§1.7 Metrics

Question: 高微用什麼工具來研究集合?
Answer: 用集合中任意兩元素的遠近距離來研究集合

Question: 那如何衡量集合中任二元素之遠近?
Answer: Let 集合為 M. $f: M\times M \to \mathbb{R}$ 之函數.
① $f(x,x)=0$, $\forall x\in M$
② $f(x,y)>0$, $\forall x\neq y$ in M.
③ $f(x,y)=f(y,x)$, $\forall x,y\in M$.
④ $f(x,y)+f(y,z)>f(x,z)$, $\forall x,y,z\in M$.

Def. Let M be a set, A distance function or a metric on M is a function
$d: M\times M\to\mathbb{R}$ s.t.
(1) $d(x,y)\geq 0$, $\forall x,y\in M$
(2) $d(x,y)=0 \iff x=y$, $\forall x,y\in M$
(nondegeneracy)
(3) $d(x,y)=d(y,x)$ (symmetry)
(4) $d(x,y)\leq d(x,z)+d(z,y)$, $\forall x,y,z\in M$
(triangle inequality).

Question: 是否在任意的群組上都能找到 distance function 呢?
Answer: Yes.
Let M be a set. Define $d(x,y)$
$= \begin{cases} 0, & \text{if } x=y \text{ in } M \\ 1, & \text{if } x\neq y \text{ in } M \end{cases}$
Show that d is a metric.

pf: Obvious, d is a function from $M \times M$ to $\mathbb{R}$ s.t. $d(x,y) \geq 0$, $\forall x, y \in M$ and $d(x,y) = 0 \iff x, y$ in M and $d(x,y) = d(y,x)$, $\forall x, y$ in M.

If $d(x,y) = 0$, $\because d(x,z) \geq 0$ and $d(z,y) \geq 0$, $\therefore 0 = d(x,y) \leq d(x,z) + d(z,y)$, $\forall z \in M$.

If $d(x,y) = 1$, then $x \neq y$.
$\because$ Either $z \neq x$ or $z \neq y$, $\forall z \in M$
$\therefore$ Either $d(z,y) = 1$ or $d(x,z) = 1$
$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$, $\forall z \in M$.
Therefore d is a metric on M.

Q.E.D.

Remark. This metric is called the discrete metric for M.

Question: Metric 的存在是否必要.
Answer: 不必要.

ex. (Bounded Metric)
If d is a metric on a set M.
Let $f(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$
then $0 \leq f(x,y) \leq 1$, $\forall x, y \in M$.
Show that f is a metric on M.

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ex2. Let M be the set of all words, each word consisting of an ordered δ-tuple of 0s and 1s. i.e.,

$M = \{(w_1, w_2, \dots, w_\delta) : w_i = 0, 1, \forall i = 1, \dots, \delta\}$

Define $d(v, w)$ = the number of places in which v and w are different. For example, $v = (1, 0, \dots, 0)$ and $w = (0, 1, 1, 1, 0, \dots, 0)$ Then $d(v, w) = 4$. Show that d is a metric on M.

pf: exercise. (Hint: $d(v,w) = \sum_{i=1}^{\delta} |v_i - w_i|$)

Def. A metric space (M, d) is a set M equipped with a metric d on M.

exercise: P70. (10), word metric in a metric.

補充題: ① 在人類社会中，兩性遠近关係不是一个距離函数 的

② metric 滿足兩边之差小於等於第三边.

Ch2 The topology of Euclidean Space

§2.1 Open Sets

度量的引進，讓人的所在空間化．它賦与元素空間感

$(\mathbb{R}^n, \|\cdot\|)$ Euclidean Space

Question：透过d引進了空間感，第一个要探討的問題為如何認知元素x的附近呢？

Answer：$\{y \in M : d(x,y) < r\}$

Def. Let (M,d) be a metric space, Let $x \in M$ and $r \in \mathbb{R}^1$. The set $D(x,r) = \{y \in M : d(x,y) < r\}$ is called r-disk about x or r-ball. about x.

Def. Let x be a point in a Metric Space (M,d). a nbh of x in M is a subset M containing a ball of x.

Def. A subset A of a metric space (M,d) is called open if for each $x \in A$, $\exists\ \varepsilon > 0$ s.t. $D(x,\varepsilon) \subset A$.

Prop. M is open in (M,d).

pf: $\because D(x,r) \subset M$, $\forall x \quad \therefore M$ is open. $\#$

Note: In $(\mathbb{R}^n, \|\cdot\|)$ For $n=1$, $D(x,1) = (x-1, x+1)$

For $n=2$, $D(x,r) = \{y \in \mathbb{R}^2 : \|x-y\| < r\}$

For $n=3$, $D(x,r) =$ 以 x 為圓心，r 為半徑

的球体.

In general, $D(x,r)$ 為一个以 x 為圆心, r 為半徑的 n 維球体

ex. Show that $(0,1)$ is open in $\mathbb{R}$, but $(0,1]$ is not open in $\mathbb{R}$.

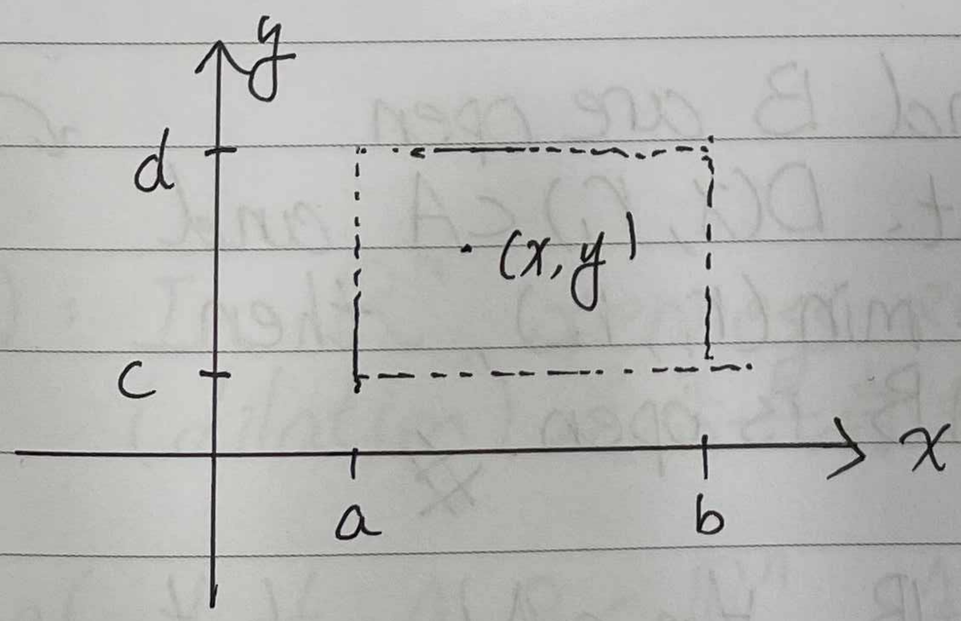
pf: ① Let $x \in (0,1)$, let $r = \min(x, |x-1|)$, then $D(x,r) \subset (0,1)$.

$\Rightarrow (0,1)$ is open.

② $\forall r > 0$, $\because D(1,r) \not\subset (0,1]$ $\therefore (0,1]$ is not open. ※

Q: ①与②的差别在哪裡?

ex. Show that $(a,b)\times(c,d)$ is open in $\mathbb{R}^2$



pf. Let $(x,y) \in (a,b)\times(c,d)$. Let $r = \min(x-a, b-x, y-c, b-y)$ then $D(x,r) \subset (a,b)\times(c,d) \Rightarrow (a,b)\times(c,d)$ is open.

Q.E.D.

ex. Show that $(0,1]$ is open in $\mathbb{R}$ with discrete metric d.

pf: Let $x \in (0,1]$. $\because D(x,1) = \{x\} \subset (0,1]$

$\therefore (0,1]$ is open in $(\mathbb{R}, d)$.

Prop. Let (M,d) be a Metric space with discrete metric, then any subset of M is open.

Convention: ① $\emptyset$, the empty set, is open.

② From now on, a n.b.h. of a point means an "open" n.b.h. of x.

Q: Openness 对子集的自然运算是否具封閉性?

A: Let A and B be open subsets of a metric spaces (M,d).

Let $x \in A \cap B$, $\because A$ and B are open.

$\therefore \exists r_1 > 0$ and $r_2 > 0$ s.t. $D(x,r_1) \subset A$ and $D(x,r_2) \subset B$. Let $r = \min(r_1, r_2)$, then $D(x,r) \subset A \cap B$. $\Rightarrow A \cap B$ is open. ※

ex. $(-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R}$, $\forall n \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not open in $\mathbb{R}$.

Thm. In a metric space (M, d)
① The union of an arbitrary collection of open subsets is open
② The finite intersection of open subsets of M is open.

pf. ① Let A_λ be an open subset of M.
$\forall \lambda \in \Lambda$. Let $x \in \bigcup_{\lambda\in\Lambda} A_\lambda$, then $\exists \lambda_0$

$\in \Lambda$ s.t. $x \in A_{\lambda_0}$.
$\because A_{\lambda_0}$ is open $\therefore \exists r>0$ s.t. $D(x,r) \subset A_{\lambda_0}$.
$\Rightarrow D(x,r) \subset \bigcup_{\lambda\in\Lambda} A_\lambda$

$\Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda$ is open.

② See

Q.E.D.

Q: In $\mathbb{R}^n$, openness 对平移 (translation) 或拉張 (dilation) 是否具有封閉性?

Def. Let $A, B \subset \mathbb{R}^n$, $A+B \triangleq \{x+y \mid x\in A, y\in B\}$ is called the translation.

ex. If A is open, is $A+B$ open?

§

pf: Let $y \in B$. To show $A+\{y\}$ is open.
Let $p \in A+\{y\}$. then $\exists\, x \in A$ s.t. $p = x+y$.
$\because A$ is open $\therefore \exists\, r>0$ s.t. $D(x,r) \subset A$. To show that $D(p,r) \subset A+\{y\}$,
let $w \in D(p,r)$, then $\|w-p\| < r$
$\Rightarrow \|w-x-y\| < r$
$\Rightarrow \|x-(w-y)\| < r$.
$\Rightarrow w-y \in D(x,r)$.
$\because w = (w-y)+y \quad \therefore w \in A+\{y\}$
$\Rightarrow D(p,r) \subset A+\{y\}$
$\Rightarrow A+\{y\}$ is open

$\because A+B = \bigcup_{y \in B} (A+\{y\}) \quad \therefore A+B$ is open

Q.E.D.

HW p108 1,3,4,5,6
p145 18,24,25,30,42

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§2-2 Interior of a set.

① $\forall$ n.b.h. of $x \Leftrightarrow \forall$ r-ball about x.
② $\exists$ n.b.h. of $x \Leftrightarrow \exists$ r-ball about x

想討論集合M中的元素与給定的子集A的关係.
從 metric d 引進的空間感来討論. 就可以依元素x与子集A的相对位置来討論. 故可分為
① 元素x在A的內部
② x在A的聚点處 (2.4)
③ x在A的边界 (2.6)
④ x在A的外部 (与①類似 ∵ x即在A之補集內).

Q: 如何認定元素x会落在子集A的內部?

Def ① Let A be a subset of a metric space (M,d). A point x is an interior pt of A if $\exists$ a n.b.h. U of x s.t. $U \subset A$.
② The interior of A, denoted by $\text{int}(A)$ (or $\mathring{A}$) is the set of all interior pts of A. ($\text{int}(A) \subset A$).

Thm. ① $\text{int}(A)$ is open.
② $\text{int}(A)$ is the largest subset of A. In other words, $\text{int}(A)$ is the union of open subsets in A.

pf. ① Let $x \in \text{int}(A)$, then $\exists r>0$ s.t. $D(x,r) \subset A$. ∵ $D(x,r)$ is open, ∴ $D(x,r)$ is a n.b.h. of any pt of $D(x,r)$.
⇒ any point of $D(x,r)$ is an interior point of A.

$\Rightarrow D(x,r) \subset int(A)$.
$\Rightarrow int(A)$ is open.

② By similar argument in the proof of ①, $U \subset int(A)$ whenever U is an open subset of A.

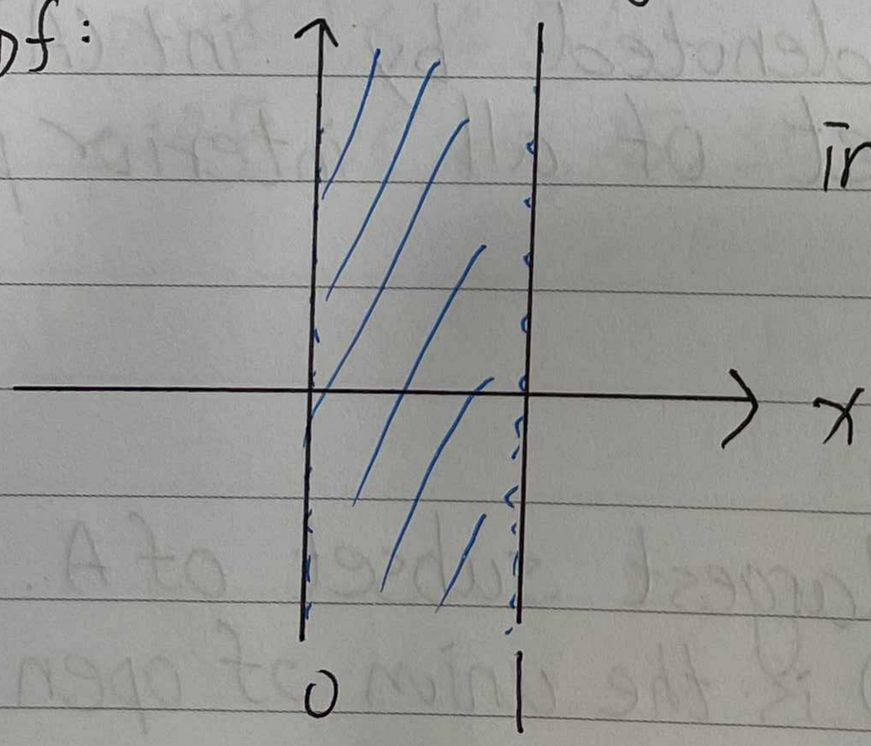
(续) Thm. ③ A is open $\Leftrightarrow A = int(A)$.

pf: ($\Leftarrow$) Obvious by ①
($\Rightarrow$) $\because A$ is open $\therefore$ by ②, $A \subset int(A)$.
$\because int(A) \subset A \therefore int(A) = A$.

Q.E.D.

ex. Let $S = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1\}$. Find $int(S)$

pf:



$int(S) = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1\}$

Q: ① $int(A)$ 与集合运算 "$\subset$" 具有封闭性与否?
② $int(A)$ 与 $\cup$ or $\cap$ 是否具有分配律?
A: ① Assume $A \subset B$, Let $x \in int(A)$, then $\exists$ a n.b.h. U of x s.t. $U \subset A$. $\because A \subset B \therefore U \subset B$
$\Rightarrow x \in int B$.

$\Rightarrow int(A) \subset int(B)$.

② $\because A, B \subset A \cup B$ $\therefore$ By ① $int(A) \subset int(A \cup B)$ and $int(B) \subset int(A \cup B)$

$\Rightarrow int(A) \cup int(B) \subset int(A \cup B)$

ex. Let $A = [0,1]$, $B = [1,2]$ in $\mathbb{R}$.
Then $int(A) = (0,1)$, $int\, B = (1,2)$.
But $A \cup B = [0,2]$, $int(A \cup B) = (0,2)$
$int(A) \cup int(B) \neq int(A \cup B)$ #.

③ From ① we know that $int(A \cap B) \subset (int(A) \cap int(B))$
To show $int(A) \cap int(B) \subset int(A \cap B)$
$\because int(A) \subset A$ and $int(B) \subset B$
$\therefore int(A) \cap int(B) \subset A \cap B$.
$\because int(A)$ and $int(B)$ are open
$\therefore int(A) \cap int(B)$ is open
$\Rightarrow int(A) \cap int(B) \subset int(A \cap B)$
Therefore $int(A \cap B) = int(A) \cap int(B)$ #

Thm. ① If $A \subset B$, then $int(A) \subset int(B)$.
② $int(A) \cup int(B) \subset int(A \cup B)$
③ $int(A \cap B) = int(A) \cap int(B)$.

ex. Is it true that, in general metric space (M, d) that $int\{y \in M : d(x,y) \leq r\} = \{y \in M : d(x,y) < r\}$? where $x \in M$ and $r > 0$ are given.

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Ans: No! In discrete metric,

$\{y \in M : d(x,y) \leq 1\} = M$

but $\{y \in M : d(x,y) < 1\} = \{x\}$.

→ 只有在 $\mathbb{R}^n$ 裡，去掉 "=" 才代表去掉邊界！

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§2.3 Closed Sets

Def. A set A in a metric space is called closed, if $M \setminus A$ is open.

Remark. In $(\mathbb{R}^n, \|\cdot\|)$, openness means 集合完全沒邊, 故其補集完全有邊. 故 closedness means 完全有邊.

Q. closed sets 是 open sets 的補集, 而 open sets 有性質, 那這些性質是否能順利遷移到它的補集去呢?

A. 討論 closedness 与「$\cup$, $\cap$」的封閉性.

① Let A_λ be a closed set, $\forall \lambda \in \Lambda$.

$$M \setminus \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \bigcap_{\lambda\in\Lambda} (M \setminus A_\lambda)$$

$\because A_\lambda$ is closed $\therefore M \setminus A_\lambda$ is open, $\forall \lambda \in \Lambda$.

Hence if Λ is a finite set,

$M \setminus \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$ is open $\Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda$ is closed.

② $\because M \setminus \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda} (M \setminus A_\lambda)$ is open

$\therefore \bigcap_{\lambda\in\Lambda} A_\lambda$ is closed.

Thm. In a metric space (M, d).

① The finite union of closed sets is closed.

② The intersection of any family of closed sets is closed

③ M and $\emptyset$ are closed.

Double A

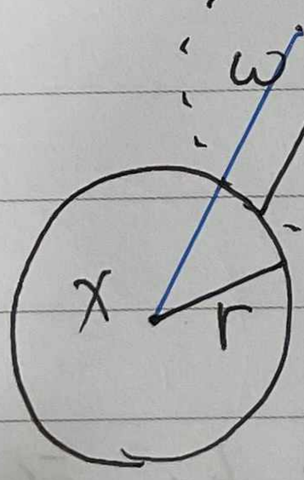
Remark. ① $\bigcup_{n=1}^{\infty}[\frac{1}{n},1]=(0,1]$ → 不開不閉

Def. The set $\{y\in M : d(x,y)\le r\} \triangleq B(x,r)$.

Prop. In a metric space (M,d)
① any single point set is closed.
② $B(x,r)$ is closed, $\forall x\in M$, $\forall r>0$.

pf: ① To show that $M\setminus\{x\}$ is open,
If $M\setminus\{x\}=\emptyset$, it's done.
If $y\in M\setminus\{x\}$, then $D(y, d(x,y))$
$\hat{x\notin}$
$\Rightarrow D(y,d(x,y))\subset M\setminus\{x\}$
$\Rightarrow M\setminus\{x\}$ is open
$\Rightarrow \{x\}$ is closed.

② Let $y\in M\setminus B(x,r)$
then $d(y,x)>r$. Let $\varepsilon=d(y,x)-r>0$.
Let $w\in D(y,\varepsilon)$, $\because d(x,w)\ge d(x,y)-d(w,y)$
$>d(x,y)-\varepsilon$
$=d(x,y)-(d(y,x)-r)$
$=r$
$\therefore w\notin B \Rightarrow w\in M\setminus B$
$\Rightarrow D(y,w)\subset M\setminus B$.
$\Rightarrow M\setminus B$ is open
$\Rightarrow B$ is closed.



Q.E.D.

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e.g. Let A be a finite set in a metric space (M, d). Let $B = \{x \in M : d(x, y) \leq 1 \text{ for some } y \in A.\}$. Show that B is closed.

pf: Let $A = \{y_1, \dots, y_n\}$, then $B = \bigcup_{i=1}^{n} B(y_i, 1)$

$\because B(y_i, 1)$ is closed $\forall i = 1, \dots, n$

$\therefore B$ is closed. ∎

p112 2, 4, 5, 6

p143 1, 20(b), 22.

補充題: Show that $A = \{\frac{1}{n} \in \mathbb{R} : n \in \mathbb{N}\} \cup \{0\}$ is closed in $\mathbb{R}$.

§2.4 Accumulation Point.

↳ 会出現的地方

Q: 如何認知 M 中的元素 x 为某一给定的子集 A 的聚点?

從 M 中去找聚点.

Def. ① A point x in a metric space M is called an accumulation point of a subset A of M if $\forall$ n.b.h. U of x, $(U \setminus \{x\}) \cap A \neq \emptyset$.

② We denote A' the set of accumulation pts of A.

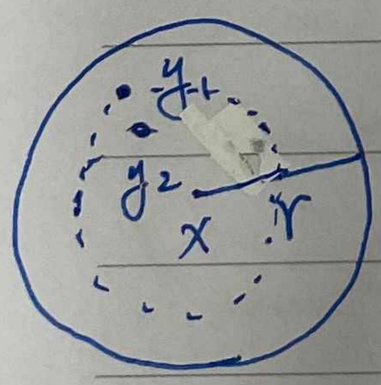
Thm. Let A be a subset of a metric space (M, d). If $x \in A'$, then every n.b.d. U of x contains infinitely many distinct elements in A.

pf: It suffices to show that it's true for $D(x,r)$, $r>0$.

$\because x \in A'$, $\therefore \exists\, y_1 \in (D(x,r) \setminus \{x\}) \cap U$.

Let $r_2 < d(x, y_1)$ (>0)

$\because x \in A'$ $\therefore \exists\, y_2 \in (D(x, r_2) \setminus \{x\}) \cap A$.



Let $r_3 = d(x, y_2) > 0$. Continuing this process (infinitely) many times), we obtain $y_1, y_2, \ldots, y_n, \ldots \in A$ that are distinct pts in $D(x,r)$.

Q.E.D.

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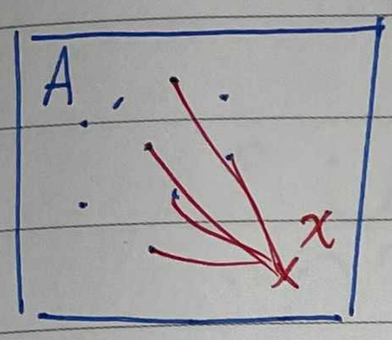
Cor. Any finite set in a Metric space has no Accumulation pts.

pf. Let $A=\{x_1, x_2, \dots, x_n\}$. Let $x \in M \setminus A$



Let $r=\min(d(x,x_1)\dots, d(x,x_n))>0$.
Then $x_i \notin D(x,r)$, $\forall i=1,\dots,n$.
$\Rightarrow D(x,r)\cap A=\emptyset \Rightarrow (D(x,r)\setminus\{x\})\cap A=\emptyset$.
$\Rightarrow x \notin A'$.

Let $x \in A$, say $x=x_i \Rightarrow \dots$

Q.E.D.

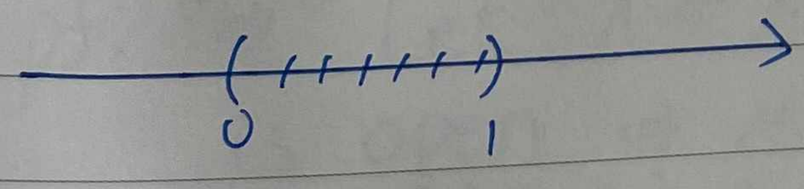
Question: infinite set A 是否一定就有 accumulation pt 呢?

Answer: No. ex. $\mathbb{Z}$、$\mathbb{N}$ in $\mathbb{R}$

ex. Find A' if ① $A=(0,1)$ ② $A=(0,1]$
③ $A=[0,1]$ in $\mathbb{R}$.

-sol-.

由图可知 A 聚集在 $[0,1]$. 因此去証 $A'=[0,1]$.



