $V(A)$ = the length of A. If $n=2$, then $V(A)$ = area of A

Prop. All rectangles $B = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ has volume $= \prod_{i=1}^{n} (b_i - a_i)$

pf. $\because L(1_B, P) = U(1_B, P) = \prod_{i=1}^{n} (b_i - a_i)$, $\forall$ partition P of B

$\therefore \underline{\int}_B 1_A = \overline{\int}_B 1_B = \prod_{i=1}^{n} (b_i - a_i)$

Q.E.D.

Double A

Q: ① 怎样的 bounded set 会有体积呢?
② 怎样的 bounded set 会有 volume zero 呢?

A: ① §8-3

② A has volume zero $\Leftrightarrow$ 1_A is integrable and $\int_A 1_A = 0$ $\Leftrightarrow$ Let S be a rectangle in $\mathbb{R}^n$ containing S. $\forall \varepsilon > 0$, $\exists$ a partition P of S s.t. $U(1_A, P) < \varepsilon$ and $L(1_A, P) > -\varepsilon$

$$\Leftrightarrow \sum_{S_i \in P} 1 \cdot V(S_i) + \sum_{S_i \in P} 0 \cdot V(S_i) < \varepsilon$$

$$\begin{cases} S_i \cap A \neq \emptyset & S_i \cap A = \emptyset \\ \sum_{S_i \in P} 1 \cdot V(S_i) + \sum_{S_i \in P} 0 \cdot V(S_i) > -\varepsilon \\ S_i \subset A & S_i \not\subset A \end{cases}$$

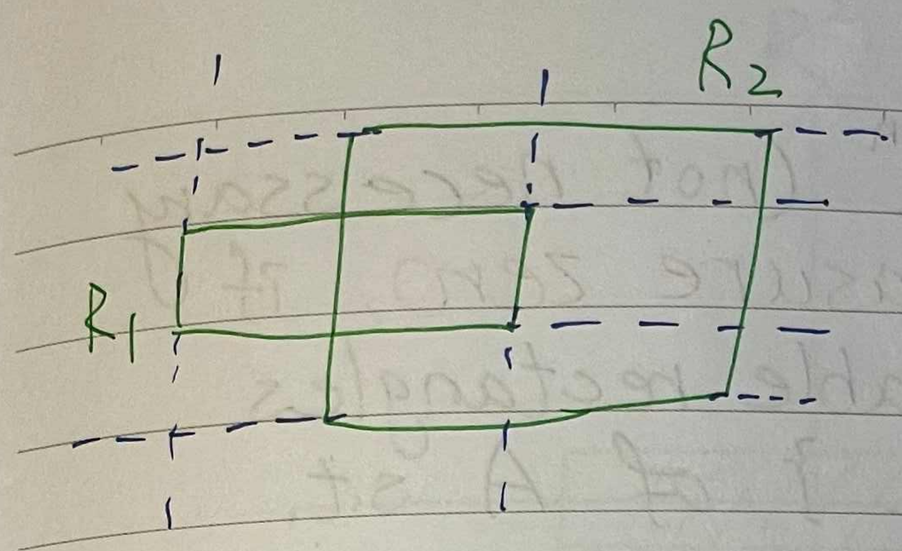
(inf 取小, 有1有0全取0)

可以有 overlap, 但可包含边

$\Rightarrow$) $\forall \varepsilon > 0$, $\exists$ a finite cover of Rectangles $R_1, \dots, R_m$ of A s.t. $\sum_{i=1}^{m} V(R_i) < \varepsilon$.

Assume $\forall \varepsilon > 0$, $\exists$ a finite cover of rectangles $R_1, \dots, R_m$ of A s.t. $\sum_{i=1}^{m} V(R_i) < \varepsilon$.

Let S be a rectangle in $\mathbb{R}^n$ containing A. Let $\varepsilon > 0$. Let $P = \{S_1, \dots, S_N\}$ be a partition of S s.t. each S_i is either contained in some R_j or



把這些 cover 的邊長拉長，變成鉄絲網，再把超出 S 處刪掉.

at most its boundary in common with some R_j's namely the partition is defined using all edges of R_j's. Then $U(1_A, P) \leq \sum_{j=1}^{m} V(R_j) < \varepsilon$.

↑ 有 overlap.