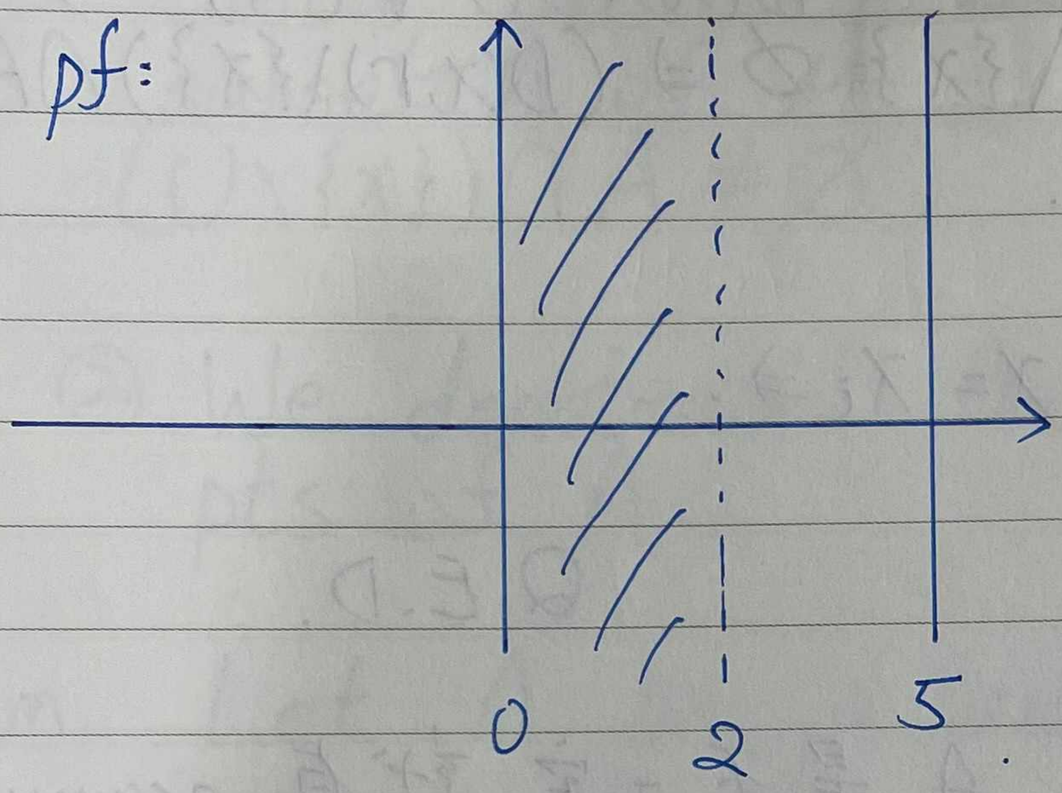
$\begin{cases} \because x \notin [0,1] \\ x=[0,1]. \end{cases}$

ex. Let $\mathbb{Q}=\{x \in \mathbb{R} : x \text{ is rational}\} \subset \mathbb{R}$.
Find $\mathbb{Q}'$ and $(\mathbb{R}\setminus\mathbb{Q})'=\mathbb{Q}^{c'}$.
pf: $\because$ Every interval in $\mathbb{R}$ contains infinitely rational and irrational numbers

$\therefore \mathbb{Q}' = \mathbb{R}$ and $(\mathbb{R} \setminus \mathbb{Q})' = \mathbb{R}$. //

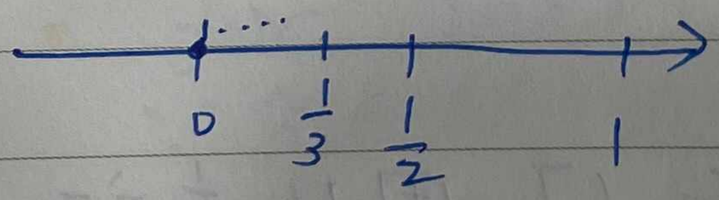
ex. Let $A = \{(x,y) \in \mathbb{R}^2 : 0 \le x < 2 \text{ or } x = 5\} \subset \mathbb{R}^2$
Find A'

pf:



由圖, $A' = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2$

ex. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ in $\mathbb{R}$. Find A'.



$A' = \{0\}$.

ex. In (M, d), is it true that $(D(x,r))' = B(x,r)$?

-sol- No. Let d be the discrete matrix.
$\because D(x, \frac{1}{2}) = \{x\} \therefore (D(x, \frac{1}{2}))' = \emptyset$. ($\because$ finite)
but $B(x, \frac{1}{2}) = \{x\}$.

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Q: 取 accumulation pt 是否对 "C" 具有封閉性呢? 与 $\cup$、$\cap$ 是否具分配律?

pf: "C" 明顯对: ∵ 若 $A \subset B$, 則 A 的聚点附近碰到A的点 → 会碰到 B 的点

"$\cup$" 也对

"$\cap$" 不对, ∵ 可能交成有限点 #

Thm. Let A, B be two subsets in a metric space (M, d), then

① If $A \subset B$, then $A' \subset B'$

② $(A \cup B)' = A' \cup B'$

③ $(A \cap B)' \subset A' \cap B'$.

pf: exercise.

Thm: Let A be a subset of a metric space (M, d), then A is closed $\Leftrightarrow A' \subset A$.

pf. "$\Rightarrow$" Let $x \in M \setminus A$. ∵ A is closed ∴ $M \setminus A$ is open $\Rightarrow M \setminus A$ is a nbd of x s.t. $((M \setminus A) \setminus \{x\}) \cap A = \emptyset$.

$\Rightarrow x \notin A'$. $\Rightarrow (M \setminus A) \cap A' = \emptyset$.

$\Rightarrow A' \subset A$.

"$\Leftarrow$" Let $x \in M \setminus A$ ∵ $A' \subset A$ ∴ $x \notin A'$.

$\Rightarrow \exists\, r > 0$ s.t. $(D(x, r) \setminus \{x\}) \cap A$

$= \emptyset \Rightarrow D(x,r) \setminus \{x\} \subset M \setminus A$.

$\because x \in M \setminus A \therefore D(x,r) \subset M \setminus A$

$\Rightarrow M \setminus A$ is open

$\Rightarrow A$ is closed. ※.

Exercise. p115. 2, 3, 4, 6

p145. 19, 28(a)(b)(c), 29, 31, 36.

Cor. If $A' = \emptyset$, then A is closed.

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§2.5 Closure of a set.

Let A be a subset of a metric space (M,d). Consider the set $A \cup A'$

Let $x \in M \setminus (A \cup A')$. $\because x \notin A'$ $\therefore \exists$ a n.b.h. U of x s.t. $(U \setminus \{x\}) \cap A = \emptyset$.

$\because x \notin A$ $\therefore U \cap A = \emptyset$.

$\because U$ is open, $\therefore \forall y \in U,\ y \notin A'$.

$\Rightarrow U \cap A' = \emptyset$. $\Rightarrow U \cap (A \cup A') = \emptyset$.

$\Rightarrow U \subset M \setminus (A \cup A')$

$\Rightarrow M \setminus (A \cup A')$ is open $\Rightarrow A \cup A'$ is closed. closed.

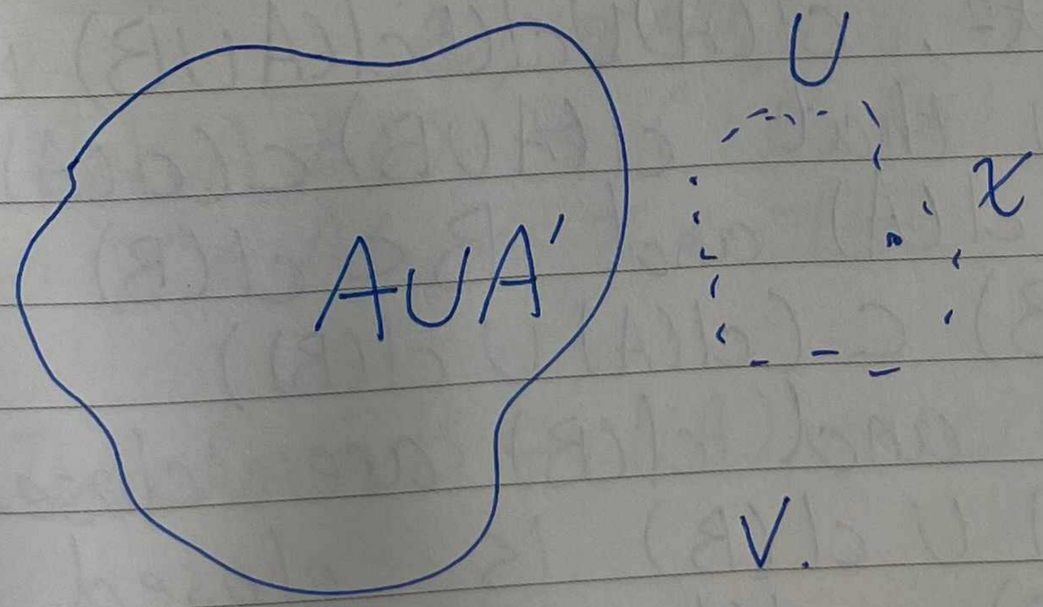
Moreover, $\because U \cap (A \cup A') = \emptyset$ $\therefore A \cup A' \subset \underline{M \setminus U}$.

Conclusively, an open set U intersecting A empty $\rightarrow (A \cup A') \subset M \setminus U$.

$\Leftrightarrow$ a closed set $V = M \setminus U$ containing A.

Therefore any closed set V containing A contains $A \cup A'$.

$\Rightarrow A \cup A'$ is the smallest closed set containing A.



Double A

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Def. Let A be a subset of a metric space, the closure of A, denoted by $cl(A)$, is the set $A \cup A'$

Thm. Let A be a subset of a metric space (M, d), then:
① $cl(A)$ is closed
② $cl(A)$ is the smallest closed set containing A.
③ $cl(A)$ = the intersection of all closed sets containing A.

Rmk. In $\mathbb{R}^n$, $cl(A)$ = 把 A 的边界補全.

Cor. A is closed $\Leftrightarrow A = cl(A)$

Q: closure 与 "$\subset$" "$\cup$" "$\cap$" 的運算关係?

pf: ① If $A' \subset B' \Rightarrow (A \cup A') \subset (B \cup B')$
$\Rightarrow cl(A) \subset cl(B)$

② 由① 可知, $cl(A) \cup cl(B) \subset cl(A \cup B)$
To show that $cl(A \cup B) \subset (cl(A) \cup cl(B))$
$\because A \subset cl(A)$ and $B \subset cl(B)$
$\therefore (A \cup B) \subset (cl(A) \cup cl(B))$
$\because cl(A)$ and $cl(B)$ are closed
$\therefore cl(A) \cup cl(B)$ is closed
$\Rightarrow cl(A \cup B) \subset cl(A) \cup cl(B)$.
Therefore $cl(A \cup B) = cl(A) \cup cl(B)$

Double A

③ 由①, $cl(A\cap B)\subset(cl(A)\cap cl(B))$.
However, $cl(A)\cap cl(B)\not\subset cl(A\cap B)$.
For example, In $\mathbb{R}$, $A=(0,1)$ and $B=(1,2)$. then $A\cap B=\emptyset$
$\Rightarrow cl(A\cap B)=\emptyset$ But $cl(A)=[0,1]$ and $cl(B)=[1,2]$ and $cl(A)\cap cl(B)=\{1\}$.

Thm. Let A and B be subsets of a metric space (M,d), then:
① $cl(A)\subset cl(B)$, if $A\subset B$.
② $cl(A\cup B)=cl(A)\cup cl(B)$
③ $cl(A\cap B)\subset cl(A)\cap cl(B)$

Prop. $cl(A)=\{x\in M:$ every n.b.h. of x meets $A\}$
交A非空.↵

pf: Let $B=\{x\in M:$ every n.b.h. of x meets $A\}$, then $A\subset B$ and $A'\subset B$
$\Rightarrow A\cup A'\subset B. \Rightarrow cl(A)\subset B.$

Let $x\in B$. Assume $x\in A$, then $x\in cl(A)$.
Assume $x\notin A$, $\because x\in B$, $\therefore\forall$ nbh U of x,
$U\cap A\neq\emptyset. \Rightarrow (U\setminus\{x\})\cap A\neq\emptyset.$
$\Rightarrow x\in A'. \Rightarrow x\in cl(A).$

Therefore $B\subset cl(A)\Rightarrow cl(A)=B.$

Q.E.D.

Double A

ex. Let $A \subset (M, d)$. Show that $x \in cl(A)$
$\iff \inf\{d(x,y) : y \in A\} = 0$.

pf: "$\Rightarrow$" Assume $x \in A$, then $\inf\{d(x,y) : y \in A\}$
$= 0$. Assume $x \in A' \setminus A$, then $d(x,y) \geq 0$,
$\forall y \in A$. $\Rightarrow 0$ is a lower bound of
$B = \{d(x,y) : y \in A\} \subset \mathbb{R}$.

Let $\varepsilon > 0$, $\because x \in A' \therefore \exists y \in (D(x,\varepsilon) \setminus \{x\}) \cap A$
$\Rightarrow y \in A$ and $d(x,y) < \varepsilon$. $\Rightarrow \varepsilon$ is not a
lower bound
$\Rightarrow \inf B = 0$.

"$\Leftarrow$" Let $r > 0$. To show $D(x,r) \cap A \neq \emptyset$,
$\because \inf B = 0 \therefore \exists y \in A$ s.t. $d(x,y) < 0 + r$.
$\Rightarrow y \in D(x,r)$
$\Rightarrow D(x,r) \cap A \neq \emptyset$.
$\Rightarrow x \in cl(A)$.

Q.E.D.

Exercise: p117. 1, 3, 4, 5
p143. 3, 14, 41, 44.

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§5 Boundary of a set.

Q: 如何認知 M 中的元素 x 為子集 A 的邊界点呢?

Def. Let A be a subset of a metric space (M, d)

① A pt $x \in M$ is a boundary pt of A if $\forall$ n.b.d. U of x, $U \cap A \neq \emptyset$ and $U \cap (M \setminus A) \neq \emptyset$.

② The boundary of A, denoted by $bd(A)$, is the set of all boundary pts of A.

Thm. ① $bd(A) = bd(M \setminus A)$ (From def.)

② $bd(A) = cl(A) \cap cl(M \setminus A)$ (by prop.).

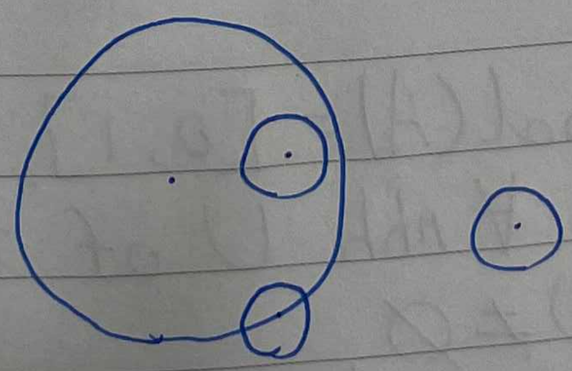
③ $cl(A) = A \cup bd(A)$. (exercise).

Cor. ① $bd(A)$ is closed.

② If A is closed, then $bd(A) \subset A$.

ex. $bd(D(x,r)) = ?$ in $(\mathbb{R}^n, \|\cdot\|)$.

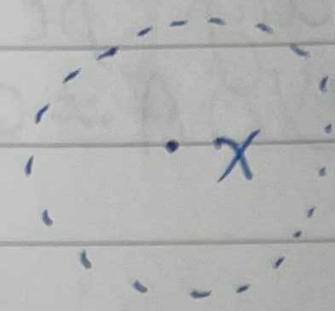
pf: $bd(D(x,r)) = \{y \in M : d(x,y) = r\}$, 討論球內, 球外 与 邊界上之点.



ex. Find $bd(A)$, if A = a single point set in $(\mathbb{R}^n, \|\cdot\|)$.

pf: 直觀來說, $bd(A) = A$.

Let $A = \{x\}$.



Cor. A is closed $\Leftrightarrow bd(A) \subset A$

pf: ($\Leftarrow$) Let $x \in M \setminus A$, $\because bd(A) \subset A$

$\therefore x \notin bd(A)$.

$\because x \notin A \Rightarrow \exists$ a n.b.h. U of x

s.t. $U \cap A = \emptyset$.

$\Rightarrow U \subset M \setminus A$.

$\Rightarrow M \setminus A$ is open.

$\Rightarrow A$ is closed.

Q.E.D.

ex. Let $A = \mathbb{Q} \cap (0,1) \subset \mathbb{R}$. Find $bd(A)$

pf. 由图形觀察可知 $bd(A) = [0,1]$.

Let $x \in [0,1]$, then $\forall$ nbh U of x,

$U \cap A \neq \emptyset$, $U \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$

$\Rightarrow x \in bd(A) \Rightarrow [0,1] \subset bd(A)$

Let $x \notin [0,1]$, let $r = \min(|x|, |x-1|)$

then $D(x,r) \cap [0,1] = \emptyset$.

$\Rightarrow D(x,r) \cap A = \emptyset$.

$\Rightarrow x \notin bd(A)$.

$\Rightarrow \mathbb{R} \setminus [0,1] \subset \mathbb{R} \setminus bd(A)$

$\Rightarrow bd(A) \subset [0,1]$

$\Rightarrow bd(A) = [0,1]$.

ex. Determine which of the following statements are true.

(a) $int(\bar{A}) = int(A)$ X, 应改为"⊃"

Let $A = \mathbb{Q}$, $int(\bar{\mathbb{Q}}) = (0,1)$, $int(\mathbb{Q}) = \emptyset$.

(b) $\bar{A} \cap A = A$ (O)

(c) $\overline{int(A)} = A$. X, A未必 closed

(d) $bd(\bar{A}) = bd(A)$. X.

$bd(\bar{\mathbb{Q}}) = bd(\mathbb{R}) = \emptyset$

$bd(\mathbb{Q}) = \mathbb{R}$.

(e) If A is open, then $bd(A) \subset M \setminus A$. (O)

(∵ A 只有内点).

Exercise:

p 120. 1 2 4 5

p 145. 2 7 9 10 13 15 28(6)

§2-7. Sequences.

Def. A sequence x_n in a metric space (M,d) converges to x in M if $\forall \varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. $\forall n>N$, $d(x_n,x)<\varepsilon$. We write $x_n\to x$.

Thm. Let (x_n) be a seq. in (M,d)
then $x_n\to x$ in M $\Leftrightarrow$ $\forall$ n.b.h U of x, $\exists N\in\mathbb{N}$ s.t. $\forall n>N$, $x_n\in U$.

pf: Exercise.

Thm. Let A be a subset of a metric space (M,d).
① $x\in cl(A)$. $\Leftrightarrow$ $\exists$ a seq. x_n in A that converges to x. (in M).
② A is closed $\Leftrightarrow$ every seq. in A that converges in M has limit in A.

pf. ① "$\Rightarrow$" $cl(A)=\{x\in M$: every n.b.h. U of x meets $A\}$.
$\because x\in cl(A)$
$\therefore \forall n\in\mathbb{N}, \exists x_n\in D(x,\frac{1}{n})\cap A$
$\Rightarrow d(x_n,x)<\frac{1}{n}$
$\Rightarrow d(x_n,x)\to 0$.
$\Rightarrow x_n\to x$

"$\Leftarrow$" Let U be a n.b.h. of x
$\because x_n\to x$
$\therefore \exists x_{n_0}\in U\cap A$ for some $n_0\in\mathbb{N}$.
(since $\exists N\in\mathbb{N}$ s.t. $\forall n>N$, $x_n\in U$)

$\Rightarrow U \cap A \neq \emptyset$.

$\Rightarrow x \in cl(A)$.

② $\because A$ is closed $\Leftrightarrow A = cl(A)$, by ①,
② holds.

Q.E.D.

ex. Let x_n be a convergent sequence in $\mathbb{R}^n$ with $\|x_n\| \leq 1$, $\forall n \in \mathbb{N}$. Show that the limit x also satisfies $\|x\| \leq 1$.

pf. $\because B(0,1)$ is closed and $x_n \in B(0,1)$
$\therefore x \in B(0,1) \Rightarrow \|x\| \leq 1$. #

ex. If we change the condition in previous example to $\|x_n\| < 1$, $\forall n \in \mathbb{N}$, do we have $\|x\| < 1$?

pf: No! $\because x \in D(0,1) \subseteq B(0,1)$
$\therefore \|x\| \leq 1$. #.

ex. Find the closure of $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$.

pf. Any seq. in A converges, if converges, to $1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, \cdots, 0$. $\Rightarrow cl(A) \subset A$, by ①. and we can construct a seq. in A converges to $\frac{1}{n}$, $\forall n \in \mathbb{N}$ or 0.
$\Rightarrow cl(A) = A$. #.

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補 1.7 Norms.

Def. Let V be a vector space, a function $\|\cdot\| : V \to \mathbb{R}$ is called a norm on V if (i) $\|v\| \geq 0$, $\forall v \in V$. (positivity)

(ii) $\|v\| = 0$ if and only if $v = 0$. (nondegeneracy)

(iii) $\|\lambda v\| = |\lambda| \cdot \|v\|$, $\forall v \in V$ and $\forall \lambda \in \mathbb{R}$ (multiplicativity)

(iv). $\|v+w\| \leq \|v\| + \|w\|$. (triangle ineq.)

Def. A vector space V with a norm $\|\cdot\|$ on it is called a normed space, written as $(V, \|\cdot\|)$

ex. $(\mathbb{R}^n, \|\cdot\|)$ is a normed space.

Prop. Let $(V, \|\cdot\|)$ be a normed space. Define a function $d : V \times V \to \mathbb{R}$ by $d(v,w) = \|v-w\|$ Then d is a metric on $\mathcal{M}$.

pf. Exercise.

Rmk. A normed space is a metric space in such way. (反之不必然!)

Def. Let v_n be a sequence in a normed space $(V, \|\cdot\|)$. We say that v_n converges to v in V, if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\|v_n - v\| < \varepsilon$.
($\Leftrightarrow \|v_n - v\| \to 0$ in $\mathbb{R}$).

Thm. Let v_n and w_n be two sequences in a normed space $(V, \|\cdot\|)$, and λ_n is a sequence in $\mathbb{R}$.
If $v_n \to v$, $w_n \to w$, and $\lambda_n \to \lambda$, then:
(i) $v_n + w_n \to v + w$
(ii) $\alpha v_n \to \alpha v$. $\forall \alpha \in \mathbb{R}$
(iii) $v_n \lambda_n \to v\lambda$
(iv) If $\lambda_n \neq 0$, $\forall n \in \mathbb{N}$ and $\lambda \neq 0$, then
$\frac{1}{\lambda_n} v_n \to \frac{1}{\lambda} v$.

pf. Let $\varepsilon > 0$. (要证 $\|\lambda_n V_n - \lambda V\| < \varepsilon$ 解 N.)
$\|\lambda_n v_n - \lambda v\| \leq \|\lambda_n v_n - \lambda_n V + \lambda_n V - \lambda V\|$
$\leq |\lambda_n| \|v_n - v\| + \|v\| |\lambda_n - \lambda|$
$|\lambda_n| \leq M$ ∵ 可解. $\|v\|$ const.
$\|v_n - v\|$
解 $< \frac{\varepsilon}{2}$
$|\lambda_n - \lambda|$
解 $< \frac{\varepsilon}{2}$ #.

Exercise: p123. 1, 2, 5.
p144. 4, 6, 8, 40, 43
p172. 16.

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上标、

Thm. Let $v_k = (v_k^1, \dots, v_k^n) \in \mathbb{R}^n$ be a seq. in $\mathbb{R}^n$ and $v = (v^1, \dots, v^n) \in \mathbb{R}^n$

Then $v_k \to v \iff v_k^i \to v^i$, $\forall i = 1, \dots, n$.

pf. Notice that $v_k \to v \iff \|v_k - v\| \to 0$.

$\overset{?}{\iff} |v_k^i - v^i| \to 0$. $\forall i = 1, \dots, n$. (Why?)

(思考 | $\because \|v_k - v\| = \sqrt{\sum_{k=1}^{n} (v_k^i - v^i)^2}$

$\therefore \|v_k - v\| \le \sum_{i=1}^{n} |v_k^i - v^i|$,)

$\iff v_k^i \to v^i$, $\forall i = 1, \dots, n$.

$\because |v_k^i - v^i| \le \|v_k - v\|$, $\forall i = 1, \dots, n$

$\therefore \frac{1}{n} \sum_{i=1}^{n} |v_k^i - v^i| \le \|v_k - v\|$.

Hence $\frac{1}{n} \sum_{i=1}^{n} |v_k^i - v^i| \le \|v_k - v\| \le \sum_{i=1}^{n} |v_k^i - v^i|$

$\Rightarrow \|v_k - v\| \to 0 \iff |v_k^i - v^i| \to 0$.

Q.E.D.

ex. Show that $v_k = (\frac{1}{k}, \frac{1}{k^2}) \to (0,0)$ in $\mathbb{R}^2$.

pf. $\because \frac{1}{k} \to 0$ and $\frac{1}{k^2} \to 0$ as $k \to \infty$

$\therefore v_k \to (0,0)$ ※.

§2.8 Completeness.

Def. A seq. x_n in a metric space (M,d) is a Cauchy sequence if $\forall \varepsilon>0$, $\exists N \in \mathbb{N}$ s.t. $\forall n,m>N$, $d(x_n,x_m)<\varepsilon$.

Q: 怎樣的收斂列 in (M,d) 會是 Cauchy seq. 呢?
A: Let $x_n \to x$ in (M,d).

Let $\varepsilon>0$. $d(x_n,x_m) \le d(x_n,x)+d(x,x_m) < \varepsilon$.
$\because x_n \to x$ $\therefore$ For $\varepsilon/2>0$, $\exists N \in \mathbb{N}$ s.t.
$\forall n>N$, $d(x_n,x)<\varepsilon/2$.
$\Rightarrow \forall m,n>N$, $d(x_n,x_m) \le \varepsilon/2+\varepsilon/2=\varepsilon$.

Therefore x_n is a Cauchy seq.

Thm. A conv. seq. in a metric space is a Cauchy seq.

Q: Is a Cauchy seq. in a metric space a convergent seq.?
A: No!
① $\because \frac{1}{n} \to 0$ in $\mathbb{R}$ $\therefore \frac{1}{n}$ is a Cauchy seq. in $\mathbb{R}$ $\Rightarrow \frac{1}{n}$ is also a Cauchy seq. in $\mathbb{R}\setminus\{0\}$.
(但此時已無處可收斂.)
but it doesn't conv. in $\mathbb{R}\setminus\{0\}$.

② Consider $(\mathbb{Q},\|\cdot\|)$, 可以造出一个 Cauchy seq. $x_n \in \mathbb{Q}$ s.t. $x_n^2 \to 2$, but it doesn't conv. in $\mathbb{Q}$.

Q: 怎樣的 metric space 其上的 Cauchy seq. 會收斂?

Def. A metric space (M,d) is called complete if every Cauchy seq. in M conges.

Q: $\mathbb{R}^n$ 上的 Cauchy seq. 它的座標主又列是否為 $\mathbb{R}$ 上的 Cauchy seq?

A: Let $x_k = (x_k^1, \cdots, x_k^n)$

$$\frac{1}{n}\sum_{i=1}^{n}|x_k^i - x_\ell^i| \le \|x_k - x_\ell\| \le \sum_{i=1}^{n}|x_k^i - x_\ell^i|$$

Thm. A seq. $x_k = (x_k^1, \cdots, x_k^n)$ in $\mathbb{R}^n$ is a Cauchy seq. iff x_k^i is a Cauchy seq. in $\mathbb{R}$, $\forall i = 1, \cdots, n$.

Rmk. $\mathbb{R}^n$ 是否 complete $\Leftrightarrow$ $\mathbb{R}$ 是否 complete.

Thm. $\mathbb{R}$ is complete. Hence $\mathbb{R}^n$ is complete, $\forall n \in \mathbb{N}$.

Def. Let x_n be a seq. in a metric space (M,d). x_n is said to be bounded if $\exists\, x_0 \in M$ and $r>0$ s.t. $x_n \in D(x_0, r)$. (or, $d(x_n, x_0) < r$).

Thm. ① A conv. seq. in (M,d) is bdd.
② A Cauchy seq. in (M,d) is bdd.

pf. ① Let $x_n \to x$ in (M,d), then $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $x_n \in D(x,1)$.
Let $r = \max(d(x_1,x), \dots, d(x_N,x))$, then $x_n \in D(x,r)$. $\forall x \in \mathbb{N}$
$\Rightarrow x_n$ is bdd. (② is similar)

Q.E.D.

Def. Let x_n be a seq. in a Metric space (M,d). x_{n_k} is called a subsequence of x_n if n_k is an increasing seq. in $\mathbb{N}$.

Thm. Every subsequence of a convergent seq. in (M,d) conv.

pf. Let $x_n \to x$ and x_{n_k} be a subseq. of x_n. Let $\varepsilon > 0$, 要從 $d(x_{n_k}, x) < \varepsilon$ 去解 N 給 n_k 用.
$\because x_n \to x$, $\therefore \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $d(x_n,x) < \varepsilon$
$\Rightarrow \forall k > N$, $d(x_{n_k}, x) < \varepsilon$. ($n_k \geq n_N \geq N$).
$\Rightarrow x_{n_k} \to x$ in (M,d).

Double A

Prop. If a subseq. of a Cauchy seq. conv. to x in (M,d), then the Cauchy seq. conv. to x.

pf: Let x_n be a Cauchy seq. in (M,d) and x_{n_k} a subseq. of x_n with $x_{n_k} \to x$.

Let $\varepsilon > 0$, $d(x_n, x) \le d(x_n, x_{n_k}) + d(x_{n_k}, x)$

$\because x_{n_k} \to x$

$\therefore$ For $\varepsilon/2$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall k > N_1$, $d(x_n, x) < \varepsilon/2$

$\because x_n$ is a Cauchy seq.

$\therefore$ For $\varepsilon/2$, $\exists N_2 \in \mathbb{N}$, $\forall m, n > N_2$ $d(x_n, x_m) < \varepsilon/2$.

Let $N = \max(N_1, N_2)$ then $\forall n > N$,

$$d(x_n, x) \le d(x_n, x_{n_{2N}}) + d(x_{n_{2N}}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore $x_n \to x$.

Q.E.D.

Thm. If (M,d) is a complete metric space, then every closed subset N of M is also complete.

pf: x_n be a Cauchy sequence in N, then x_n is a Cauchy sequence in M.

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$\because M$ is complete $\therefore x_n \to x$ in M.
$\because N$ is closed
$\therefore x \in N. \Rightarrow N$ is complete.

Q.E.D.

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Ch3. Compact and Connected Sets

在本章我們要討論一个子集A的元素在空間X中的分佈狀況：

① 它元素的分佈的幅員大小、(3-1)

② 它的元素分佈是連接成一片 or 散步呢？(3-5)

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§3.1 Compactness.

為何要研究子集A的元素在其中分佈的幅員大小呢?

Q: 那什麼24子集A元素分佈的幅員呢?

Def. Let A be a subset of a metric space (M,d)
① A cover of A is a collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of sets in M whose union contains A. (i.e. $A \subset \bigcup_{\alpha\in\Lambda} U_\alpha$)
② It's an open cover if for each U_α is open.

Def. A subcover of a given cover is a subcollection of $\{U_\alpha\}_{\alpha\in\Lambda}$ whose union contains A.

Q: 怎樣的子集A會被認為幅員小呢?
A: 用來覆蓋A的子集了越(少)→有限了
不管用什麼方法覆蓋它,都能找到有限了子集君來覆蓋住.

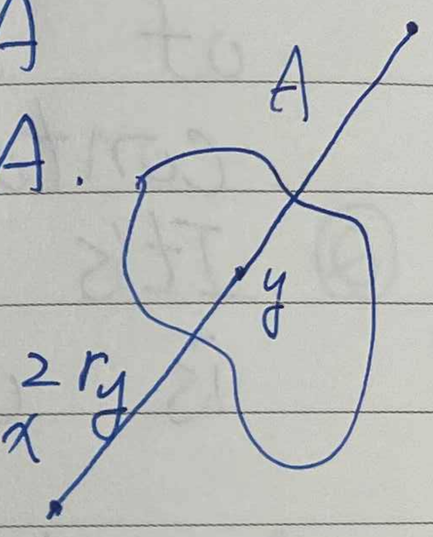
Def. A subset A of a metric space (M,d) is called compact, if every open cover of A has a finite subcover.

Double A

Q: 除了 finite set 以外，怎樣的子集 A 会 compact 呢?

Thm. If a set A in a metric space (M,d) is compact, then A is closed.

pf: Let $x \in M \setminus A$. Let $d(x,y) = 2r_y$, $\forall y \in A$.
then $r_y > 0$, $\forall y \in A$.
$\because B(x, r_y)$ is closed, $\forall y \in A$
$\therefore M \setminus B(x, r_y)$ is open, $\forall y \in A$.
$\Rightarrow \{M \setminus B(x, r_y) : y \in A\}$ is an cover of A.
$\because A$ is compact
$\therefore \exists$ finite subcover $\{M \setminus B(x, r_{y_1}), \cdots, M \setminus B(x, r_{y_n})\}$ of $\{M \setminus B(x, r_y) : y \in A\}$.



$$\Rightarrow A \subset \bigcup_{i=1}^{n} (M \setminus B(x, r_{y_i}))$$

$$= M \setminus \underbrace{\bigcap_{i=1}^{n} B(x, r_{y_i})}_{\text{包心球}}$$

Let $r = \min(r_{y_1}, \cdots, r_{y_n})$, then $A \subset M \setminus B(x, r)$.
$\Rightarrow B(x,r) \subset M \setminus A \Rightarrow D(x,r) \subset M \setminus A$.
$\Rightarrow M \setminus A$ is open
$\Rightarrow A$ is closed.

Q.E.D.

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Thm. Any closed subset A of a compact metric space (M, d) is compact.

pf. Let $\{U_\alpha\}$ be an open cover of A.
$\because A$ is closed
$\therefore M \setminus A$ is open $\Rightarrow \{U_\alpha, M \setminus A\}$ is an open cover of M.
$\because M$ is compact
$\therefore \exists$ a finite subcover $\{U_{\alpha_1}, \dots, U_{\alpha_n}, M \setminus A\}$ of $\{U_\alpha, M \setminus A\}$
$\Rightarrow \{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ covers A.
$\Rightarrow A$ is compact

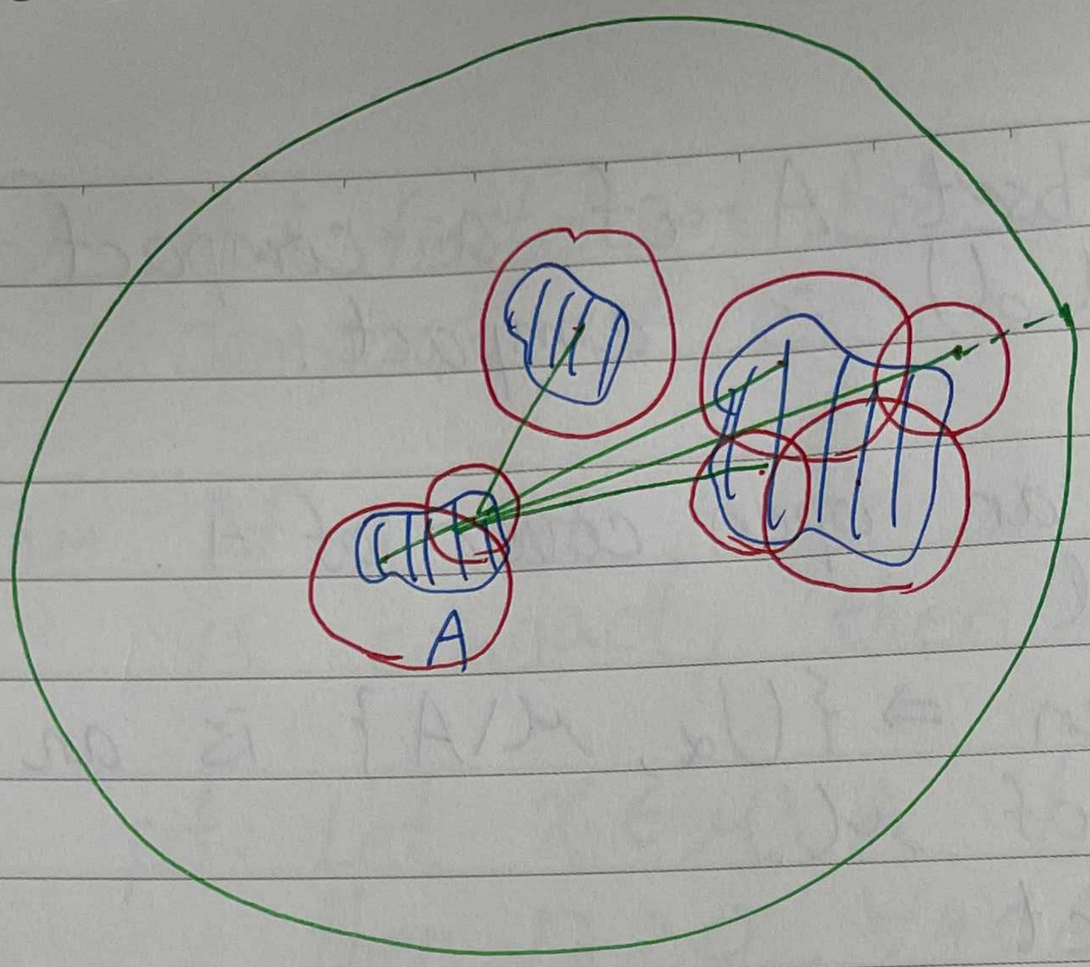
Q.E.D.

Def. A subset A of a metric space (M, d) is bounded, if $\exists\ x \in M$ and $r > 0$ s.t. $A \subset D(x, r)$.

Thm. If a set A is compact in (M, d), then A is bounded.

pf: $\because \{D(x, 1) : x \in A\}$ is an open cover of A, and A is compact
$\therefore \exists$ a finite subcover $\{D(x_1, 1), \dots, D(x_n, 1)\}$ of $\{D(x, 1) : x \in A\}$

Let $r = \max(d(x_1, x_2), d(x_1, x_3), \dots, d(x_1, x_n))$,
then by Δ ineq, $D(x_i, 1) \subset D(x_1, r+1)$,
$\forall\ i = 1, \dots, n$



$\Rightarrow$ A is bounded.

Q.E.D.

Def. A set $A \subset M$ is called totally bounded if $\forall r > 0$, $\exists$ a finite subsets $\{x_1, \dots, x_n\}$ of M s.t.
$A \subset \bigcup_{i=1}^{n} D(x_i, r)$

$\Rightarrow$ 任何半徑，一定可找到有限个球蓋住A.

Thm. ① If A is a compact subset of a metric space (M, d), then A is totally bounded

② Totally bounded $\Rightarrow$ bounded.

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Def. A subset A of a metric space M is said to be sequentially compact if every sequence in A has a subsequence that converges in A. (i.e. to a pt in A).

Thm. If a subset A of a metric space M is sequentially compact, then A is closed and bounded.

pf: ① Let x_n be a sequence in A that converges to x (in M).
$\because A$ is sequentially compact
$\therefore \exists$ a subsequence of x_n converges in A.
$\Rightarrow x \in A. \Rightarrow A$ is closed

② bounded $\Rightarrow$ 反證法
Assume A is unbounded. Let $x \in M$
$\because A$ is unbounded
$\therefore \exists x_n \in A$ s.t. $d(x_n, x) \geq n$, $\forall n \in \mathbb{N}$.
$\Rightarrow$ Every subsequence of x_n is unbounded.
$\Rightarrow$ Every subseq. of x_n doesn't converge.
$\rightarrow\leftarrow$ A is sequentially compact,
Therefore A is bounded.

Q.E.D.

Thm. (Bolzano Weierstrass Thm.)
A subset A of a metric space (M, d) is compact $\Leftrightarrow$ A is sequentially compact.

pf: ($\Rightarrow$) Let x_n be a sequence in A. Assume that x_n has no convergent subsequence. Then x_n has infinitely many distinct points
$\Rightarrow$ x_n has a subsequence $y_1, \dots, y_n, \dots$ of distinct element.

Claim: $\forall k \in \mathbb{N}$, $\exists$ an open n.b.h. U_k of y_k s.t. U_k contains no other y_i

pf: Otherwise, $\exists k_0 \in \mathbb{N}$ s.t. $\forall$ open n.b.h. U of y_{k_0}, U contains other y_i. In particular, it means that $\forall r > 0$, $D(y_{k_0}, r)$ contains infinitely many other y_i.

Let $y_{m_1} \in D(y_{k_0}, 1)$. Let $r_2 = \frac{d(y_{k_0}, y_{m_1})}{2}$.

then $\exists m_2 > m_1$ s.t. $y_{m_2} \in D(y_{k_0}, r_2)$.
Let $r_3 = \frac{d(y_{k_0}, y_{m_2})}{3}$, ..., and so on.
Then $r_n \to 0$ as $n \to \infty$ and $y_{m_1}, y_{m_2}, \dots$, is a subsequence

of y_n that converges y_{k_0}.
($\rightarrow\leftarrow$)
Therefore the claim holds. #

$\because$ x_n has no convergent subseq.
$\therefore$ " y_n "
$\Rightarrow \{y_1, y_2, \cdots, y_n, \cdots\}' = \emptyset$.
$\Rightarrow \{y_1, y_2, \cdots, y_n, \cdots\}$ is closed.

$\because$ A is compact,
$\therefore \{y_1, \cdots, y_n \cdots\}$ is compact.
$\therefore \{U_k\}$ collected from claim is an open cover of $\{y_1, \cdots, y_n, \cdots\}$ that has no finite subcover. ($\rightarrow\leftarrow$)

Therefore x_n has a convergent subseq.
$\because$ A is closed
$\therefore$ This convergent subsequence of x_n converges in A.

($\Leftarrow$) Let $\{U_\alpha\}$ be an open cover of A.

Lemma1: $\exists\, r>0$ s.t. $\forall y \in A$. $D(y,r) \subset U_\alpha$ for some α. such r is called a Lebesgue number for the cover $\{U_\alpha\}$

pf: If not, $\forall n \in \mathbb{N}$, $\exists y_n \in A$ s.t. $D(y_n, \frac{1}{n}) \not\subset U_\alpha$ for all α.

$\because A$ is sequentially compact
$\therefore \exists$ a subseq. y_{n_k} of y_n converges to $z \in A$.
$\because \{U_\alpha\}$ is an open cover of A,
$\therefore \exists \alpha_0$ s.t. $z \in U_{\alpha_0}$.
$\because U_{\alpha_0}$ is open
$\therefore \exists \varepsilon > 0$ s.t. $D(z, \varepsilon) \subset U_{\alpha_0}$.
$\because y_{n_k} \to z$
$\therefore$ For $\frac{\varepsilon}{2}$, $\exists K \in \mathbb{N}$ s.t. $\forall k > K$, $d(z, y_{n_k}) < \varepsilon/2$.
Let $k_0 > K$ and and $\frac{1}{n_{k_0}} < \frac{\varepsilon}{2}$
then by Δ ineq, $D(y_{n_{k_0}}, \frac{1}{n_{k_0}}) \subset D(z, \varepsilon)$.

$\Rightarrow D(y_{n_{k_0}}, \frac{1}{n_{k_0}}) \subset U_{\alpha_0}$. $\not\Rightarrow$
Hence $\exists r > 0$ s.t. $\forall y \in A$, $D(y, r) \subset U_\alpha$ for some α. $\not\Rightarrow$

Lemma 2. A is totally bounded.

pf. If not, then $\exists r > 0$ s.t. we can't cover A by finite r-balls.
Let $y_1 \in A$, then $\{D(y_1, r)\}$ can't cover A. $\Rightarrow \exists y_2 \in A \setminus D(y_1, r)$
$\therefore \{D(y_1, r), D(y_2, r)\}$ can't cover A.
$\Rightarrow \exists y_3 \in A \setminus (D(y_1, r) \cup D(y_2, r))$ ...

Continue this process, we obtain a seq. y_n in A s.t. $d(y_m, y_n)$

$\geq r$, $\forall m, n \in \mathbb{N}$. $\Rightarrow$ Every subseq. of y_n is not a Cauchy sequence. of y_n doesn't converges. ($\rightarrow\leftarrow$ A is sequentially compact!)

Therefore, A is totally bounded. #.

To continue our proof of ($\Leftarrow$), Let r be a Lebesgue number for the open cover $\{U_\alpha\}$.

$\because A$ is totally bounded by Lemma 2,

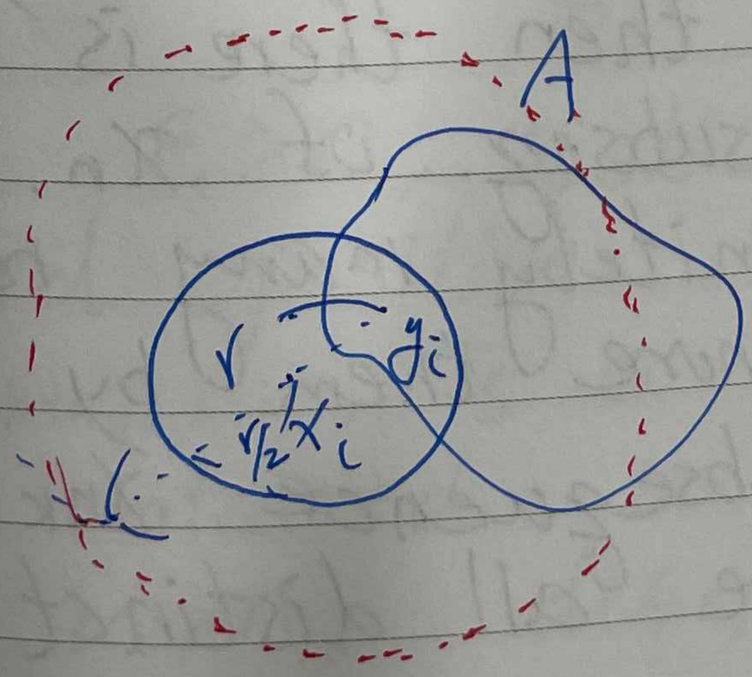
$\therefore$ For $r/2$, $\exists\ x_1, \ldots, x_N \in M$ s.t.

$$A \subset \bigcup_{i=1}^{N} D(x, \frac{r}{2}).$$

We may assume $D(x_i, r/2) \cap A \neq \emptyset, \forall i = 1, \ldots, N$! Let $y_i \in D(x_i, r/2) \cap A$, $\forall i = 1, \ldots, N$.

Then $D(x_i, r/2) \subset D(y_i, r)$, $\forall i = 1, \ldots, N$.

$\Rightarrow \{D(y_1, r), D(y_2, r), \ldots, D(y_N, r)\}$ covers A.



$\because D(y_i, r) \subset U_{\alpha_i}$ for some α_i, $\forall i = 1, \dots, N$
$\therefore \{U_{\alpha_1}, \dots, U_{\alpha_N}\}$ covers A.
$\Rightarrow A$ is compact.

Q.E.D.

Thm. A metric space M is compact
$\Leftrightarrow$ it is complete and totally bounded.

pf: "$\Rightarrow$": $\because M$ is compact
$\therefore M$ is totally bounded.
Let x_n be a Cauchy seq. in M, $\because M$ is compact, $\therefore$ by B-W Thm, M is sequentially compact
$\Rightarrow x_n$ has a convergent subseq.
$\Rightarrow x_n$ converges $\Rightarrow M$ is complete

"$\Leftarrow$" Let x_n be a seq. in M
If x_n has infinitely many repetitions, then there is a trivially convergent subseq. of x_n. If x_n has only finitely many repetitions, we may ignore them by choosing a proper subsequence x_{n_k} of x_n s.t. x_{n_k} are all distinct.

By that, we may assume x_n are all distinct.

$\because M$ is totally bounded

$\therefore$ For $r=1$, $\exists y_1, \dots, y_n \in M$ s.t.

$M \subset \bigcup_{i=1}^{n} D(y_i, 1)$

$\Rightarrow \exists$ a subseq. x_{1n} of x_n contained in one of the ball.

For $r = 1/2$, $\exists$ a subseq. x_{2n} of x_{1n} contained in a fixed ball of radius $1/2$

Repeating this process, $\exists$ a subseq. x_{kn} of $x_{(k-1)n}$ contained in a ball of radius $1/k$

Choose the diagonal sequence x_{nn}.
Then from the way of construction, x_{nn} is a subseq. of x_n.

$x_{1n}: x_{11}, x_{12}, x_{13}, x_{14}$
$x_{2n}: x_{21}, x_{22}, x_{23}, x_{24}$
$x_{3n}: x_{31}, x_{32}, x_{33}, x_{34}$

Let $\varepsilon > 0$, Notice that for any fixed positive integer k, $x_{kk}, x_{k+1\,k+1}, x_{k+2\,k+2}, \dots$, is a subseq. of x_{kn} and is contained in a ball of radius $1/k$

For $m \geq n$, $d(x_{nn}, x_{mm}) \leq 2 \cdot \frac{1}{n} < \varepsilon$

$\Rightarrow n > 2/\varepsilon$

Choose $N = [2/\varepsilon] + 1$, then $\forall m, n > N$,

$d(x_{nn}, x_{mm}) < \varepsilon$.

$\Rightarrow x_{nn}$ is a Cauchy sequence.

$\because M$ is complete

$\therefore x_{nn}$ converges

Therefore M is seqcentially compact

by B-W thm, M is compact.

Q.E.D.

ex. ① The Euclidean space $\mathbb{R}^n$ is not compact.

($\because \mathbb{R}^n$ is unbounded).

② $\{D(n,1) : n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$, but it has no finite subcover

($\because D(n,1)$ 中只有 n 一个整数点 $\Rightarrow$ 如果 compact, 代表 $\mathbb{R}$ 中只有有限个整数点).

③ Let $A=(0,1]$ in $\mathbb{R}$, Find an open cover with no finite subcover.

-sol-: $\{(\frac{1}{n}, 2) : n \in \mathbb{N}\}$ is an open cover of A. ($\because \bigcup_{n=1}^{\infty} (\frac{1}{n}, 2) \supset (0,1]$)

but clearly it has no finite subcover.

ex. Give an example of a closed and bounded set that is not compact.

pf. Let M be an infinite set with discrete metric. Then $M \subset D(x,2)$ for some $x \in M$.
$\Rightarrow M$ is bounded. M is closed clearly.

$\{D(x, \frac{1}{2}) : x \in M\}$ is an open cover of M.
$\because D(x, \frac{1}{2}) = \{x\}$ and M is infinite
$\therefore \{D(x, \frac{1}{2}) : x \in M\}$ has no finite subcover.
Therefore M is not compact #.

Rmk: Compact $\Rightarrow$ closed and bounded
$\not\Leftarrow$

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§ 3.2 The Heine-Borel Thm.