$\because L(1_A, P) \geq 0 > -\varepsilon$

$\therefore$ By Riemann's condition, 1_A is integrable with $\int_A 1_A = 0$

$\Rightarrow V(A) = 0$

Prop. A bounded set A in $\mathbb{R}^n$ has volume zero $\Leftrightarrow \forall \varepsilon > 0$, $\exists$ a finite cover of rectangles $R_1, \dots, R_m$ of A s.t. $\sum_{i=1}^{m} V(R_i) < \varepsilon$

Cor. ① If A has volume zero, then any subset of A has volume zero

② Any finite union of volume zero sets also has volume zero

Prop. A bounded set $A \subset \mathbb{R}^n$ has volume not zero $\Leftrightarrow$ $bd(A)$ has volume zero.

pf. exercise.

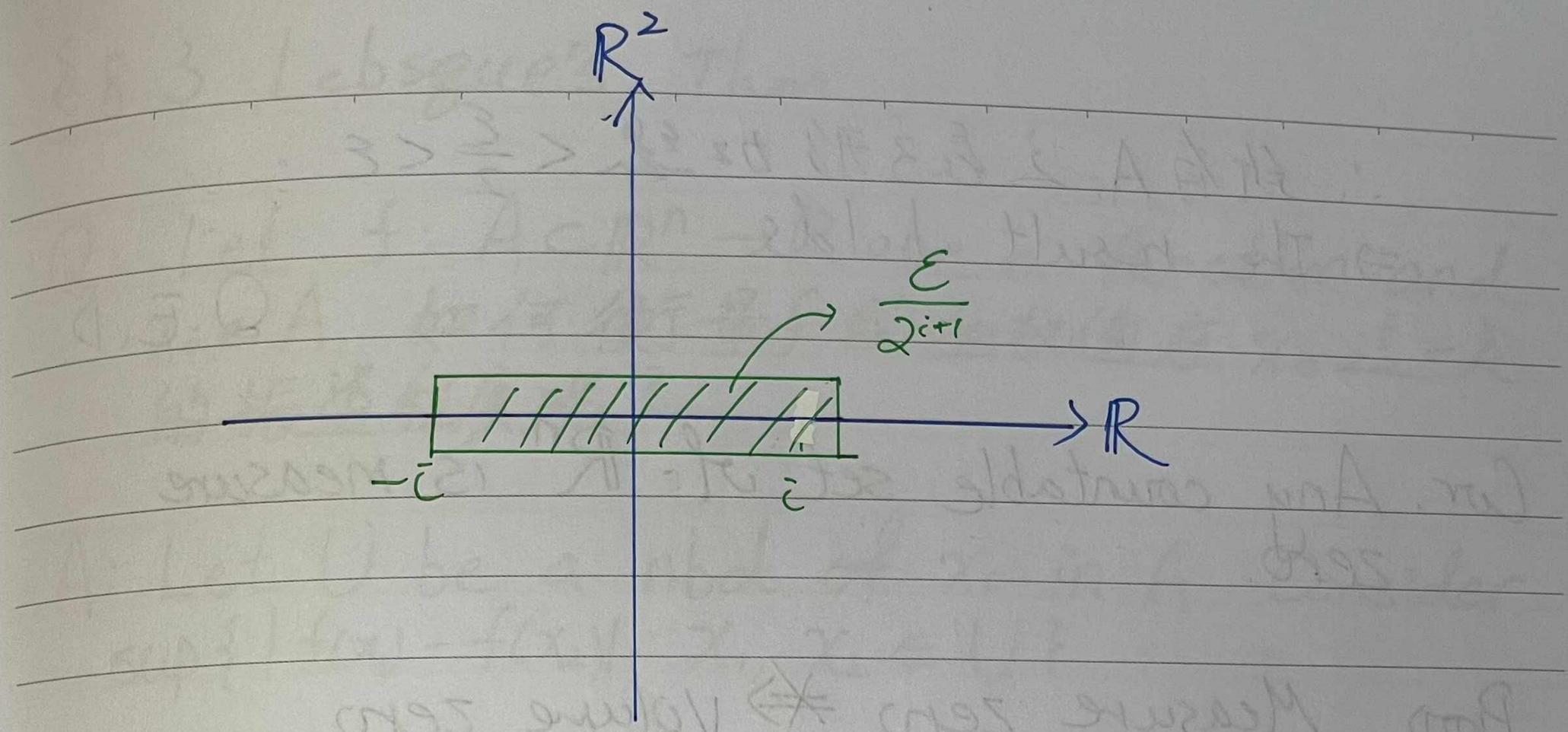
Def. A set $A \subset \mathbb{R}^n$ (not necessary bounded) has measure zero, if $\forall \varepsilon > 0$, $\exists$ a countable rectangles cover $\{R_1, \dots, R_m, \dots\}$ of A s.t.
$\sum_{i=1}^{\infty} V(R_i) < \varepsilon$.

Prop. ① Any subset of measure zero set is also measure zero.
② A set in $\mathbb{R}^n$ having volume is not measure zero

思考
② 如果有体积，可考虑 - partition 之 lower sum (不考虑 upper sum).
lower sum 取到 1 ⇒ 有一小长方形包在 A 裡頭
⇒ 盖 A 的 cover 不会比比长方形小.

③ If A has volume zero, then A is measure zero
④ Any single point set in $\mathbb{R}^n$ has volume zero and measure zero.
⑤ The real line, regarded as a subset of $\mathbb{R}^2$ has measure zero, but as a subset of $\mathbb{R}$, it does not.
非=volume zero
因为 unbound!

pf. ⑤ Consider the real line is the x-axis.



Let $\varepsilon > 0$. Let $R_i = [-i, i] \times \left[\frac{-\varepsilon}{i \cdot 2^{i+3}}, \frac{\varepsilon}{i \cdot 2^{i+3}}\right]$

Then $V(R_i) = \frac{\varepsilon}{2^{i+1}}$

$\therefore$ Union of R_i's is a cover, and

$$\sum_{i=1}^{\infty} V(R_i) = \frac{\varepsilon}{2} < \varepsilon \quad \square$$

$\therefore$ $\mathbb{R}$ has measure zero as a subset of $\mathbb{R}^2$

Q.E.D.

Thm. Suppose that the sets $A_1, A_2, \ldots$ have measure zero in $\mathbb{R}^n$. Then the union $\bigcup_{i=1}^{\infty} A_i$ has measure zero.

pf. 对每个 A_i，都有可数个长方形 cover，其总体积要多小有多小。设小于 $\frac{\varepsilon}{2^i}$

$\therefore$ countable union is countable

∴ 所有 A_i 之長方形力2總和 $< \frac{\varepsilon}{2} < \varepsilon$.
⇒ The result holds.

Q.E.D.

Cor. Any countable set in $\mathbb{R}^n$ is measure zero.

Prop. Measure zero $\not\Leftrightarrow$ Volume zero

pf. Let $A = \mathbb{Q} \cap [0,1]$. Then A is countable
$\Rightarrow$ A is measure zero.

$\because \int_A 1_A = \underline{1}$ and
↑ ↳每小區間必包含有理數
等[0,1]做細分

$\underline{\int_A 1_A} = 0$ and $1 \neq 0$

$\therefore 1_A$ is not integrable
$\Rightarrow A$ has no volume
$\Rightarrow A$ is not of volume zero

Q.E.D.

Prop. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be integrable and bounded on the bounded set A. Then the graph of $f = \{(x, f(x)) \mid x \in A\} \subset \mathbb{R}^{n+1}$ has volume zero.

pf. exercise.

§8.3 Lebsgue's Thm.

Q: Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function and $x_0 \in A$. 如何衡量 f 的函数值在 x_0 這一点的振盪程度呢?

A: Let U be a nbd of x_0 in A. Consider $\sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in U\}$

Def. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function and $x_0 \in A$. We define the oscillation of f at x_0, written by $\omega(f, x_0) = \inf\{\sup\{|f(x_1) - f(x_2)| \; x_1, x_2 \in U\} \mid U \text{ is a n.b.d. of } x_0\}$, or

$$\omega(f, x_0) = \inf_{\text{nbd } U \text{ of } x_0} \sup_{x_1, x_2 \in U} |f(x_1) - f(x_2)|$$

Prop. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function and $x_0 \in A$. Then f is cont. at $x_0 \iff \omega(f, x_0) = 0$.

pf. f is cont. at $x_0 \iff \forall \varepsilon > 0$, $\exists$ a nbd U of x_0 s.t. $\forall x_1, x_2 \in U$, $|f(x_1) - f(x_2)| < \varepsilon$.

$\iff \forall \varepsilon > 0$, $\exists$ a nbd U of x_0 s.t. $\sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in U\} \leq \varepsilon$.

$\iff \boxed{\forall \varepsilon > 0}$, $\inf\{\sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in U\} \mid U \text{ is a nbd. of } x_0\} \boxed{\leq \varepsilon}$ (≥ 0).

↳ 因为取的是 inf, 故可以这样说

对 ε 取极限

$\iff \inf_{\text{nbd } U \text{ of } x_0} \sup_{x_1, x_2 \in U} |f(x_1) - f(x_2)| = 0$.

$\iff \omega(f, x_0) = 0$