Thm. $[a,b]$ is compact in $\mathbb{R}$.

pf. Let $\{U_\alpha\}$ be an open cover of $[a,b]$. Let $S = \{x \in [a,b] : [a,x]$ can be covered by a finite subcover of $\{U_\alpha\}\}$. Then $S \neq \emptyset$ ($\because a \in S$)

$\because$ S is bounded above by b

$\therefore \sup S \leq b$.

Let $c = \sup S$, then $c \in [a,b]$

$\because \{U_\alpha\}$ covers $[a,b]$, $\therefore \exists \alpha_0$ s.t. $c \in U_{\alpha_0}$.

$\because U_{\alpha_0}$ is open.

$\therefore \exists r > 0$ s.t. $(c-r, c+r) \subset U_{\alpha_0}$.

$\because c = \sup(S)$

$\therefore \exists x \in S$ s.t. $c - r < x$

$\Rightarrow \exists \alpha_1, \ldots, \alpha_N$ s.t. $\{U_{\alpha_1}, \ldots, U_{\alpha_N}\}$ covers $[a,x]$.

$\Rightarrow \{U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_N}\}$ covers $[a,c]$

$\Rightarrow c \in S$

Assume $c < b$.

then $\exists K \in \mathbb{N}$ s.t. $c + \frac{r}{K} < b$,

$\Rightarrow \{U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_N}\}$ covers $[a, c + \frac{r}{K}]$

$\Rightarrow c + \frac{r}{K} \in S$ ($\rightarrow\leftarrow$ $c = \sup(S)$)

Therefore $c=b \Rightarrow [a,b]$ can be covered by finite subcover of $\{U_\alpha\}$.
$\Rightarrow [a,b]$ is compact.

Q.E.D.

Note: Why does this proof fail for $[a,b)$ or $[a,\infty)$, $(a,b]$,

$[a,b)$ ↓ 只能推到 $c \le b$.

$[a,\infty)$ ⇓ sup S不存在

$(a,b]$ ↓ S可能会 $=\emptyset$

Thm. (Heine-Borel Thm) 只有在$\mathbb{R}^n$对!
A subset A of $\mathbb{R}^n$ is compact
$\Leftrightarrow$ A is closed and bounded.

pf: ($\Rightarrow$) 已证.

($\Leftarrow$) Let $x_k = (x_k^1, x_k^2, \dots, x_k^n)$ be a sequence in A.

$\because$ A is bounded
$\therefore \exists\, r>0$ s.t. $\|x_k\| < r$. $\forall k \in \mathbb{N}$.
$\Rightarrow x_k^i$ is a sequence in $[-r,r]$, $\forall i=1,\dots,n$
$\because [-r,r]$ is compact
$\therefore [-r,r]$ is sequentially compact,
x_k^1 has a convergent subseq. $x^1_{f_1(k)}$
$\because [-r,r]$ is compact
$\therefore x^2_{f_1(k)}$ has a convergent

subsequence $x^2_{f_2(k)}$.

Continuing this process, we obtain a convergent subsequence $x^n_{f_n(k)}$ of $x^n_{f_{n-1}(k)}$

$\Rightarrow x_{f_n(k)} = (x^1_{f_n(k)}, \cdots, x^n_{f_n(k)}$ is a convergent subsequence of x_k.

$\therefore$ A is closed

$\therefore x_{f_n(k)}$ converges in A

$\Rightarrow$ A is sequentially compact

$\Rightarrow$ A is compact.

Q.E.D.

Thm. Let A be a compact set in $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^m$ then $A \times \{x_0\}$ and $\{x_0\} \times A$ is compact $\mathbb{R}^{n+m}$

$\Rightarrow$ ① 可以把 A 的元素放到 $\mathbb{R}^{n+m}$ 中 ex. $1 \in \mathbb{R}^1$ 放到 $\mathbb{R}^2$,
取 $x_0 = 2 \to (1, 2) \in \mathbb{R}^2$ (補坐標)

② 可以有其他補法 ex. $(1,1,1) \to (1,2,1,2,1)$

Prop. Let A and B be subsets of a metric space (M, d). Then

① If A is open and B is closed, then $A \setminus B$ is open

② If A is closed and B is open, then $A \setminus B$ is closed

pf: exercise

ex. Determine the compactness of the following:

(a) $\{x \in \mathbb{R} \mid x \geq 0\}$ in $\mathbb{R}$ (X)

(b) $[0,1] \cup [2,3]$ in $\mathbb{R}$ (O)

(c) $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ (X)

(d) $\mathbb{R}^n$ in $\mathbb{R}^n$ (X)

ex. Let x_k be a sequence in $\mathbb{R}^n$ with $\|x_k\| < 3$, $\forall k \in \mathbb{N}$. Show that x_k has a convergent subsequence

pf. $\because$ $B(0,3)$ is closed and bounded in $\mathbb{R}^n$

$\therefore$ By H-B Thm. $B(0,3)$ is compact

$\Rightarrow$ $B(0,3)$ is sequentially compact.

$\Rightarrow$ x_k has a convergent subsequence (in $B(0,3)$). #

ex. Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$. Show directly that A is compact

pf. Let $\{U_\alpha\}$ be an open cover of A. Then $\exists \alpha_0$ s.t. $0 \in U_{\alpha_0}$ (0附近的 cover 可把其他人吃进去).

$\because \frac{1}{n} \to 0$

$\therefore$ For U_{α_0}, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $\frac{1}{n} \in U_{\alpha_0}$

$\because \{U_\alpha\}$ is a cover of A

$\therefore \exists \alpha_1, \dots, \alpha_N$ s.t. $\frac{1}{i} \in U_{\alpha_i}$, $\forall i = 1, \dots, N$

$\Rightarrow \{U_{\alpha_0}, U_{\alpha_1}, \dots, U_{\alpha_N}\}$ is a finite subcover of $\{U_\alpha\}$

$\Rightarrow$ A is compact #

§3.3 Nested Set Property.

Q: If F_n be a nonempty subsets of a metric space (M,d) with $F_{n+1} \subset F_n$, $\forall n \in \mathbb{N}$. Is $\bigcap_{n=1}^{\infty} F_n$ nonempty?

A: ① Let $F_n = (n, \infty)$, $\forall n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} \overline{F_n} = \emptyset$

pf. Assume $x \in \bigcap_{n=1}^{\infty} F_n$, then $\exists N \in \mathbb{N}$ s.t. $x < N$
$\Rightarrow x \notin F_N \Rightarrow x \notin \bigcap_{n=1}^{\infty} F_n$ ($\rightarrow\leftarrow$)
$\Rightarrow \bigcap_{n=1}^{\infty} F_n = \emptyset$

② Let $F_n = [n, \infty)$, then $\bigcap_{n=1}^{\infty} F_n = \emptyset$

③ $(1, 2]$
$\longleftarrow$ 往open的地方缩，就能缩到没有
$< \infty$

Thm. Nested Set Property
Let F_n be a sequence of compact nonempty sets in a metric space (M,d) s.t. $F_{n+1} \subset F_n$, $\forall n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$

pf. Let $x_n \in F_n$, $\forall n \in \mathbb{N}$. $\because F_{n+1} \subset F_n$, $\forall n \in \mathbb{N}$
$\therefore x_n$ is a seq. in F_1
$\because F_1$ is compact
$\therefore$ By B-W Thm, F_1 is sequentially compact
$\Rightarrow \exists$ a subsequence of x_n converges to $x \in F_1$. We denote this subsequence also by x_n.

By removing x_1 from x_n, and denoting it (also) by x_n, x_n is a sequence in F_2 that converges to x.

$\because F_2$ is compact

$\therefore F_2$ is closed

$\Rightarrow x \in F_2$. Continuing the same argument,

$x \in F_n, \forall n \in \mathbb{N}$

$\Rightarrow x \in \bigcap_{n=1}^{\infty} F_n$

$\Rightarrow \bigcap_{n=1}^{\infty} F_n \neq \emptyset$

Q.E.D.

Cor. Let U_n be open sets with $U_n \subset U_{n+1}$ ($\overline{F_n}$), $\forall n \in \mathbb{N}$ in a metric space (M, d). If $M \setminus U_n$ is compact, $\forall n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} U_n \neq M$

pf. $\because U_n \subset U_{n+1}, \forall n \in \mathbb{N}$

$\therefore M \setminus U_{n+1} \subset M \setminus U_n, \forall n \in \mathbb{N}$

$\because M \setminus U_n$ is compact, $\forall n \in \mathbb{N}$

$\therefore$ By N.S.P., $\bigcap_{n=1}^{\infty} (M \setminus U_n) \neq \emptyset$.

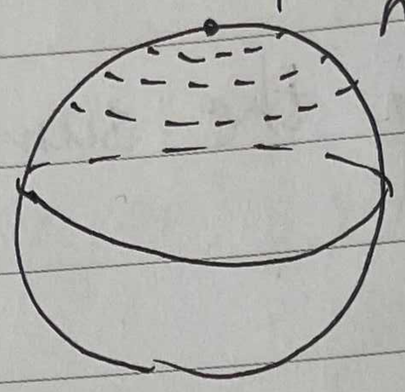
$\because \bigcap_{n=1}^{\infty} (M \setminus U_n) = M \setminus \left(\bigcup_{n=1}^{\infty} U_n \right)$

$\therefore \bigcup_{n=1}^{\infty} U_n \neq M$.

Q.E.D.

ex. Let M be the unit sphere in $\mathbb{R}^3$, i.e. $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with the standard metric. Let U_n be the portion of M strictly below latitude

$90° - \frac{1}{n}$, $\forall n \in \mathbb{N}$. Prove that $\bigcup_{n=1}^{\infty} U_n \neq M$.



小心：$M \neq \mathbb{R}^3$，M也不是歐氏空間；我們想要 (M, U_n) 「在 M 上 ampact」

pf. $\because U_n$ is open, $\forall n \in \mathbb{N}$

$\therefore M \setminus U_n$ is closed, $\forall n \in \mathbb{N}$.

(M is bdd by $D(0,2)$) Let $x_n = (x_n^1, x_n^2, x_n^3)$ be a sequence in M that converges to $x = (x^1, x^2, x^3)$ in $\mathbb{R}^3$

Then $x_n^1 \to x^1$, $x_n^2 \to x^2$ and $x_n^3 \to x^3$

$\Rightarrow \lim_{n\to\infty} \left((x_n^1)^2 + (x_n^2)^2 + (x_n^3)^2 \right) = (x^1)^2 + (x^2)^2 + (x^3)^2$

$\because (x_n^1)^2 + (x_n^2)^2 + (x_n^3)^2 = 1$, $\forall n \in \mathbb{N}$

$\therefore (x^1)^2 + (x^2)^2 + (x^3)^2 = 1$

$\Rightarrow x \in M$

$\Rightarrow M$ is closed.

$\because M$ is bdd by $D(0,2)$

$\therefore$ by H-B Thm, M is compact

$\because M \setminus U_n$ is closed, $\forall n \in \mathbb{N}$

$\therefore M \setminus U_n$ is compact, $\forall n \in \mathbb{N}$.

$\because U_n$ is open $\forall n \in \mathbb{N}$ and $U_n \subset U_{n+1}$, $\forall n \in \mathbb{N}$

$\therefore$ By cor, $\bigcup_{n=1}^{\infty} U_n \neq M$

#

§3-4 Path Connected Sets

Let $f:[a,b]\to\mathbb{R}^n$ be a function. We say that $\lim_{t\to t_0} f(t)=L\ (\in\mathbb{R}^n)$ if $\forall\varepsilon>0$, $\exists\delta>0$ s.t. $\forall t$ in $0<|t-t_0|<\delta$, $\|f(t)-L\|<\varepsilon$

Def. Let $[a,b]$ be an interval of $\mathbb{R}$ and (M,d) a metric space. Let $\varphi:[a,b]\to M$ be a map (or a function). Let $t_0\in(a,b)$, and $p\in M$.

① We say that $\lim_{t\to t_0}\varphi(t)=p$ if $\forall\varepsilon>0$, $\exists\delta>0$ s.t. $\forall t$ in $0<|t-t_0|<\delta$, $d(\varphi(t),p)<\varepsilon$

② We say that $\lim_{t\to a^+}\varphi(t)=p$ if $\forall\varepsilon>0$, $\exists\delta>0$ s.t. $\forall t\in(a,a+\delta)$, $d(\varphi(t),p)<\varepsilon$

③ We say that $\lim_{t\to b^-}\varphi(t)=p$ if $\forall\varepsilon>0$, $\exists\delta>0$ s.t. $\forall t\in(b-\delta,b)$, $d(\varphi(t),p)<\varepsilon$

④ We say that φ is conti. at $t_0\in(a,b)$ (cf a or b), if $\lim_{t\to t_0}\varphi(t)=\varphi(t_0)$ (cf. $\lim_{t\to a^+}\varphi(t)=\varphi(a)$ or $\lim_{t\to b^-}\varphi(t)=\varphi(b)$)

⑤ We say that φ is conti. on $[a,b]$, if it is conti. at every pt of $[a,b]$.

Note: $\neq(M,d)=(\mathbb{R}^n,\|\cdot\|)$, φ is called a vector-valued function which can be viewed as a space curve

Def. Let p and q be two points in a metric space (M,d)

① A path joining p and q is a map $\varphi:[a,b]\to M$ s.t. $\varphi(a)=p$ and $\varphi(b)=q$.

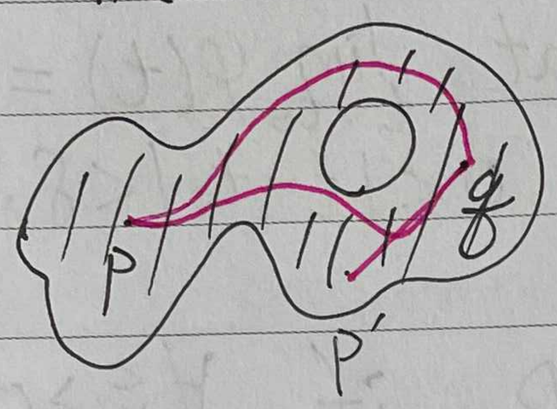
② A conti. path is a path which is conti.

③ A path $\varphi:[a,b]\to M$ is said to lie

in a subset A of M if $\varphi([a,b]) \subset A$

Def. A set $A \subset (M,d)$ is called path-connected if every two points of A can be joined by a conti. path lying in A.

ex. $A \subset \mathbb{R}^n$



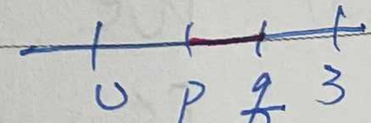
ex. Which of the following sets are path-connected?

(a) $[0,3]$

(b) $[1,2] \cup [3,4]$

(c) $\{(x,y) \in \mathbb{R}^2 : 0 \le x < 1\}$

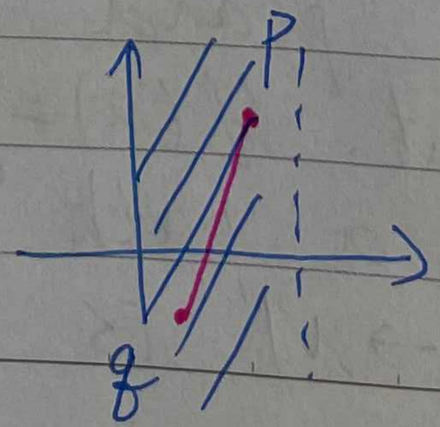
(d) $\{(x,y) \in \mathbb{R}^2 : 0 < x^2+y^2 < 1\}$



pf. (a) Yes. Let $p, q \in [0,3]$.
Define $\varphi: [0,1] \to [0,3]$ by $\varphi(t) = p + t(q-p)$
Then φ is conti. path joining from p to q lying in $[0,3]$.
$\Rightarrow [0,3]$ is path connected.

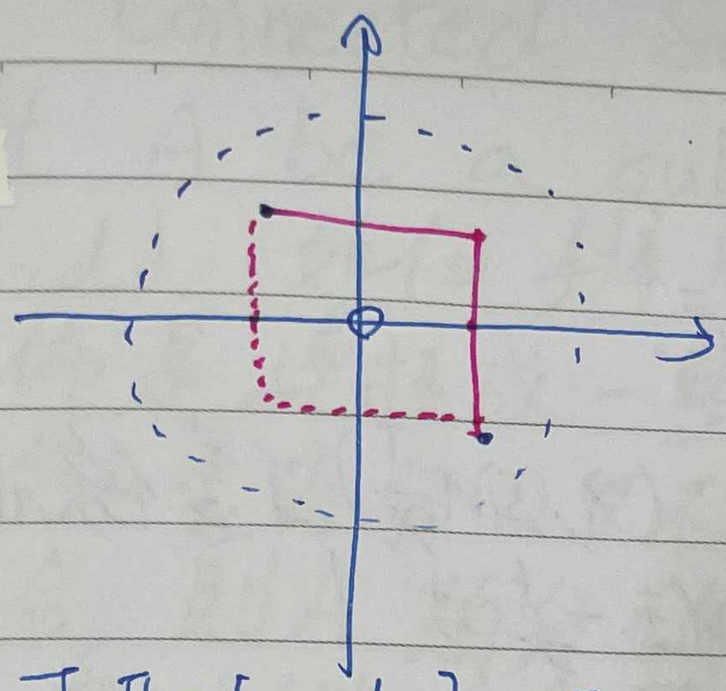
(b). No. 難証!

(c).

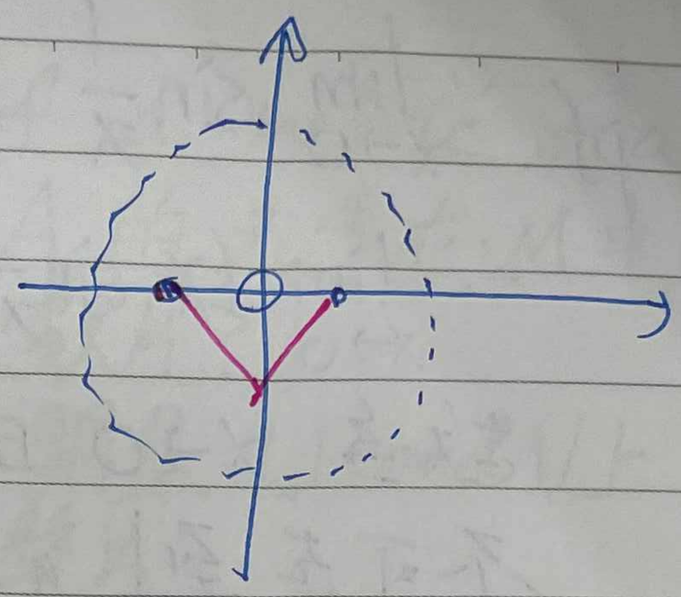


Yes (類似(a).)

(d)



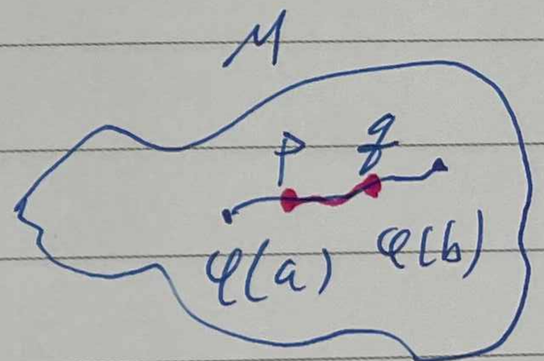
or



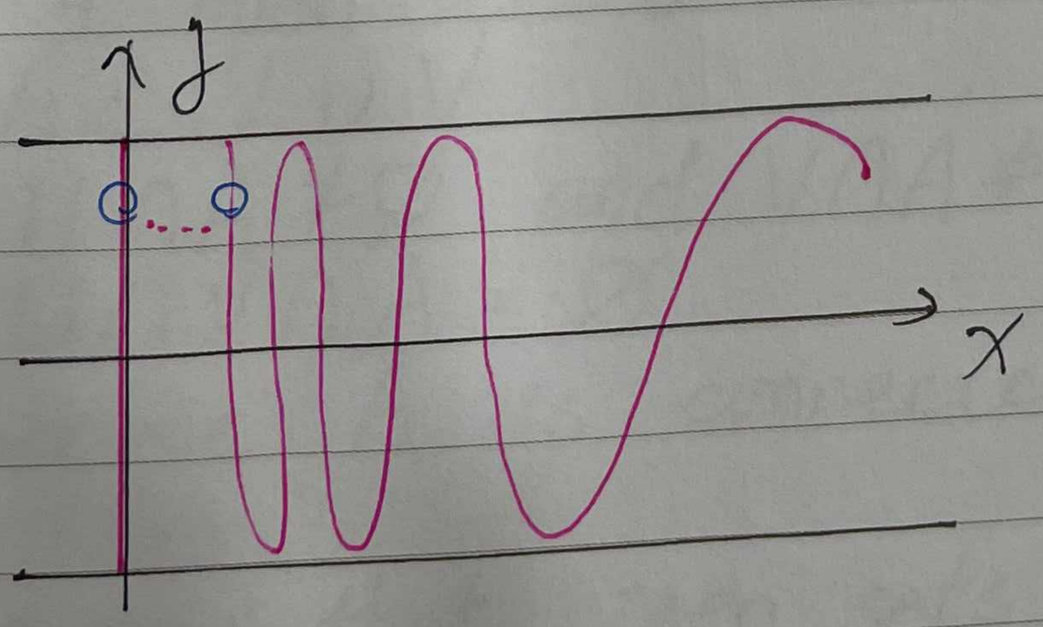
$\exists$ 某 $\mathbb{R}[a,b] = [0,2]$ (exercise) #

ex. Let $\varphi : [a,b] \to M$ be a conti. map. Prove or disprove that $\varphi([a,b])$ is path-connected.

pf: Let $p, q \in \varphi([a,b])$. Then $\exists\, t_0, t_1 \in [a,b]$ s.t. $\varphi(t_0) = p$, and $\varphi(t_1) = q$. W.L.O.G. we may assume $t_0 < t_1$. Then $\varphi|_{[t_0, t_1]}$ is a conti. path joining p and q lying in $\varphi([a,b]) \Rightarrow \varphi([a,b])$ is path-connected. #



ex. Is the set $\{(x, \sin\frac{1}{x}) \mid x>0\} \cup \{(0,y) \mid -1 \le y \le 1\}$ path connected?



pf. $\because \lim_{x\to 0} \sin\frac{1}{x}$ D.N.E.

$\therefore \lim_{x\to 0} (x, \sin\frac{1}{x})$ D.N.E.

故当 $x\to 0$ 時沿著 $(x, \sin\frac{1}{x})$ 這條曲線就不可走到 y 軸上的任一点去！

$[-1, \hat{1}]$

故 it is not path-connected. #

§3.5 Connected Sets

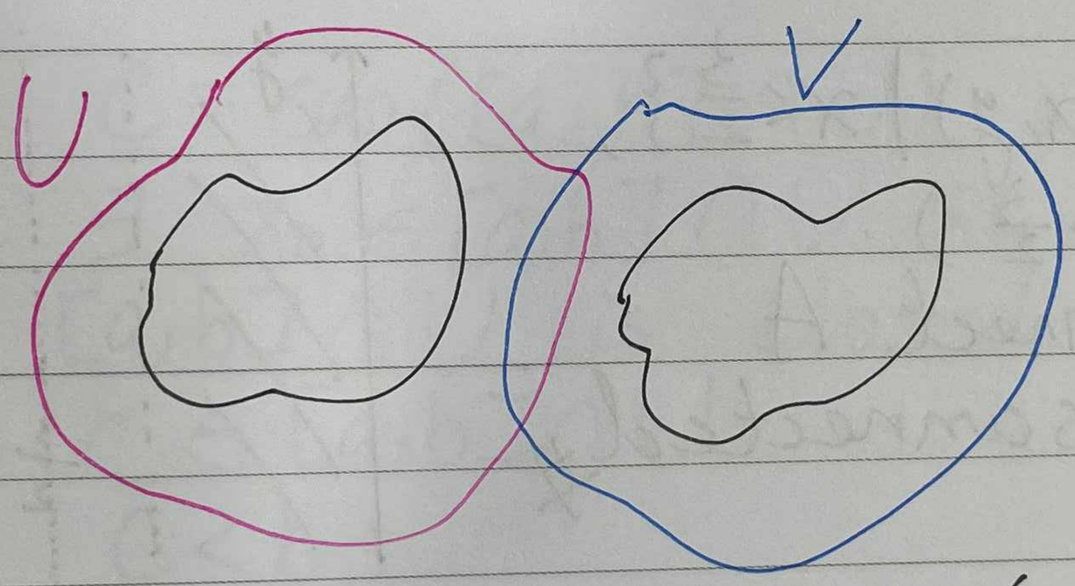
Q: Let A be a subset of a metric space (M,d). 如何判别子集 A 的元素在 M 中的分佈是連接成一整片或數片?

A: A 会不会落在2个不相交的開集 U and V 上呢? 若会, 則 A 的分佈不是一整片.

Q: 什麼叫做 A 会落在不相交的2開集 U and V 呢?

A: 不相交表 $U\cap V=\emptyset$, A 会落在表 $A\subset U\cup V$,
分別落在表 $A\cap U\neq\emptyset$ and $A\cap V\neq\emptyset$
→ U 与 V 將 A 分成 $U\cap A$ and $V\cap A$
若 $(U\cap A)\cap(V\cap A)=\emptyset$, 則此2片是分開的

$U\cap V\cap A$ //

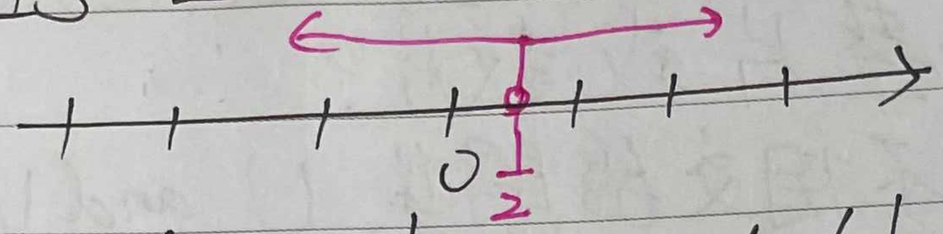


⇒ $U\cap V$ 未必一定要是 $\emptyset$.

Def. Let A be a subset of a metric space (M,d). We say that A is disconnected, if $\exists$ open set U and V s.t. ①
① $A\subset U\cup V$
② $U\cap A\neq\emptyset$ and $V\cap A\neq\emptyset$,
and ③ $U\cap V\cap A=\emptyset$
Otherwise A is connected

Def. We say that open sets U and V disconnect A if U and V satisfy ①, ② and ③

ex. Is $\mathbb{Z} \subset \mathbb{R}$ connected or disconnected?



pf. $\because (-\infty, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$ disconnect $\mathbb{Z}$

These two open set

$\therefore \mathbb{Z}$ is disconnected. #

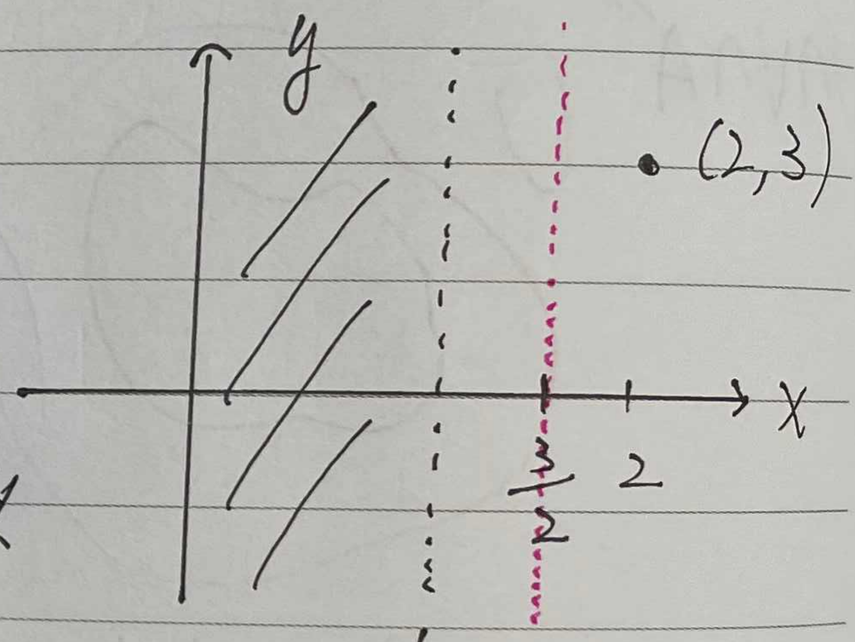
ex. Let $A = \{(x,y) \mid 0 \le x \le 1\} \cup \{(2,3)\} \subset \mathbb{R}^2$. is A connected or disconnected?

pf. Let $U = \{(x,y) \mid x < \frac{3}{2}\}$,

$V = \{(x,y) \mid x > \frac{3}{2}\}$.

$\because U, V$ disconnect A

$\therefore$ A is disconnected. #



Thm. Let $A \subset \mathbb{R}$. A is connected $\Leftrightarrow$ A is an interval

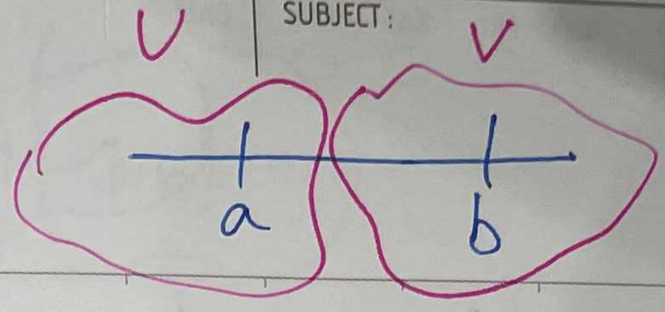
pf. ($\Rightarrow$) If not, $\exists\, x, z \in A$ and $y \in \mathbb{R} \setminus A$ s.t. $x < y < z$. Then open set (y, ∞) and $(-\infty, y)$ disconnect A

$\Rightarrow$ A is disconnected $\rightarrow\leftarrow$

Therefore A is interval.

We only prove for $[a,b]$.

($\Leftarrow$) Suppose $[a,b]$ is disconnected. Then $\exists$ open sets U and V disconnected A.



$\Rightarrow A \subset U \cup V$

Assume $b \in V$. $\because [a,b] \cap U \neq \emptyset$

$\therefore [a,b] \cap U$ is bounded above by b.

Let $c \triangleq \sup([a,b] \cap U)$. Then $c \leq b$.

Claim: $[a,b] \cap U$ and $[a,b] \cap V$ are closed

pf of claim: $\because \mathbb{R} \setminus ([a,b] \cap U) = \mathbb{R} \setminus ([a,b] \setminus ([a,b] \cap V)) = \underbrace{(\mathbb{R} \setminus [a,b])}_{\text{open}} \cup \underbrace{V}_{\text{open}}$ is open

$\therefore [a,b] \cap U$ is closed. #

$\because c = \sup([a,b] \cap U)$

$\therefore \exists\ x_n \in [a,b] \cap U$ s.t. $x_n \to c$

$\because [a,b] \cap U$ is closed

$\therefore c \in [a,b] \cap U$

$\Rightarrow c < b$

$\because U$ is open

$\therefore \exists\ r > 0$ s.t. $(c-r, c+r) \subset U$ and $(c-r, c+r) \subset [a,b]$

$\Rightarrow c + \frac{r}{2} \in [a,b] \cap U$

$\Rightarrow c + \frac{r}{2} < c$ $\to\leftarrow$

Therefore $[a,b]$ is connected. Q.E.D.

→ $\mathbb{R}$ 上有 sup 和 inf 是很好用的工具，高维度就没有这麼好的東西了

Remark. Using the similar proof of this thm. one can prove that the set $\{(x, \sin\frac{1}{x}) \mid x>0\} \cup \{(0,y) \mid -1 \le y \le 1\}$ is connected.

Cor. There are no nonempty disjoint closed subsets C and D of $[a,b]$ whose union in $[a,b]$

pf. Otherwise let $U = \mathbb{R} \setminus C$ and $V = \mathbb{R} \setminus D$.
then U and V is open in $\mathbb{R}$
$\therefore U \cap V \cap [a,b] = (\mathbb{R} \setminus [a,b]) \cap [a,b] = \emptyset$

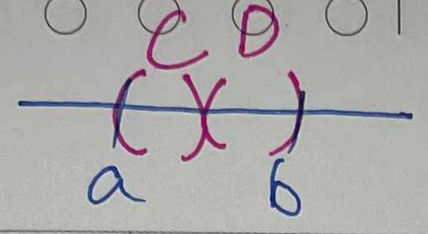
$\because C \cup D = [a,b]$
$\therefore U \cap [a,b] \neq \emptyset$ and $V \cap [a,b] \neq \emptyset$ and $[a,b] \subset U \cup V \Rightarrow [a,b]$ is disconnected $\to\leftarrow$.

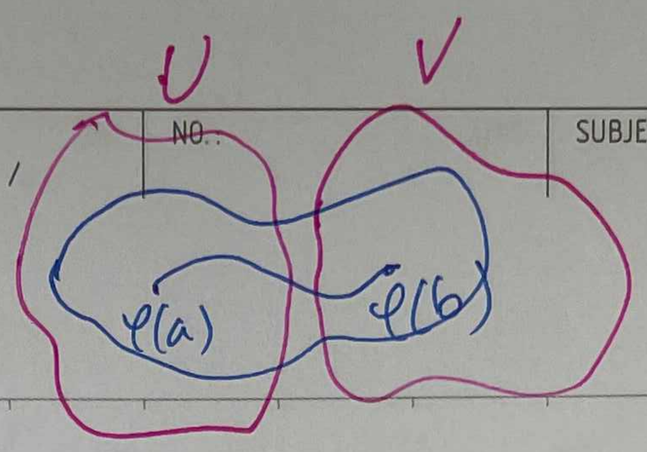
Therefore there are no such sets C and D
Q.E.D.

Q: Path-connected and connected 有何关係呢?

Thm. Path-connected sets in (M,d) is connected

pf. Let A be a path-connected subset of M.
Assume A is disconnected by open sets U and V in M.
Let $x \in U \cap A$, $y \in V \cap A$. $\because A$ is path-connected $\therefore \exists$ a cont. path $\varphi : [a,b] \to A$ s.t. $\varphi(a) = x$ and $\varphi(b) = y$.





Let $C = \varphi^{-1}(U \cap A)$ and $D = \varphi^{-1}(V \cap A)$.
Then $a \in C$ and $b \in D$. $\because (U \cap A) \cap (V \cap A) = \emptyset$, $\therefore C \cap D = \emptyset$
$\because A \subset U \cup V$
$\therefore C \cup D = [a, b]$

Claim: C and D are closed in $\mathbb{R}$ $\rightarrow\leftarrow$.
Hence A is connected

pf of claim: Let t_k be a seq. in C that converges to t in $\mathbb{R}$.

$\because [a,b]$ is closed in $\mathbb{R}$,
$\therefore t \in [a,b] \Rightarrow \varphi(t) \in U \cap A$ or $\varphi(t) \in V \cap A$.
Assume $\varphi(t) \in V \cap A$.
Let $\varepsilon > 0$, $\because \varphi$ is conti. at t, $\therefore$ For this ε, $\exists \delta > 0$ s.t. $\forall u$ in $|u - t| < \delta$, $d(\varphi(u), \varphi(t)) < \varepsilon$. $\because t_k \to t$ $\therefore$ For this δ $\exists N \in \mathbb{N}$ s.t. $\forall k > N$, $|t_k - t| < \delta \Rightarrow d(\varphi(t_k), \varphi(t)) < \varepsilon \to \varphi(t_k) \to t \Rightarrow \exists N \in \mathbb{N}$ s.t. $\forall k > N$, $\varphi(t_k) \in V$.
$\Rightarrow t_{N+1} \in D$.
$\Rightarrow C \cap D = \emptyset$ $\rightarrow\leftarrow$

Therefore C is closed. Similarly D is closed
Q.E.D.

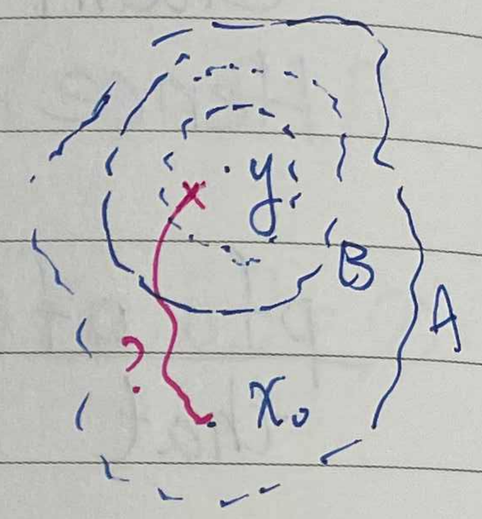
Remark. The converse is not true. For example, $\{(x, \sin\frac{1}{x}) \mid x > 0\} \cup \{(0, y) \mid -1 \le y \le 1\}$ is connected but not path connected.

Thm. If A is open ① connected ② subset of $\mathbb{R}^n$ ③, then A is path-connected

→ we want to prove A=B

pf. Let $x_0 \in A$. Let $B = \{y \in A \mid x_0$ and y can be joined by a conti. path in $A\}$, then $x_0 \in B$ (非空).

Let $y \in B$. ∵ A is open, ∴ $\exists r > 0$ s.t. $D(y,r) \subset A$



⇒ $\forall z \in D(y,r)$, there is a conti. path in A joining x_0 and z first along the conti. path from x to y, then followed by line segment from y to z.

⇒ $z \in B$, $\forall z \in D(y,r)$

⇒ $D(y,r) \subset B$

⇒ B is open

Let $y \in (cl(B) \cap A)$. Then ∃ a sequence y_k in B s.t. $y_k \to y$

∵ A is open

∴ $\exists r > 0$ s.t. $D(y,r) \subset A$

∵ $y_k \to y$

∴ For this r, $\exists N \in \mathbb{N}$ s.t. $\forall k > N$, $\|y_k - y\| < r$

⇒ $y_{2N} \in D(y,r)$

∴ The conti. path in A joining x_0 to y_{2N}, then followed by the

line segment from y_{2N} to y joins x_0 to y

$\therefore y \in B$.

$\Rightarrow cl(B) \cap A \subset B$ (B本來就 $\subset cl(B)$, $B \subset A$)

$\Rightarrow cl(B) \cap A = B$

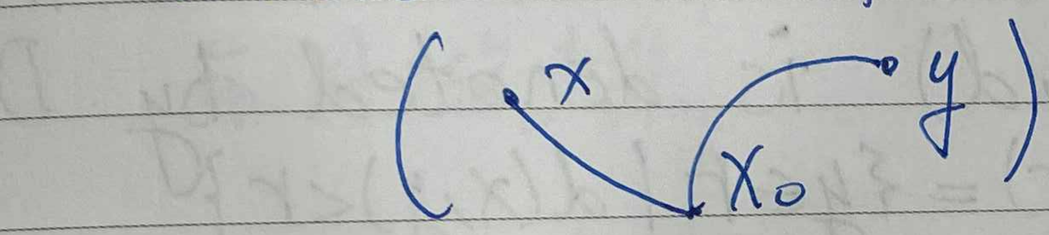
If $B \subsetneq A$, then $A \setminus B = A \setminus (cl(B) \cap A) = A \setminus cl(B)$

$\Rightarrow A \setminus B$ is open

$\because$ B and $A \setminus B$ disconnect A

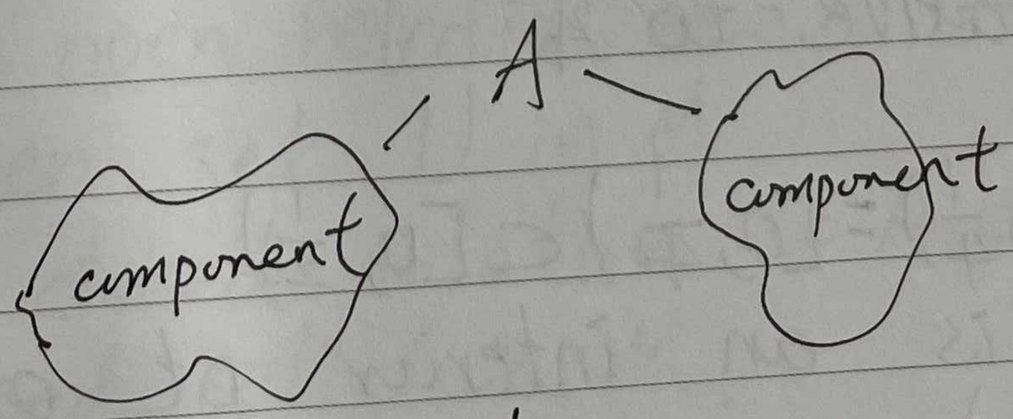
$\therefore$ A is disconnected $\rightarrow\leftarrow$

Therefore $B = A \Rightarrow$ A is path connected.



Q.E.D.

Def. Let A be a subset of a metric space (M, d). A component of A is a connected subset $A_0 \subset A$ s.t. there is no connected set in A containing A_0 other than A_0 itself (i.e. a maximal connected subset of A is called a component!)



(Note: 可定義 path-component!)

ex. How many components does $\mathbb{Z} \subset \mathbb{R}$ have?

Ans: Infinitely many

Ch4. Continuous Mappings

§4.1 Continuity.

Let A be a subset of a metric space (M, d). Then $d|_{A\times A} : A\times A \to \mathbb{R}$ is a metric on A. $\Rightarrow (A, d|_{A\times A})$ is a metric space. 將 $(A, d|_{A\times A})$ 記成 (A, d), and (A, d) 被稱、(M, d) 的一个 subspace.

Def. ① Let $x \in A$. The r-ball about x in the subspace (A, d) is denoted by $D_A(x, r)$ i.e. $D_A(x,r) = \{y \in A \mid d(x,y) < r\}$
(Then $D_A(x,r) = D(x,r) \cap A$)

② The open set C in the subspace (A, d) is called open relative to A, and the closed set in (A, d) is called closed relative to A.

ex. Let $A = [0, 1) \subset \mathbb{R}$. Show that $[0, \frac{1}{2})$ is open relative to A.

pf. $\because D_A(0, \frac{1}{4}) = [0, \frac{1}{4}) \subset [0, \frac{1}{2})$
$\therefore$ 0 is an interior pt of $[0, \frac{1}{2})$ in (A, d)
Let $x \in (0, \frac{1}{2})$, Let $r = \min(x, \frac{1}{2} - x)$, then
$\because D_A(x,r) = D(x,r) \subset [0, \frac{1}{2})$
$\therefore [0, \frac{1}{2})$ is relatively open to A. #

Q: Let $A \subset (M,d)$. $\forall x \in A$, $D_A(x,r) = D(x,r) \cap A$.
Open relative to A and open in M 有什麼關係?

A: Let $C \subset (A,d)$. C is open relative to A
$\Leftrightarrow C = \bigcup_{x \in C} D_A(x, r_x) = \bigcup_{x \in C} (D(x,r) \cap A)$
$= \left(\bigcup_{x \in C} D(x,r) \right) \cap A$
↓ ball 跟 A 做交集
open set 也跟 A 做交集
$\Leftrightarrow C = U \cap A$ for some open U in M.
→ 小空間的 open set 完全來自於大空間的 open set 與 A 做交集

若 $C = U \setminus A$, Then $A \setminus C = (M \setminus U) \cap A$. 故 $A \setminus C$ is close relative to A $\Leftrightarrow$ C is open relative to A
$\Leftrightarrow C = U \cap A$ for some open set in M
$\Leftrightarrow A \setminus C = (M \setminus U) \cap A = D \cap A$ for some closed set in M.

Thm. Let A be a subset of a metric space (M,d).
① $C \subset A$ is open relative to A $\Leftrightarrow C = U \cap A$ for some open set U in M
② $D \subset A$ is closed relative to A $\Leftrightarrow D = V \cap A$ for some closed set V in M.

Cor. ① If A is an open subset of (M,d), then $C \subset A$ is open relative to A $\Leftrightarrow$ C is open in M
② If A is a closed subset of (M,d), then D is closed relative to A $\Leftrightarrow$ C is closed in M

Double A

MON TUE WED THU FRI SAT SUN

①

Thm. A is a compact (respectively, connected, path ③ connected) subset of a metric space $(M,d) \Leftrightarrow$ A is a compact (respectively, connected, path-connected) subspace ②

pf. ① $\{U_\alpha\}$ is an open cover of A in (M,d) $\Leftrightarrow$ $\{U_\alpha \cap A\}$ is an open cover of A in (A,d)

② Open sets U and V disconnect A in (M,d) $\Leftrightarrow$ open relative to A subsets $U \cap A$ and $V \cap A$ disconnect A in (A,d)

③ $\varphi : [a,b] \to A$ is conti. on (M,d) $\Leftrightarrow$ φ: is conti. on (A,d).

Q.E.D.

Cor. Let $B \subset A$ be subsets of a metric space (M,d). B is compact in the subspace (A,d) $\Leftrightarrow$ B is compact in the metric space (M,d) (respectively, connected, path-connected)

pf: 跟 Thm 一樣 (把 M 改成 A, A 改成 B)

Remark. Compactness, connectness, path-connectness 与子集所處的 metric space 無关

Thm. A set $A \subset M$ is connected $\Leftrightarrow$ $\emptyset$ and A are the only subsets of A that are open and closed.

pf. Exercise

能撞A中的点靠近/聚集到x_0。
accumulatim pt 附近

Def. Let (M,d) and (N,ρ) be two metric spaces, $A \subset M$ and $f: A \to N$ a mapping

① Suppose that $x_0 \in A'$, we say that $b \in N$ is a limit of f at x_0 if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in A$ with $0 < d(x, x_0) < \delta$, $\rho(f(x), b) < \varepsilon$.

② We say that f is conti. at $x_0 \in A$, if either $x_0 \notin A'$ or $\lim_{x \to x_0} f(x) = f(x_0)$

只对有定义的点讨论是否连续

③ We say that f is conti. on the set $B \subset A$ if it is conti. at every pt of B.

④ We say that f is conti. if it is conti. on its domain.

Thm. Let $f: A \subset (M,d) \to (N,\rho)$ be a mapping

① f is conti. at $x_0 \Leftrightarrow \forall$ nbd V of $f(x_0)$ in N, $\exists$ a nbd U of x_0 in A s.t. $f(U) \subset V$

② If f is conti. at x_0, then $\forall x_k \to x_0$ in A, $f(x_k) \to f(x_0)$ in N

pf. ① $\Rightarrow$) $\because V$ is open
$\therefore \exists \varepsilon > 0$ s.t. $D(f(x_0), \varepsilon) \subset V$
$\because f$ is conti.

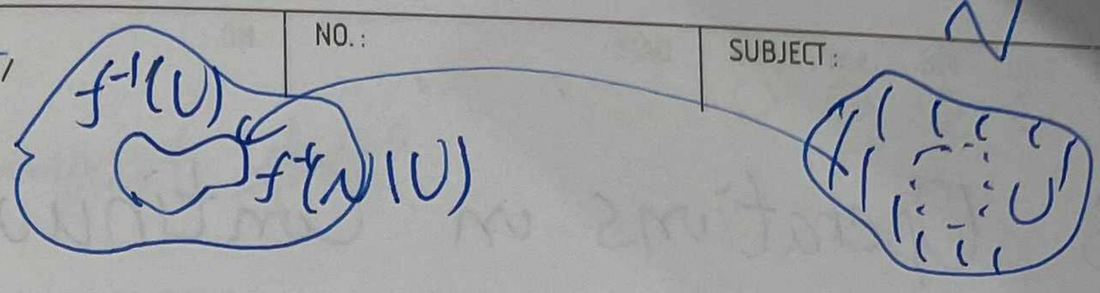
For this ε, $\exists\ \delta>0$ s.t. $\forall x\in A$ with
$d(x,x_0)<\delta$, $\rho(f(x),f(x_0))<\varepsilon$
$\Rightarrow f(x)\in D(f(x_0),\varepsilon)$
$\Rightarrow f(x)\in V$
$\Rightarrow \forall x\in D(x_0,\delta),\ f(x)\in V$
$\Rightarrow f(D(x_0,\delta))\subset V$.

(⇐ 略
② 上次已證

Q.E.D.

Thm. Let $f:A\subset(M,d)\to(N,\rho)$ be a function then the followings are equivalent.
① f is cont. on A
② For each sequence $x_k\to x$ in A, $f(x_k)\to f(x)$ in N
③ For each open set U in N, $f^{-1}(U)$ is open relative to A.
④ For each closed set V in N, $f^{-1}(V)$ is closed relative to A

pf: ① ⇒ ② (由上方 thm ② 所得)
② ⇒ ④ Let x_k be a sequence in $f^{-1}(V)$ that converges to $x\in A$. By ②, $f(x_k)\to f(x)$.
$\because f(x_k)\in V$ and V is closed
$\therefore f(x)\in V$
$\Rightarrow x\in f^{-1}(V)$
$\Rightarrow f^{-1}(V)$ is closed.



④⇒③ ∵ U is open in N
∴ $N\setminus U$ is closed in N
By ④ $f^{-1}(N\setminus U)$ is closed relative to A
∵ $A\setminus f^{-1}(N\setminus U)=f^{-1}(U)$
∴ $f^{-1}(U)$ is open relative to A

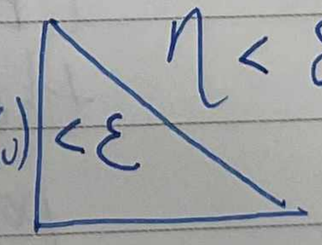
③⇒① By 上面的 thm ①.

Q.E.D.

Thm. Let (M,d) and (N,ρ) be two metric spaces. Then the projections $\pi_1: M\times N\to M$ and $\pi_2: M\times N\to N$ are conti. where $M\times N$ is equipped with the product metric $\eta((x_1,y_1),(x_2,y_2))=(d(x_1,x_2)^2+\rho(y_1,y_2)^2)^{\frac{1}{2}}$

投影到 M

pf. Let $(x_0,y_0)\in M\times N$. Let $\varepsilon>0$, $d(\pi_1(x,y),\pi_1(x_0,y_0))$
$=d(x,x_0)\le\eta((x,y),(x_0,y_0))$. Choose $\delta=\varepsilon$,
Then $\forall(x,y)\in M\times N$ with
$\eta((x,y),(x_0,y_0))<\delta$, $d(\pi_1(x,y),\pi_1(x_0,y_0))<\varepsilon$. Therefore π_1 is conti. at $(x_0,y_0)\Rightarrow\pi_1$ is conti. on $M\times N$.



Q.E.D.

ex. Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a conti. function. Show that the set $\{x\in\mathbb{R}^n\mid\|f(x)\|<1\}$ is open

pf: $\{x\in\mathbb{R}^n\mid\|f(x)\|<1\}=f^{-1}(D(0,1))$
∵ f is conti. and $D(0,1)$ is open
∴ $\{x\in\mathbb{R}^n\mid\|f(x)\|<1\}$ is open ※

§4.3 Operations on Continuous mapping

→ 与 $\mathbb{R}^n$ 相同，自行 Review

§4.2 Images of Compact and connected set

Q：連續函数是否会把 open set 送到 open set，或把 closed set 送到 closed set 呢？

A：No! see exercise §4.1 ③ on p182 与 ⑲ on p233. 但可把收斂的数列送到收斂的数列，連续函数可以把很逼近的兩点送到很逼近的之点

不会把 $\mathbb{R}^n$ 中的函数图形拙折→送过去还是连成一片（path connected 中的 path 还是连续的 path）

compact → 有收斂子数列 $\xrightarrow{\text{conti. fun.}}$ 收斂子数列

⇒ compact 送到 compact

Thm. Let $f: A \subset (M,d) \to (N,\rho)$ be a conti. mapping. Let $B \subset A$. If B is compact (respectively, connected, path-connected,) then $f(B)$ is compact (respectively, connected, path-connected).

pf. ①. (法I) Let y_n be a sequence in $f(B)$. Then $\exists\, x_n \in B$ s.t. $f(x_n) = y_n$ $\forall n \in \mathbb{N}$.

∵ B is compact

∴ By B-W Thm, B is sequentially compact.

⇒ x_n has a subsequence $x_{n_k} \to x$ in B

∵ f is conti.

∴ $f(x_{n_k}) \to f(x) \Rightarrow y_{n_k} \to f(x)$

法I利用 seq. compact 來做

$\Rightarrow f(B)$ is sequentially compact
By B-W Thm, $f(B)$ is compact.

(法II) Let $\{U_\alpha\}$ be an open cover of $f(B)$.
Then $\{f^{-1}(U_\alpha)\}$ is a cover of B.
$\because f$ is cont. and U_α is open, $\forall \alpha$.
$\therefore \{f^{-1}(U_\alpha)\}$ is an open cover of B.
$\because B$ is compact
$\therefore \exists$ a finite subcover $\{f^{-1}(U_{\alpha_1}), \dots, f^{-1}(U_{\alpha_n})\}$ of $\{f^{-1}(U_\alpha)\}$ that covers B
$\Rightarrow \{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ covers $f(B)$.
$\Rightarrow f(B)$ is compact

② Assume $f(B)$ is disconnected by open sets U and V. Then $B \subset f^{-1}(U) \cup f^{-1}(V)$ and $B \cap f^{-1}(U) \neq \emptyset$, $B \cap f^{-1}(V) \neq \emptyset$, $f^{-1}(U) \cap f^{-1}(V) \cap B \neq \emptyset$
$\because f$ is conti. U and V are open
$\therefore f^{-1}(U)$ and $f^{-1}(V)$ are open
$\Rightarrow B$ is disconnected $\rightarrow\leftarrow$
Therefore $f(B)$ is conneeted.

③ Let $y_1, y_2 \in f(B)$ Then $\exists x_1, x_2 \in B$ s.t. $f(x_1)=y_1$ and $f(x_2)=y_2$
$\because B$ is path-conn.,
$\therefore \exists$ a cont. path $\gamma: [a,b] \to B$ s.t. $\gamma(a)=x_1$ and $\gamma(b)=x_2$
$\Rightarrow f \circ \gamma : [a,b] \to f(B)$ is a conti. path s.t. $f \circ \gamma(a)=y_1$ and

$f\circ\gamma(b)=y_2$.
$\Rightarrow f(B)$ is path connected

Q.E.D.

Remark. 此定理的逆敘述皆是錯的

ex. ① $f(x)=\sin x$. $\because [-1,1]$ is compact, but $f^{-1}([-1,1])$ $=\mathbb{R}$ is not compact.

② $f(x)=x^2$. $\{1\}$ is connected. but $f^{-1}(\{1\})$ $=\{1,-1\}$ is disconnected

③ $f:(0,2\pi)\cup\{4\pi\}\to\mathbb{R}$ $[-1,1]$ is path-connected but $f^{-1}((0,2\pi))$, but $f^{-1}([-1,1])=(0,2\pi)\cup\{4\pi\}$ is not connected hence not path connected. it is

ex. Let $K\subset\mathbb{R}^2$ be compact. Prove that $A=\{x\in\mathbb{R}\mid \exists y\in\mathbb{R} \text{ s.t. } (x,y)\in K\}$ is compact

pf. Let $\pi_1:\mathbb{R}^2\to\mathbb{R}$ be the projection s.t. $\pi_1(x,y)=x$.
Then $\pi_1(K)=A$.
$\because \pi_1$ is conti. and K is compact
$\therefore A$ is compact

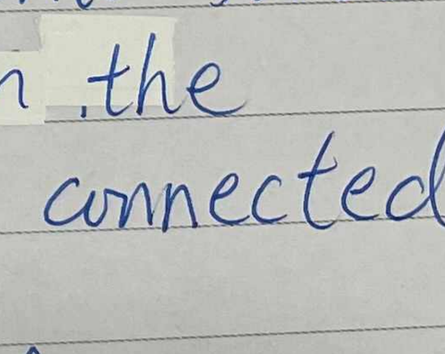
ex. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a conti. function and let $A=\{f(x)\mid \|x\|=1\}$. Show that A is a closed interval.

pf. Let $S=\{x\in\mathbb{R}^2\mid \|x\|=1\}$. Then $f(S)=A$.
$\because S$ is path-connected

∴ S is connected. ∵ f is conti,
∴ A is connected in R
⇒ A is an interval
∵ S ⊂ D(0,2)
∴ S is bounded.
Let $g(x,y)=x^2+y^2$, then g is conti. at $\mathbb{R}^2$
and $S=g^{-1}(\{1\})$. ∵ $\{1\}$ is closed,
∴ S is closed. By H-B Thm, S is compact
↳ S要在compact之下透过f才能证A是closed.
∵ f is conti.
∴ A is compact
⇒ A is closed. Therefore A is a closed interval. ※

Thm. Let $f: A \subset (M,d) \to (N,\rho)$ be a cont. function, and $K \subset A$. If K is connected, then the graph of $f=\{(x,f(x)) \mid x \in K\}$ is connected.

pf. Let $B=\{(x,f(x)) \mid x \in K\}$. Define $g: K \to M \times N$ be a function by $g(x)=(x,f(x))$. Then $g(K)=B$. Let $x \in K$. Let $\varepsilon>0$. Let η be the product matrix on $M \times N$.
$\eta(g(y),g(x))=\eta((y,f(y)),(x,f(x))$
$=(d(y,x)^2+\rho(f(y),f(x))^2)^{\frac{1}{2}}$
$\leq d(y,x)+\rho(f(y),f(x))<\varepsilon$



∵ f is conti. at x
∴ For $\varepsilon/2$, $\exists \delta_1>0$ s.t. $\forall y \in K$ with $d(y,x)<\delta_1$, $d(f(y),f(x))<\varepsilon/2$.

Let $\delta = \min(\varepsilon/2, \delta_1)$, then $\forall y \in K$ with
$d(y,x) < \delta$, $\eta(g(y), g(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$
$\Rightarrow$ g is conti. at x
$\Rightarrow$ g is conti. on K.

$\therefore$ K is connected
$\therefore$ B is connected.

Q.E.D.

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4-4

§ The Boundedness of Continuous functions on compact set

Def. Let $A \subset (M,d) \to (N,\rho)$ be a mapping.
We say that f is bounded on $K \subseteq A$, if the set $\{f(x) \mid x \in K\}$ is bounded in N.

We know that if $N = \mathbb{R}$, the continuity of f fails to guarentee the boundedness of f. Even if f is bounded f may not assume its maximum or minimum.

Thm. Maximum and Minimum Thm.
Let $f: A \subset (M,d) \to \mathbb{R}$ be a conti. function. If $K \subset A$ is compact, then f is bounded on K and $\exists\, x_1, x_2 \in K$ s.t. $f(x_1) = \inf\{f(x) \mid x \in K\}$ and $f(x_2) = \sup\{f(x) \mid x \in K\}$

pf. $\because$ f is conti. and K is compact,
$\therefore$ $f(K)$ is compact
$\Rightarrow$ $f(K)$ is bounded
$\Rightarrow$ f is bounded on K

$\because$ $f(K)$ is bounded
$\therefore$ $\inf(f(K))$ and $\sup(f(K))$ exist
$\Rightarrow$ $\inf(f(K)), \sup(f(K)) \in cl(f(K))$
$\because$ $f(K)$ is compact
$\therefore$ $f(K)$ is closed
$\Rightarrow$ $\inf(f(K)), \sup(f(K)) \in f(K)$.
$\Rightarrow$ $\exists\, x_1, x_2 \in K$ s.t. $f(x_1) = \inf(f(K))$ and $f(x_2) = \sup(f(K))$ Q.E.D.

Double A

§4.5 The Intermediate value Thm

Thm. Let M be a metric space $A \subset M$, and $f: A \to \mathbb{R}$ a conti. function. Suppose $K \subset A$ is connected, and $x, y \in K$. If $C \in \mathbb{R}$ s.t. $f(x) < C < f(y)$, then $\exists z \in K$ s.t. $f(z) = C$.

pf. <法I> Suppose that no such z exists.
Then let $U = (-\infty, C)$ and $V = (C, \infty)$.
Then U and V are disjoint open sets in $\mathbb{R}$.

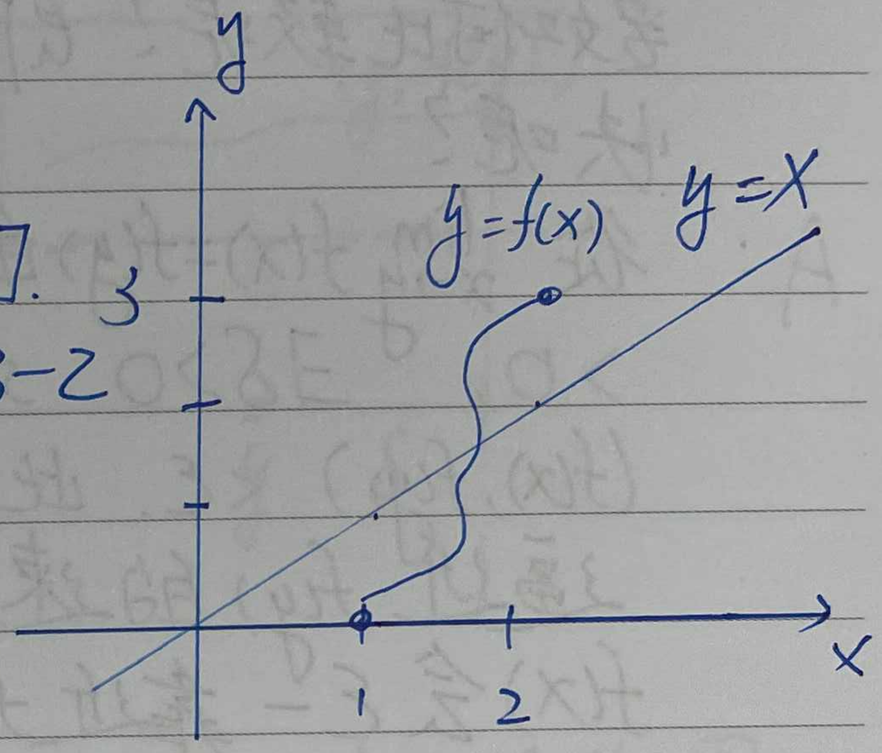
$\because f$ is conti.
$\therefore f^{-1}(U)$ and $f^{-1}(V)$ are open relative to A and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Moreover, $K \subset f^{-1}(U) \cup f^{-1}(V)$ and $x \in K \cap f^{-1}(U)$ and $y \in K \cap f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) \cap K = \emptyset$
$\Rightarrow K$ is disconnected in the subspace A.
$\Rightarrow K$ is disconnected (in M) $\rightarrow\!\!\!\!\!\!\times$
Therefore $\exists z \in K$ s.t. $f(z) = C$.

<法II> $\because f$ is conti. on K. and K is connected
$\therefore f(K)$ is connected in $\mathbb{R}$.
$\Rightarrow f(K)$ is an interval
$\because f(x), f(y) \in f(K)$
$\therefore [f(x), f(y)] \subset f(K)$
$\Rightarrow C \in f(K)$
$\Rightarrow \exists z \in K$ s.t. $f(z) = C$.

Q.E.D.

ex. Let $f:[1,2]\to[0,3]$ be a conti. function with $f(1)=0$ and $f(2)=3$. Show that f has a fixed point. (i.e. $\exists\ c\in[1,2]$ s.t. $f(c)=c$).

pf. Let $h(x)=f(x)-x$.
Then h is cont. on $[1,2]$.
$h(1)=0-1<0$ and $h(2)=3-2>0$.
$\because [1,2]$ is connected,
By I.V.T. $\exists\ c\in[1,2]$ s.t. $h(c)=0 \Rightarrow f(c)-c=0$
$\Rightarrow f(c)=c$. ∎



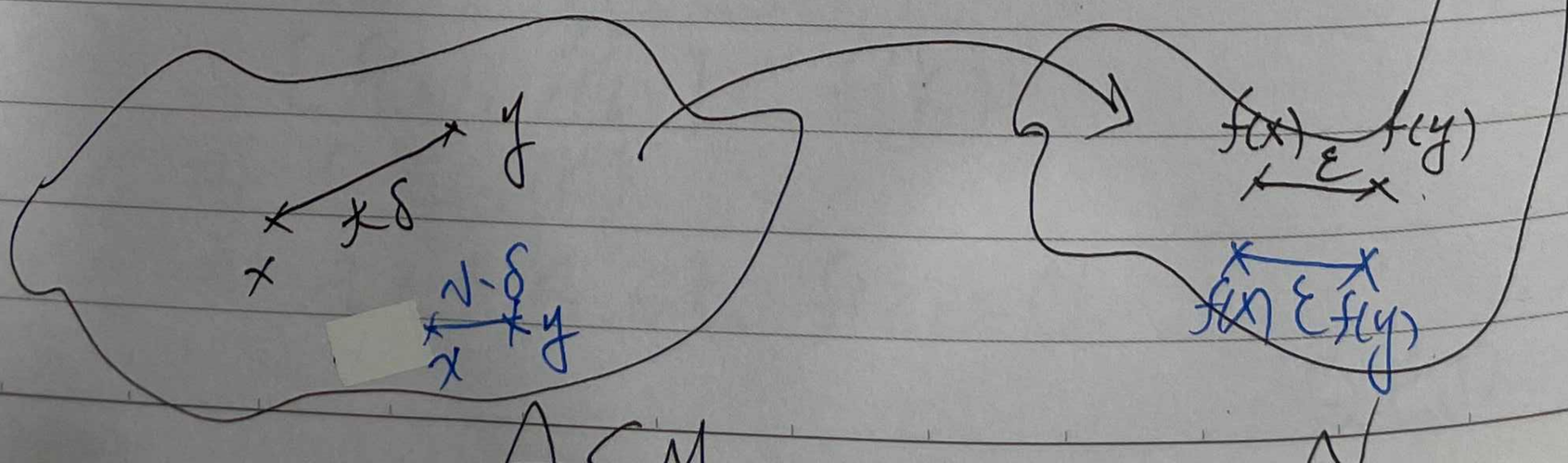
§4-6 Uniform Continuity.

Q: Let $f: A \subset (M,d) \to (N,\rho)$ be a conti. function. Then $\forall y \in A'$, $\lim_{x\to y} f(x) = f(y)$, 即当 x 在 A 中逼近 y 時, $f(x)$ 会逼近 $f(y)$. 現在想問讓 y 在 A' 中变动時, 其 $f(x)$ 逼近 $f(y)$ 的逼近速度能比較嗎? 要如何比較呢? 即哪个 y 其 $f(x)$ 逼近 $f(y)$ 的速度較快呢?

A: 從 $\lim_{x\to y} f(x) = f(y)$ 的权字建模來思維此問題。$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in A$ with $d(x,y) < \delta$, $\rho(f(x), f(y)) < \varepsilon$. 此定義中那一部份是在描述 $f(x)$ 逼近 $f(y)$ 的速度呢? $\rho(f(x), f(y)) < \varepsilon$, 描述的是 $f(x)$ 会 ε- 靠近 $f(y)$, 而 $\exists \delta > 0$ s.t. $\forall x \in A$ with $d(x,y) < \delta$」描述的事是 $f(x)$ 会 ε- 靠近 $f(y)$ 這件事是由 A 中每 y 关距离在 δ- 靠近的关 x 所達成的, 所做到的.

那当 y 在 A' 中变动時比較不同的 y 之 $f(x)$ 逼近 $f(y)$ 的速度如何比? 应該对每个 ε- 靠近, 去比較不同的 y 所得 δ 的大小? 為何呢?

因当 δ 越大時, 表示的是当 x 離 y 很遠時, 就開始讓 $f(x)$ ε- 靠近 $f(y)$ 了, 而 δ 愈小時, 表示在 x 要一直走到离 y 很近時, 才能讓 $f(x)$ ε- 靠近 $f(y)$. ⇒ δ 大時表示 $f(x)$ ε- 靠近 $f(y)$ 的速度 tt 較快



找不到伙伴了, 找到才要好脸

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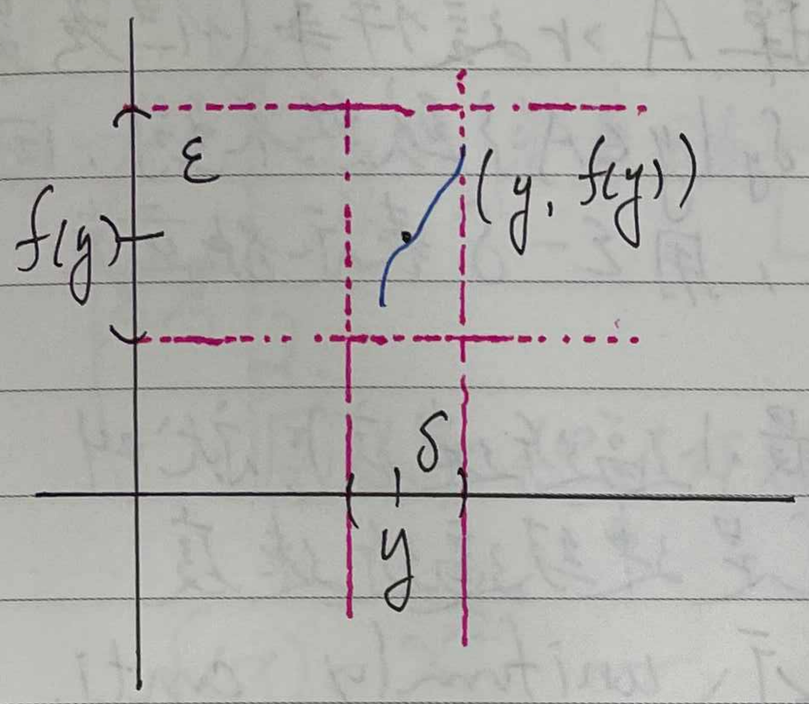
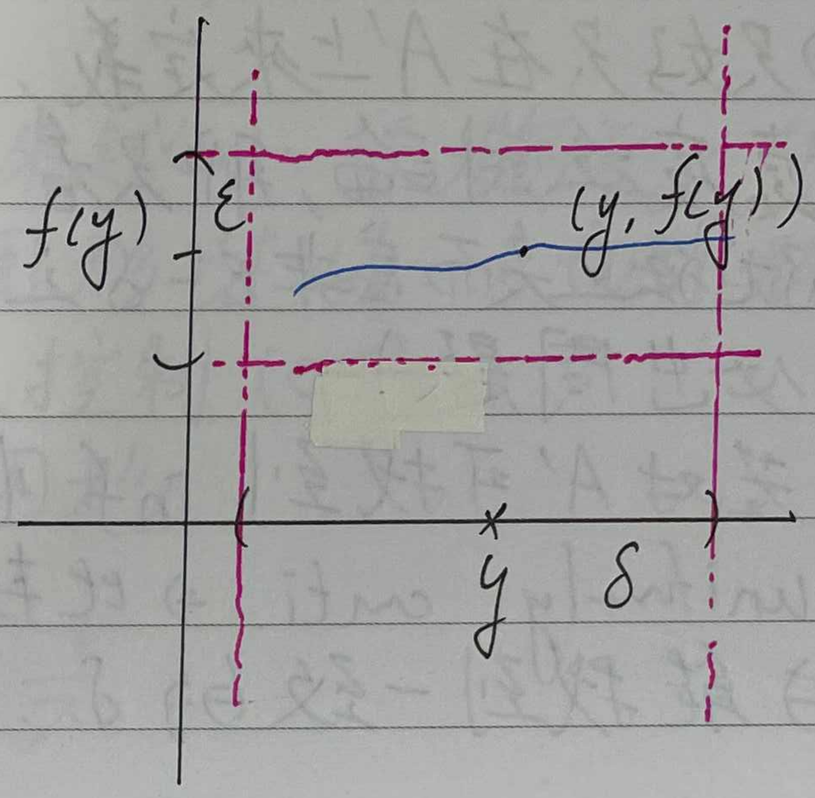
δ小表示 $f(x)$ ε-靠近 $f(y)$ 的速度較慢

若 $(M,d)=(N,\rho)=\mathbb{R}$，在 f 的图形上來看這邊討論的事情：





→ 在固定 ε-靠近之下，δ愈大的 y 点，f 的图形在 $(y, f(y))$ 附近較平坦，而 δ 愈小的 y 点，對應的图形較崎嶇.

f 之图形在 $(y, f(y))$ 附近

Q：对所有的 y 在 A' 中，分別对每一个 ε-靠近，可否找到最小的連續逼近速度呢？

A：$\forall \varepsilon>0$，需要考慮 $\inf\{\delta_y \mid y\in A'\}\geq 0$ 与否？（→必为正数→下界=0→inf存在）

若 $\inf\{\delta_y \mid y\in A'\}>0$ 時，則此 inf 可以生成所有 y 在 A' 中最小的 ε-靠近速度（→y是独立点）

Q：若 $y\in A\setminus A'$ 時，即 $\exists r>0$ s.t. $D_A(y,r)\cap A=\{y\}$（連续点在 metric space 裡分成兩類，一類是被 A 中的元素所聚集，一類是附近只有他自己一个点（孤立点）的狀況）→对→找 inf，其他点怎麼辦？

找不到就算了，找到才要炒腦

A：若 $\inf\{\delta_y \mid y\in A'\}=\delta>0$. If $\delta\leq r$ 時，okay.

（就只有 x 一点了，当然 $<\varepsilon$）

$\rho(f(x),f(y))$

Double A

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對於這兩來講

If $\delta > r$ 時 $D(y,\delta) \supset D_A(y,r)$

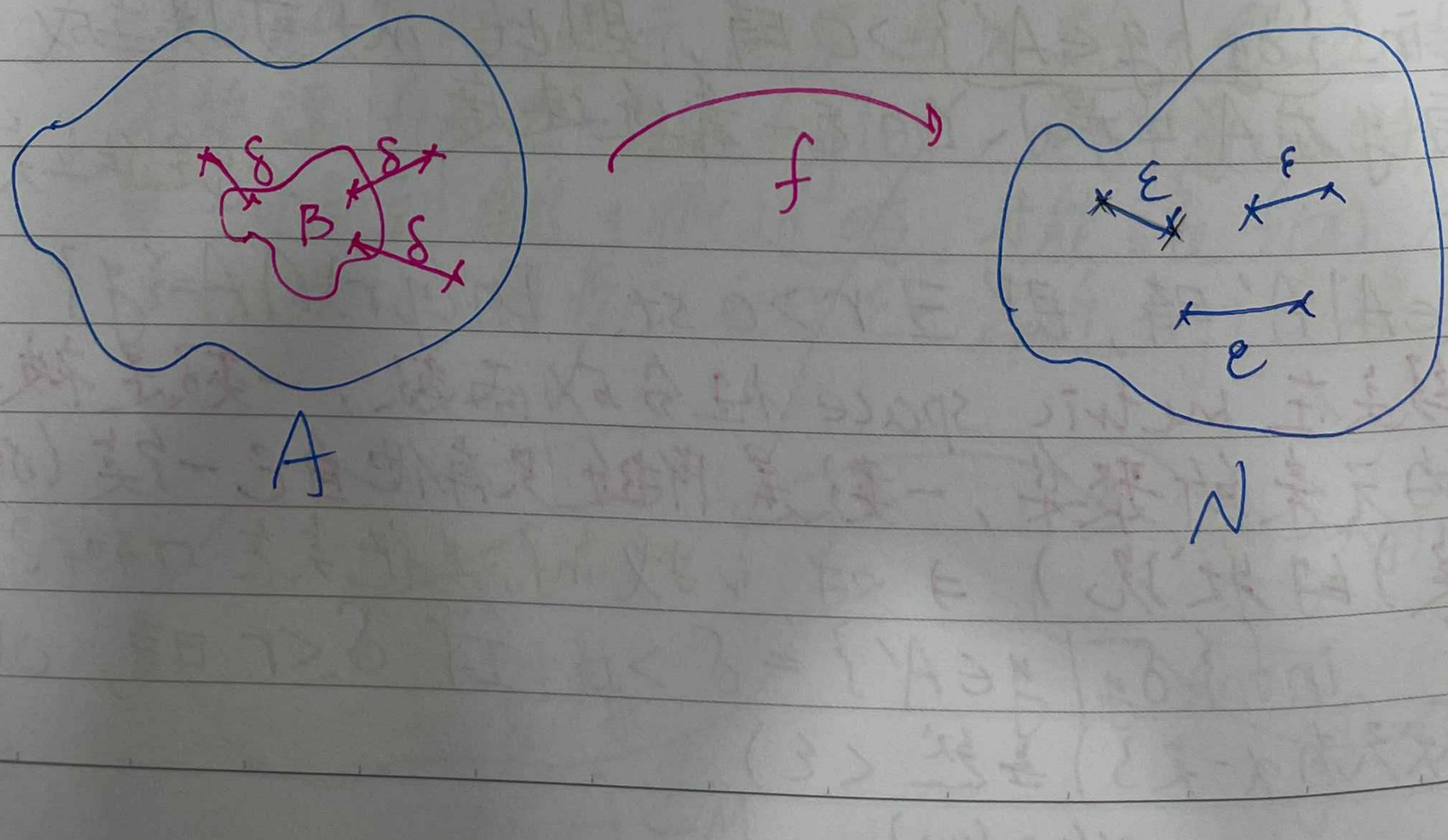
↳不知裡面有沒有東西，可不知其中函數
但与 $f(y)$ 的差值 $\Rightarrow \rho(f(x), f(y))$ 無法被保證

⇒只好只在 A' 上來定義，可避掉 $\delta > r$ 這件事(但如果要度沒討論，那只看 $\inf\{\delta_y \mid y \in A'\}$ 雖然不對，因為就独立点而言非 ε-δ 之形式，用 ε-δ 表示独立点必出問題)

若对 A' 可找到 inf 恒正(最小逼近速度)，就叫 uniformly cnti. ⇒比較的 δ 是連续逼近速度
⇒能找到一致的 δ，就称為 uniformly cnti.

Def. Let (M,d) and (N,ρ) be metric spaces, $B \subset A$ and $f: A \to N$ is a (cnti.) function. We say that f is uniformly cnti. on B if $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x \in A$, $\forall y \in B$ with $d(x,y) < \delta$, $\rho(f(x), f(y)) < \varepsilon$

① 不会有人靠近了还有人沒靠近 → 不隨 y 而改變
② 大家共用的 δ



Prop. Uniformly cont. on $B \Rightarrow$ cont. on B.

ex. 給一个 $f: I \subset \mathbb{R} \to \mathbb{R}$ 的連續函数，但 f 不会 uniformly cont. 的例子.

pf: (思考 | 1. δ 与 y 变动 ↳ 越来越小 → 趋近 0 (几乎不見) ↳ 图形陡峭 (切线斜率到無法控制 (∞)) 2. 圓滑但不可微的图形)

$f(x) = x^2$ ⇓ $x \to \infty, f'(x) \to \infty$ or $f(x) = \frac{1}{x}$ ↓ $x \to 0, f'(x) \to \infty$ //

ex. Show that $f(x) = x^2$ is not .

(思考 | 由求證想起要去証 $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x, y \in \mathbb{R}$ with $|x-y| < \delta$ s.t. $|x^2 - y^2| \geq \varepsilon$. Note: 証不是 uniformly cont. 喜欢用反證法)

Assume f is uniformly on $\mathbb{R}$. Then for $\varepsilon = 1$, $\exists \delta > 0$ s.t. $\forall x, y \in \mathbb{R}$, $|x-y| < \delta$, $|f(x) - f(y)| < 1$.

方便. → 要与 $x\delta$ 靠近，随便取了 $\frac{\delta}{2} + x$

(思考 | 取 $x > 0$, $y = x + \frac{\delta}{2}$ ↑ 要解 $x \Rightarrow$ $(x + \frac{\delta}{2})^2 - x^2 = x\delta + \frac{\delta^2}{4} \geq 1$ ∵ $x\delta + \frac{\delta^2}{4} > x\delta$, 令 $x\delta \geq 1$ $\Rightarrow x \geq \frac{1}{\delta}$, 取 x 为 $\frac{1}{\delta}$ ($\frac{2}{\delta}$ 也可))

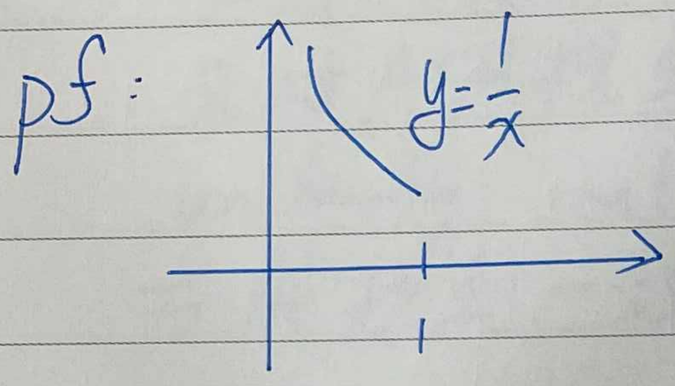
Huwever if $x \geq \frac{1}{\delta}$ and $y = \frac{1}{\delta} + \frac{\delta}{2}$,

$$\left|\left(\frac{1}{\delta}\right)^2-\left(\frac{1}{\delta}+\frac{\delta}{2}\right)^2\right| \geq \frac{1}{\delta}\cdot\delta = 1 \quad \rightarrow\leftarrow$$

Therefore f is not uniformly conti. on $\mathbb{R}$ #

ex. Show that $f(x)=\frac{1}{x}$ is not uniformly conti. on $(0,1]$

pf:



Assume f is uniformly conti. on $(0,1]$. Then $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $\forall x,y\in(0,1)$ with $d(x,y)<\delta$, $\left|\frac{1}{x}-\frac{1}{y}\right|<\varepsilon$. Let $\varepsilon=1$. Let $n\in\mathbb{N}$ s.t. $\frac{1}{n}<\delta$ (此時兩点即 $\frac{1}{n}$, $\frac{1}{n}+\frac{1}{n}$). Then $\left|\frac{2}{n}-\frac{1}{n}\right|<\delta$ and $\left|\frac{n}{2}-n\right|=\frac{n}{2}>1$ if $n>2$.

(or Let $n\in\mathbb{N}$ s.t. $\frac{1}{2n}<\delta$. Then $\left|\frac{1}{n}-\frac{1}{2n}\right|=\frac{1}{2n}<\delta$ and $|n-2n|=n\geq 1$) $\rightarrow\leftarrow$

Therefore f is not uniformly conti. on $(0,1]$

ex. Prove or disprove that $f(x)=\sin(x^2)$ is uniformly conti. on $\mathbb{R}$

pf. 看多項式程度 → 研究其微分

$\because f'(x)=2x\cos(x^2)$

$\therefore f'(\sqrt{2n\pi}) = 2\sqrt{2n\pi} \rightarrow \infty$ as $n\rightarrow\infty$ → 總有些点陡到不行

↓

使 cos 成為定值.

Assume f is uniformly conti. on $\mathbb{R}$. Let $\varepsilon=\frac{1}{2}$
Then $\exists\,\delta>0$ s.t. $\forall\, x,y\in\mathbb{R}$ with $|x-y|<\delta$,
$|\sin(x^2)-\sin(y^2)|<\frac{1}{2}$.

(思考 | 取 $\sqrt{2n\pi}$, $\sqrt{2n\pi+\frac{\pi}{2}}$, n 很大時, 開 δ 把它們包起來能誤差距 $<\delta$)

Let $n\in\mathbb{N}$ s.t. $|\sqrt{2n\pi}-\sqrt{2n\pi+\frac{\pi}{2}}|<\delta$
But $|\sin(2n\pi+\frac{\pi}{2})-\sin(2n\pi)|=1-0=1\geq\frac{1}{2}$

$\rightarrow\leftarrow$
Therefore f is not uniformly conti. on $\mathbb{R}$. #

ex. Prove or disprove that $f(x)=x^2$ is uniformly on $[a,b]$

pf. Let $\varepsilon>0$. $|x^2-y^2|=|x+y||x-y|\leq 2(|a|+|b|)$ 能$<\varepsilon$
$\Rightarrow |x-y|<\dfrac{\varepsilon}{2(|a|+|b|)}$.

Choose $\delta=\dfrac{\varepsilon}{2(|a|+|b|)}$. Then $\forall\, x,y\in[a,b]$ with $|x-y|<\delta$, $|x^2-y^2|<\varepsilon$. Therefore f is uniformly conti. on $[a,b]$. #

討論: 若 $f: I\to\mathbb{R}$ 可微, $|f'|\leq M$ on I $\rightarrow$ 會 uniformly conti. on I; 若 $|f'|$ 無界, 則不在 I 上 uniformly conti. on I?

Thm. Let $f: I \to \mathbb{R}$ be a diff. function. If $|f'(x)| \leq M$, $\forall x \in I$

pf: (思考) 1. 從求證想起：Let $\varepsilon > 0$, 從 $|f(x)-f(y)| < \varepsilon$ 去找 $|x-y|$ 的解

Let $\varepsilon > 0$. By M.V.T. $|f(x)-f(y)| = |f'(c)||x-y|$
where $c \in (x,y) \Rightarrow |f(x)-f(y)| \leq M|x-y| < \varepsilon$
$\Rightarrow |x-y| < \varepsilon/M$.

Choose $\delta = \frac{\varepsilon}{M}$. Then $\forall x, y \in I$ with $|x-y| < \delta$, $|f(x)-f(y)| < \varepsilon$. Therefore f is uniformly conti. on I.

Q.E.D.

Def. Let $A \subset (M,d) \to (N,\rho)$ be a function. We say that f is Lipschitz (on A) if $\exists\, M > 0$ s.t. $\forall x, y \in A$, $\rho(f(x), f(y)) \leq M d(x,y)$.

Thm. A Lipschitz function is a uniformly cont. function. Hence is conti. (follow by Thm. above)

Thm. The distance function $d: M \times M \to \mathbb{R}$ is a Lipschitz function

pf. Let $(x_1, y_1), (x_2, y_2) \in M \times M$. (先拆開)
$\because |d(x_1,y_1) - d(x_2,y_2)| \leq |d(x_1,y_1) - d(x_1,y_2)| + |d(x_1,y_2) - d(x_2,y_2)| \leq d(y_1,y_2) + d(x_1,x_2) \leq (d(x_1,x_2)^2 + d(y_1,y_2)^2)^{\frac{1}{2}}$
$\therefore d$ is Lipschitz

Q.E.D.

Thm. Let $f: A \subset (M,d) \to (N,\rho)$ be a conti. function
If $K \subset A$ is compact, then f is uniformly conti. on K.

沒有在改半徑半徑

pf. Let $\varepsilon > 0$. $\because f$ is conti. on K $\therefore \forall y \in K$,
$\exists \delta_y > 0$ s.t. $\forall x \in A$ with $d(x,y) < \delta_y$, $\rho(f(x), f(y)) < \varepsilon/2$
$\because \{D(y, \delta_y) \mid y \in K\}$ is an open cover of K.
and K is compact
$\therefore \exists$ a finite subcover $\{D(y_1, \delta_{y_1}), \dots, D(y_n, \delta_{y_n})\}$ of $\{D(y, \delta_y) \mid y \in K\}$

用 $\rho(f(x), f(y)) < \varepsilon$ 去解 $d(x,y)$

Let $x \in A$, $y \in K$
Then $\exists i \in \{1, \dots, n\}$ s.t. $y \in D(y_i, \delta_{y_i})$.
$\rho(f(x), f(y)) \le \rho(f(x), f(y_i)) + \rho(f(y_i), f(y))$

若 $d(x,y_i) < \delta_{y_i}$ 則 $\rho(f(x), f(y_i)) < \varepsilon$ $\because y \in D(y_i, \delta_{y_i})$ $\therefore$ 此項 $< \varepsilon$.

(思考 | 1. 因 y_i 是由 y_i 決定的，故要求作到 $d(x,y_i) < \delta_{y_i}$
這件事情可由改 $d(y,y_i)$ 來得到，即 $d(x,y_i) \le$
$d(x,y) + d(y,y_i) < \delta_{y_i}$
$< \frac{\delta_{y_i}}{2}$ $\le \frac{\delta_{y_i}}{2}$ ⇒但只保證 $< y$，怎辦？
→ 縮小、原本 cover 半徑)

Let $\delta = \min\left(\frac{\delta_{y_1}}{2}, \dots, \frac{\delta_{y_n}}{2}\right)$. Then $\forall x \in A$
and $y \in K$ with $d(x,y) < \delta$, $\rho(f(x), f(y)) < \frac{\varepsilon}{2}$
$+ \frac{\varepsilon}{2} = \varepsilon$ ∎

$\Rightarrow f$ is uniformly conti. on K.

Q.E.D.

ex. Prove or disprove that $\sin x$ is uniformly cont. on $\mathbb{R}$

pf: $\because |(\sin x)'| = |\cos x| \leq 1, \ \forall x \in \mathbb{R}$

$\therefore \sin x$ is uniformly conti. on $\mathbb{R}$

Q.E.D.

§4.7 Differentiation of function of one variable → 與微積分一樣 自行 review

§4-8 Integration of function of one variable

Let $f:[a,b]\to\mathbb{R}$ be a bounded function with $|f(x)|\le M$, $\forall x\in[a,b]$.

Let $P=\{a=x_0<x_1<\cdots<x_n=b\}$ be a partition of $[a,b]$. $\because |f(x)|\le M$, $\forall x\in[a,b]$ $\therefore$ $\sup_{[x_{i-1},x_i]} f(x)$ and $\inf_{[x_{i-1},x_i]} f(x)$ exist, $\forall i=1,\dots,n$.

Let $M_i=\sup\{f(x)\mid x\in[x_{i-1},x_i]\}$ and $m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}$, $\forall i=1,\dots,n$.

Def. The upper sum $U(f,P)=\sum_{i=1}^{n} M_i\Delta x_i$ and the lower sum $L(f,P)=\sum_{i=1}^{n} m_i\Delta x_i$.

Prop. $-M(b-a)\le L(f,P)\le U(f,P)\le M(b-a)$. $\forall P$.

Def. Let P and P' be partitions of $[a,b]$. If $P\subset P'$, then P' is called a refinement of P. (→ 分得較細)

Prop. If P' is a refinement of P, then $L(f,P')\ge L(f,P)$ and $U(f,P')\le U(f,P)$.

pf: exercise

Def. $\inf\{U(f,P)\mid P \text{ is a partition of } [a,b]\}$ is denoted by $\int_a^b f(x)dx$, read the upper integral of f on $[a,b]$.
$\sup\{L(f,P)\mid P \text{ is a partition of } [a,b]\}$ is denoted by

$\underline{\int_a^b} f(x)dx$, read the lower integral of f on $[a,b]$.

Thm. $\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx$. $\forall f$ bounded on $[a,b]$.

pf. Let P_1 and P_2 be two partitions of $[a,b]$. Then $P_1 \cup P_2$ is a refinement of P_1 and P_2 respectively.

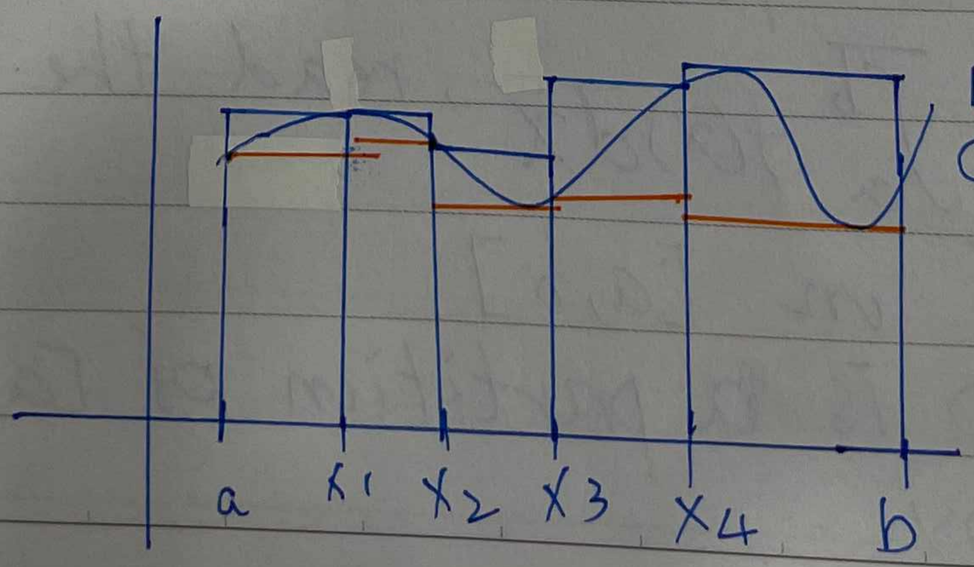
$\Rightarrow L(f,P_1) \le L(f,P_1\cup P_2) \le U(f,P_1\cup P_2) \le U(f,P_2)$

Let P_2 vary. $L(f,P_1) \le \overline{\int_a^b} f(x)dx$. Let P_1 vary, $\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx$.

Q.E.D.

Cor. $L(f,P_1) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le U(f,P_2)$, $\forall P_1, P_2, \forall f$.

(圖形): Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Let $P=\{a=x_0<\cdots<x_n=b\}$ be a partition of $[a,b]$



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Prop. $L(f,P_1) \le U(f,P_2)$, $\forall P_1, P_2$ partitions of $[a,b]$

pf: $L(f,P_1) \le L(f, P_1 \cup P_2) \le U(f, P_1 \cup P_2) \le U(f,P_2)$

Q.E.D.

Def. Let $f: [a,b]$ be a bounded function. We say that f is (Riemann) integrable over $[a,b]$ if $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$.

We thus write $\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$

Remark. ① When we say f is integrable over $[a,b]$, we always assume f is bounded on $[a,b]$.

② $0 \le \overline{\int_a^b} - \underline{\int_a^b} f(x)\,dx \le U(f,P_2) - L(f,P_1)$

$\forall P_1, P_2$ partitions of $[a,b]$.

Thm. A bounded function $f: [a,b] \to \mathbb{R}$ is integrable over $[a,b]$ $\iff$ $\forall \varepsilon > 0$, $\exists$ partitions P_1 and P_2 of $[a,b]$ s.t. $U(f,P_1) - L(f,P_2) < \varepsilon$.

$\iff$ $\exists$ a partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$

pf: $\Rightarrow$) Let $\varepsilon > 0$

$\because f$ is integrable over $[a,b]$

$\therefore \int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$

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and P_2

For $\varepsilon/2$, $\exists$ a partition P_1 of $[a,b]$ s.t.

$$\begin{cases} U(f,P_1) \le \int_a^b f(x)dx + \frac{\varepsilon}{2} \\ L(f,P_2) \ge \int_a^b f(x)dx - \frac{\varepsilon}{2} \end{cases}$$

$$\Rightarrow U(f,P_1) - L(f,P_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

($\Leftarrow$ $\because L(f,P_1) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le U(f,P_2)$,

$\forall P_1, P_2$

$\therefore 0 \le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le U(f,P_2) - L(f,P_1)$,

$\forall P_1, P_2$.

By assumption, $0 \le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le \varepsilon$, $\forall \varepsilon > 0$.

$$\Rightarrow \overline{\int_a^b} f(x) - \underline{\int_a^b} f(x)dx = 0$$

$$\Rightarrow \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$$

Therefore f is integrable over $[a,b]$.

Q.E.D.

Q: 怎樣的 bounded function 會是可積函數呢?

Thm. ① Any monotone function on $[a,b]$ is integrable

② If $f:[a,b] \to \mathbb{R}$ is bounded and is conti. at all but finitely many pts

of $[a,b]$, then f is integrable over $[a,b]$.

pf: ① (For increasing functions)
Let $\varepsilon>0$. Let $P=\{a=x_0<x_1<\cdots<x_n=b\}$ be a partition of $[a,b]$. Then $U(f,P)-L(f,P)$

找-P使此者$<\varepsilon$ $\overset{\Delta}{=}\sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))\Delta x_i$ (把Δx_i取成一樣才可細分)

Let $\Delta x_i=\frac{b-a}{n}$. Then $U(f,P)-L(f,P)$
$=\frac{b-a}{n}(f(b)-f(a))<\varepsilon \Rightarrow n>\frac{(b-a)(f(b)-f(a))}{\varepsilon}$

Choose $n=2\left\lceil\frac{(b-a)(f(b)-f(a))}{\varepsilon}\right\rceil$ Then

$U(f,P)-L(f,P)<\varepsilon$.
Therefore f is integrable.

Q.E.D.

② Suppose that f is conti. on $[a,b]$ except at $a_1,a_2,\ldots,a_k$ and $|f(x)|\le M$, $\forall x\in[a,b]$.
Let $\varepsilon>0$, Let $P=\{a=x_0<\cdots<x_n=b\}$ be a partition of $[a,b]$.

such

$U(f,P)-L(f,P)=\sum_{i=1}^{n}(M_i-m_i)\Delta x_i$ 只能控制Δx_i的長度
↷ 夠小來控制

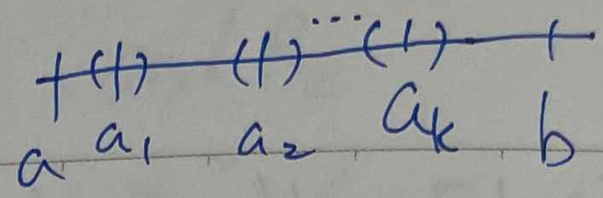
$=\sum_{i \text{ with some } a_j\in[x_{i-1},x_i]}(M_i-m_i)\Delta x_i + \sum_{i \text{ with } f \text{ is conti. on } [x_{i-1},x_i]}(M_i-m_i)\Delta x_i$

$(M_i-m_i)\le 2M$

by Min-Max Thm

$=f(x_i^*)-f(x_i^{**})$

∵f conti ∴ 只要Δx_i夠小，就會夠靠近

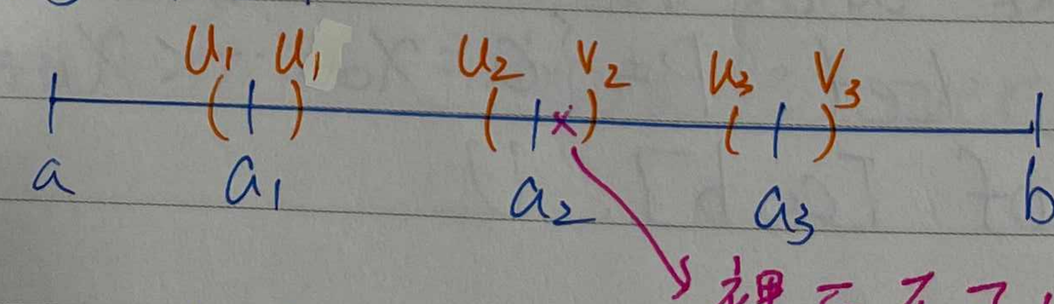


$\triangleq I + II$.

We cover $\{a_1, \dots, a_k\}$ by disjoint subintervals $[u_j, v_j]$ of $[a,b]$ s.t. $a_i \in (u_j, v_j)$ and $v_j - u_j < \frac{\varepsilon}{4M}$, then $I < \frac{\varepsilon}{2}$. Remove these open subinterval (u_j, v_j) from $[a,b]$. The remaining set K is closed and bounded in $\mathbb{R} \Rightarrow K$ is compact.

$\because f$ is conti. on K
$\therefore f$ is uniformly conti. on K.
For $\frac{\varepsilon}{2(b-a)}$, $\exists \delta > 0$ s.t. $\forall x, y \in K$ with $|x-y| < \delta$, $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$.

Now form a partition $P = \{a = x_0 < \cdots < x_n = b\}$ of $[a,b]$ s.t. each $u_j \in P$ and each $v_j \in P$, no points of any subinterval (u_j, v_j) occurs in P



subinterval
and each $[x_{i-1}, x_i]$ besides (u_j, v_j) has length $\Delta x_i < \delta$. Then $II < \frac{\varepsilon}{2(b-a)} \cdot \underbrace{(b-a)}_{\Delta x_i \text{ 總和}} = \frac{\varepsilon}{2}$
不大於 $(b-a)$ 小.

$$\Rightarrow U(f,P) - L(f,P) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore f is integrable over $[a,b]$

Q.E.D.

Prop. ① If f is integrable over $[a,b]$, then αf is integrable over $[a,b]$, $\forall \alpha \in \mathbb{R}$,

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$$

② If f and g are integrable over $[a,b]$, then $f+g$ is integrable over $[a,b]$, and

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

③ If f and g are integrable over $[a,b]$, and $f(x) \leq g(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

④ If f is integrable over $[a,b]$ and $[b,c]$, then f is integrable over $[a,c]$ and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

⑤ If f is integrable over $[a,b]$, then $\forall c \in [a,b]$, f is integrable over $[a,c]$ and $[c,b]$ and $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

pf: ① If $\alpha > 0$, then $U(\alpha f, P) = \alpha U(f, P)$
and $L(\alpha f, P) = \alpha L(f, P)$

$\Rightarrow \overline{\int_a^b} \alpha f = \alpha \overline{\int_a^b} f$ and $\underline{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f$

$\because f$ is integrable

$\therefore \overline{\int_a^b} f = \underline{\int_a^b} f$

$\Rightarrow \overline{\int_a^b} \alpha f = \underline{\int_a^b} \alpha f$

Therefore αf is integrable and

$\int_a^b \alpha f = \alpha \int_a^b f$

② Let $P = \{a = x_0 < \cdots < x_n = b\}$ be a partition of $[a, b]$. Then

$$U(f+g, P) = \sum_{i=1}^{n} M_i(f+g)\Delta x_i$$

→ 最大值未必在同一点

$$\leq \sum_{i=1}^{n} (M_i(f) + M_i(g))\Delta x_i$$

$$\leq U(f, P) + U(f, P)$$

$\Rightarrow \overline{\int_a^b} f+g \leq U(f, P) + U(g, P)$, $\forall P$

For $\frac{\varepsilon}{2} > 0$, $\exists$ partitions P_1 and P_2 of $[a, b]$ s.t. $U(f, P_1) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}$, and

$U(g, P_2) < \overline{\int_a^b} g(x)\,dx + \frac{\varepsilon}{2}$

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$$\Rightarrow U(f, P_1 \cup P_2) < \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2} \quad \text{and}$$

$$U(g, P_1 \cup P_2) < \overline{\int_a^b} g(x)dx + \frac{\varepsilon}{2}$$

$$\Rightarrow \overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g + \varepsilon$$

$$= \int_a^b f + \int_a^b g + \varepsilon$$

$\because$ f, g are integrable

$$\Rightarrow \overline{\int_a^b}(f+g) \leq \int_a^b f + \int_a^b g$$

Similarly, $\underline{\int_a^b}(f+g) \geq \int_a^b f + \int_a^b g$

$\because \overline{\int_a^b}(f+g) \geq \underline{\int_a^b}(f+g)$

$\therefore \overline{\int_a^b}(f+g) = \underline{\int_a^b}(f+g) = \int_a^b f + \int_a^b g$

Therefore $f+g$ is integrable and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

③ $\because L(f,P) \leq L(g,P)$, $\forall P$: partition of $[a,b]$.

$\therefore L(f,P) \leq \underline{\int_a^b} g$

$\because g$ is integrable over $[a,b]$

$\therefore L(f,P) \leq \int_a^b g$, $\forall P$.

$\Rightarrow \underline{\int_a^b} f \leq \int_a^b g$. $\because f$ is integrable

Double A

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$$\therefore \int_a^b f \le \int_a^b g$$

④ If P_1 is a partition of $[a,b]$ and P_2 is a partition $[b,c]$, then $P_1 \cup P_2$ is a partition of $[a,c]$. Moreover,

$$L(f,[a,b],P_1) + L(f,[b,c],P_2)$$
$$= L(f,[a,c],P_1 \cup P_2) \quad , \forall P_1, P_2$$
$$\Rightarrow \underline{\int_a^c} f \ge L(f,[a,b],P_1) + L(f,[b,c],P_2), \forall P_1, P_2$$

For $\frac{\varepsilon}{2} > 0$, $\exists$ a partition $\tilde{P}_1$ of $[a,b]$ and $\tilde{P}_2$ of $[b,c]$ s.t.

$$L(f,[a,b],\tilde{P}_1) > \underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2}, \text{ and}$$
$$L(f,[b,c],\tilde{P}_2) > \underline{\int_b^c} f(x)dx - \frac{\varepsilon}{2}$$
$$\Rightarrow \underline{\int_a^c} f > \underline{\int_a^b} f(x)dx + \int_a^c f(x)dx - \varepsilon$$
$$= \uparrow \int_a^b f(x)dx + \int_b^c f(x)dx - \varepsilon$$

↑ f is integ. over $[a,b]$ and $[b,c]$

$$\Rightarrow \underline{\int_a^b} f \ge \underline{\int_a^b} f + \underline{\int_b^c} f$$

Similarly, $\overline{\int_a^c} f \le \overline{\int_a^b} f + \overline{\int_b^c} f$

$\because \overline{\int_a^c} f \ge \underline{\int_a^c}$

$$\therefore \overline{\int_a^c} f = \underline{\int_a^c} f = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$\Rightarrow$ f is integrable over $[a,c]$ and

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

⑤ $\because$ f is integrable over $[a,b]$

$\therefore \forall \varepsilon > 0$, $\exists$ a partition of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

$\because U(f, P\cup\{c\}) - L(f, P\cup\{c\}) < U(f,P) - L(f,P)$

$\therefore U(f, P\cup\{c\}) - L(f, P\cup\{c\}) < \varepsilon$

$\because U(f, P\cup\{c\}|_{[a,c]}) - L(f, P\cup\{c\}|_{[a,c]})$

$\leq U(f, P\cup\{c\}) - L(f, P\cup\{c\})$

$\therefore U(f, P\cup\{c\}|_{[a,c]}) - L(U, P\cup\{c\}|_{[a,c]}) < \varepsilon$.

Therefore f is integrable over $[a,c]$.

Similarly f is integrable over $[c,b]$.

By ④, $\int_a^b f = \int_a^c f + \int_c^b f$

Q.E.D.

Thm. If f is integrable over $[a,b]$, then

① $|f|$ is integrable over $[a,b]$ and

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx$$

② The function $F: [a,b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f(t)dt$ is a conti. function.

pf: ① $\because M_i(|f|) - m_i(|f|) \le M_i(f) - m_i(f)$

$\therefore U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P)$

$\because -|f| \le f \le |f|$ on $[a,b]$

$\therefore -\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$

$\to \left|\int_a^b f\right| \le \int_a^b |f|$

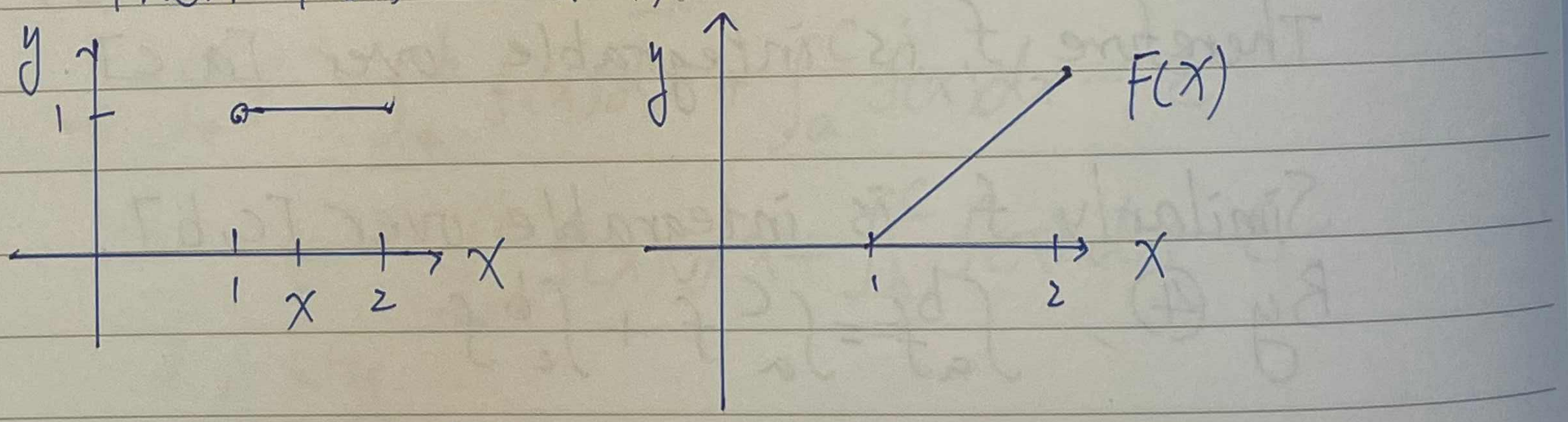
② Exercise

Q.E.D.

Question: Is $F(x) = \int_a^x f(t)\,dt$ differentiable?

Ans: No! For example, let $f(x) = \begin{cases} 0, & \forall x \in [0,1] \\ 1, & \forall x \in (1,2] \end{cases}$

Then $F(x) = \begin{cases} 0, & \text{if } x \in [0,1] \\ x-1, & \text{if } x \in (1,2] \end{cases}$



F is not diff. at 1.

Thm. Composition Thm

Suppose that $f: [a,b] \to \mathbb{R}$ is integrable and $g: [c,d] \to \mathbb{R}$ is cont. and $f([a,b]) \subset [c,d]$. Then $g \circ f: [a,b] \to \mathbb{R}$ is integ.

pf: Let $\varepsilon > 0$. Let $K = \sup\{|g(x)| \mid x \in [c,d]$

∵ g is conti. on $[c,d]$

∴ g is uniformly conti. on $[c,d]$ (and $\delta<\varepsilon'$)

$\Rightarrow \forall \varepsilon'>0$, ($\varepsilon'$為一个待定常数) $\exists \delta>0$, s.t. $\forall x,y \in [c,d]$ with $|x-y|<\delta$, $|g(x)-g(y)|<\varepsilon'$

Let $P=\{a=x_0<x_1<\cdots<x_n=b\}$ be a partition of $[a,b]$. Let m_i and M_i be the infimum and the supremum of f on $[x_{i-1},x_i]$ and let $\tilde{m}_i$ and $\tilde{M}_i$ be the infimum and the supremum of $g\circ f$ on $[x_{i-1},x_i]$.

f未必conti.

① If $M_i-m_i<\delta$, then $f([x_{i-1},x_i]) \subset [m_i,M_i]$

∵ g is conti. on $[m_i,M_i]$

∴ By Max-Min Thm, $\exists\ x_i^*$ and $x_i^{**} \in [m_i,M_i]$ s.t. $g(x_i^*)=\tilde{m}_i$ and $g(x_i^{**})=\tilde{M}_i$

∵ $|x_i^*-x_i^{**}| \le M_i-m_i<\delta$

∴ $\tilde{M}_i-\tilde{m}_i<\varepsilon'$

② If $M_i-m_i \ge \delta$, then $\tilde{M}_i-\tilde{m}_i \le 2k$. Let $A=\{i\in\{1,2,\dots,n\} \mid M_i-m_i<\delta\}$ and $B=\{i\in\{1,2,\dots,n\} \mid M_i-m_i\ge\delta\}$

(1/2 U(g∘f,P))

$$\sum_{i\in A}(\tilde{M}_i-\tilde{m}_i)\Delta x_i < \varepsilon'\sum_{i\in A}\Delta x_i \le \varepsilon'(b-a)$$

$$\sum_{i\in B}(\tilde{M}_i-\tilde{m}_i)\Delta x_i \le 2K\sum_{i\in B}\Delta x_i \le 2K\sum_{i\in B}\frac{(M_i-m_i)}{\delta}\Delta x_i$$

$\delta \to 1$.

$$\le \frac{2K}{\delta}\sum_{i\in B}(M_i-m_i)\Delta x_i$$

∵f可積 ∴要多小有多小

$$\leq \frac{2K}{\delta}\left(U(f,P)-L(f,P)\right)$$

∵ f is integrable over $[a,b]$

∴ For δ^2, $\exists$ a partition P of $[a,b]$ s.t.

$U(f,P)-L(f,P)<\delta^2$

$$\Rightarrow \sum_{i\in B}(\tilde{M}_i-\tilde{m}_i)\Delta x_i \leq 2K\delta < 2K\varepsilon'$$

$$U(g\circ f,P)-L(g\circ f,P)\leq \varepsilon'(b-a)+2K\varepsilon'=(b-a+2K)\varepsilon'$$

$$<\varepsilon \Rightarrow \varepsilon' < \frac{\varepsilon}{b-a+2K}$$

Choose $\varepsilon' = \frac{\varepsilon}{b-a+2K}$, Then $U(g\circ f,P)-L(g\circ f,P)<\varepsilon$. Therefore $g\circ f$ is integrable over $[a,b]$.

Q.E.D.

$g(x)=x^n$, $f^n=g\circ f$.

Cor. ① If f is integrable over $[a,b]$, then f^n is integrable over $[a,b]$, $\forall n\in\mathbb{N}$

② If f and g are integ. over $[a,b]$, then fg is integrable over $[a,b]$.

pf: ②: $f\cdot g=\frac{1}{2}\left((f+g)^2-f^2-g^2\right)$

Q.E.D.

* 補充題:

① Let $f: [a,b] \to \mathbb{R}$ be cont. and $f(x) \geq 0$ $\forall x \in [a,b]$. Prove that if $L(f,P) = 0$ $\forall$ partition P of $[a,b]$, then $f(x) = 0$, $\forall x \in [a,b]$.

② Let $f: [a,b] \to \mathbb{R}$ be a bounded function. Suppose that for any $c \in (a,b)$, the restriction of f to $[c,b']$ is integrable. Show that f is integrable over $[a,b]$ and $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$.

Ch5 Uniform convergence

§5-1 Pointwise and uniform convergence

Def. Let (M,d) and (N,ρ) be metric spaces and $A\subset M$. Let $f_k: A\to N$ be a seq. of functions $\forall k\in\mathbb{N}$, and $f: A\to M$ be a function

① We say that f_k converges (pointwise) to f if for each $x\in A$, $f_k(x)\to f(x)$.
Write $f_k\to f$,
(Namely, $\forall x\in A$, $\forall \varepsilon>0$, $\exists K\in\mathbb{N}$ s.t. $\forall k>K$ $\rho(f_k(x), f(x))<\varepsilon$. When x varies, K varies, i.e. $K=K(x,\varepsilon)$).

② We say that f_k converges on A if $\exists$ a function $f: A\to N$ s.t. $f_k\to f$.
⇒ seq. of function 是 seq. of pts 之推廣.
些數列同時收不收斂.

Q: 对 seq. of functions 可以討論怎樣的問題?
A: ① 会不会收敛呢?
② 若 f_k converges, f 与 f_k 的关係? (e.x. 若 f_k cont, bdd, $\forall k\in\mathbb{N}$, f 是否 cont? bdd?)
③ 可否找到共有的收敛速度呢?

ex. Let $f_k(x)=\begin{cases}0, & \text{as } x\geq\frac{1}{k}\\ -kx+1, & \text{as } 0\leq x<\frac{1}{k}\end{cases}$
Find $\lim_{k\to\infty} f_k(x)=?$ $\forall x\in[0,\infty)$

MON TUE WED

pf:

Remark.

Def.

Remark

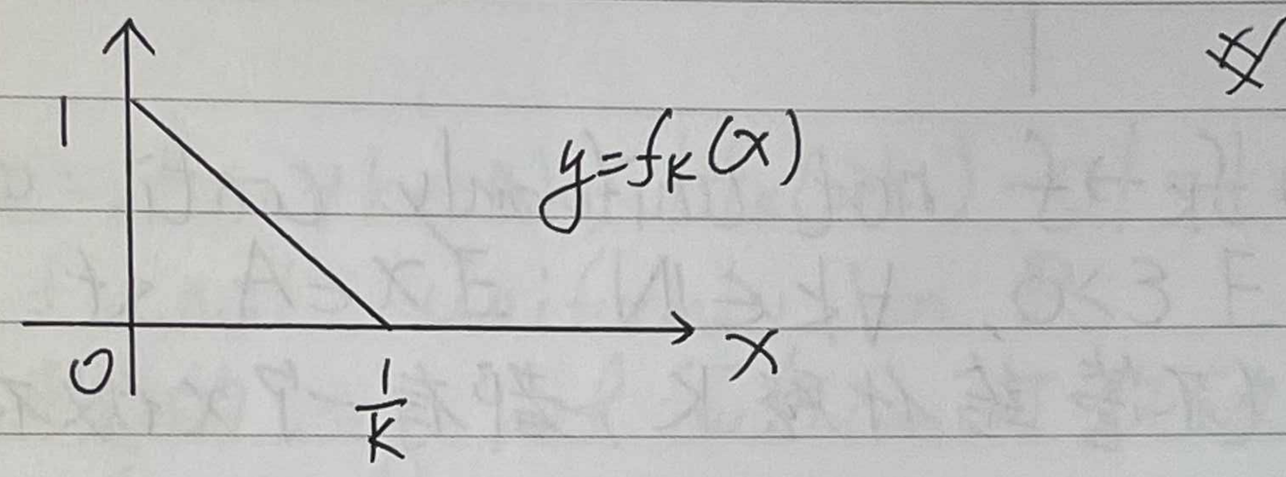
Nam
"ε-
k>

pf: $\because f_k(0)=1,\ \forall k\in\mathbb{N},\ \therefore \lim_{k\to\infty} f_k(0)=1$.

Let $x\in(0,\infty)$, then when $k>\frac{1}{x}$, $f_k(x)=0$

$\Rightarrow \lim_{k\to\infty} f_k(x)=0$

Hence $\lim_{k\to\infty} f_k(x)=\begin{cases}0, & \text{as } x>0\\ 1, & \text{as } x=0\end{cases}$.



Remark. ① f_k is conti. on $[0,\infty)$, $\forall k\in\mathbb{N}$, but its limit function is not conti.

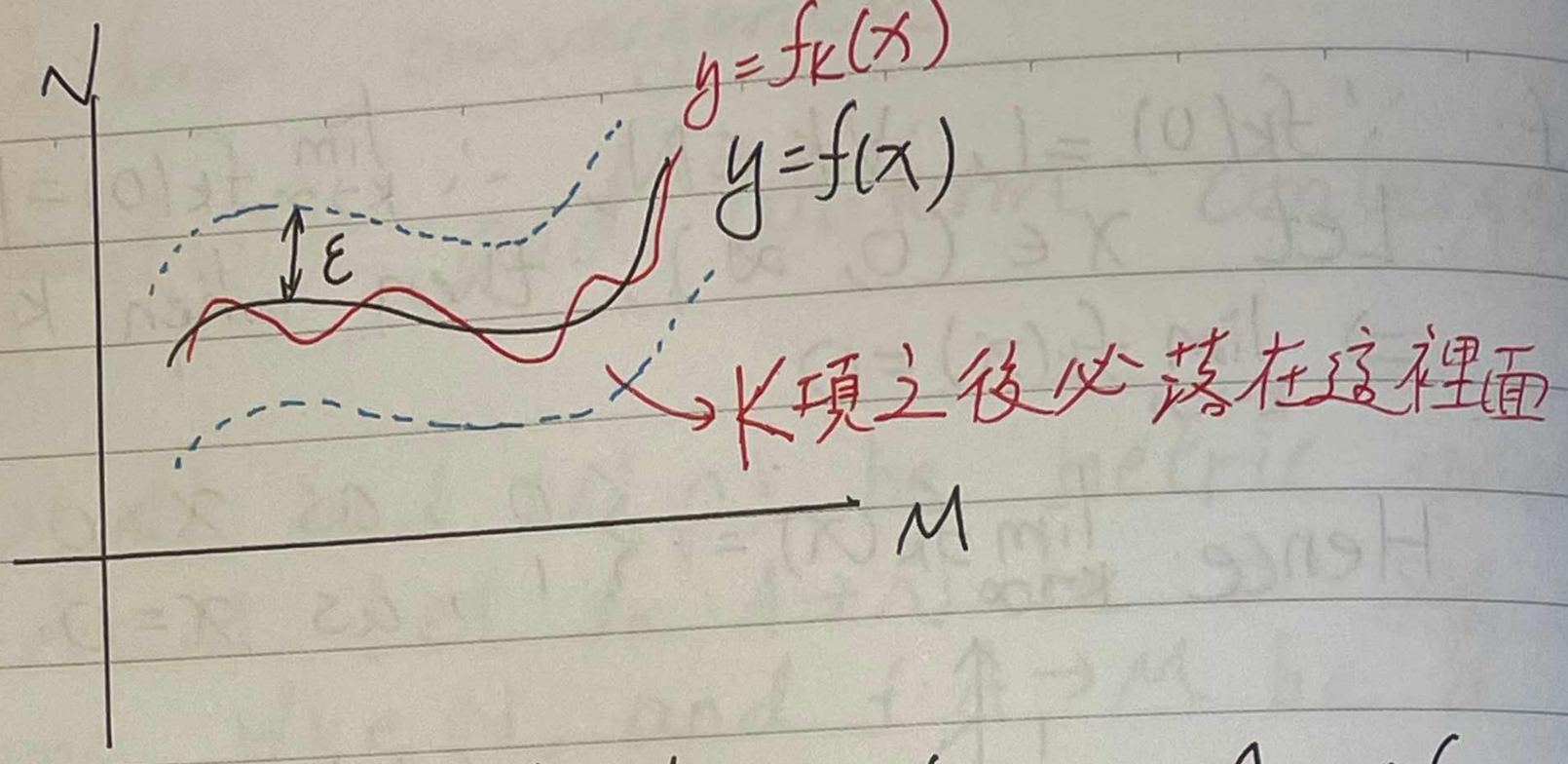
② $\sup\{Kx\geq 0 \mid x\in A\}\geq 0$, when $f_k(x)\to f(x)$ $\forall\varepsilon>0$, "K"越大，收斂速度越慢 ⇒ 找不到最小收斂速度 ∴ inf 跑到 ∞

Def. We say that $f_k\to f$ uniformly on A, if $\forall\varepsilon>0$, $\exists K\in\mathbb{N}$ s.t. $\forall k>K$, $\rho(f_k(x),f(x))<\varepsilon$, $\forall x\in A$.

Remark. ① Uniform convergence ⇒ convergence

② The condition "$\rho(f_k(x),f(x))<\varepsilon$, $\forall x\in A$" means $f_k(x)$ is always within the distance ε to $f(x)$ whenever $k>K$.

Namely, the graph of f_k must lie within "ε-tube" around the graph of f whenever $k>K$.

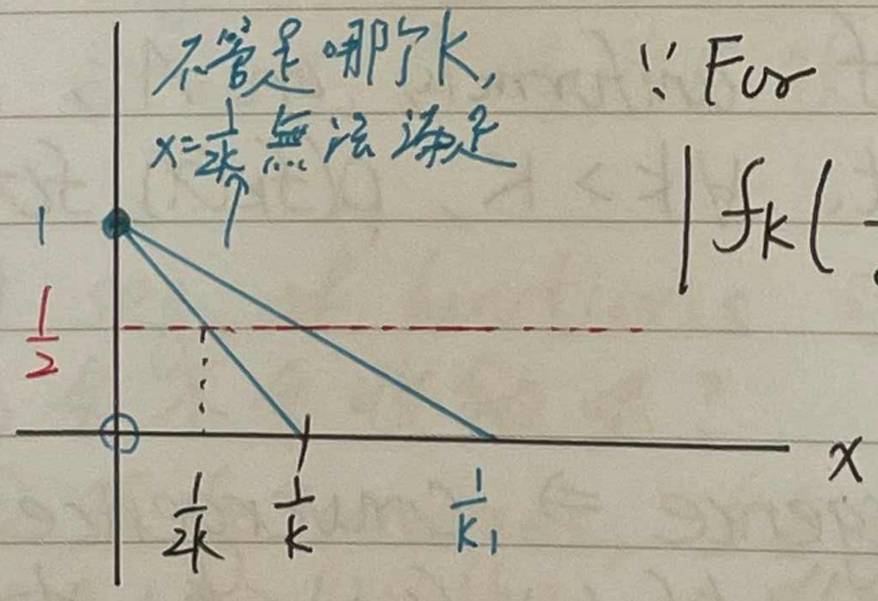


③ $f_k \to f$ not uniformly conti. on A if
$\exists\, \varepsilon>0,\ \forall k\in\mathbb{N},\ \exists x\in A$ s.t. $\rho(f_k(x), f(x))\geq\varepsilon$
(不管給什麼K，都有一個x做不到)

ex. $f_k(x)=\begin{cases}0 & \text{, if } x\geq\frac{1}{k}\\ -kx+1 & \text{, if } 0\leq x<\frac{1}{k}\end{cases}$

Let $f(x)=\begin{cases}0, & x>0\\ 1, & x=0\end{cases}$, Then $f_k\to f$. Show that $f_k\to f$ not uniformly on $[0,\infty)$

pf:



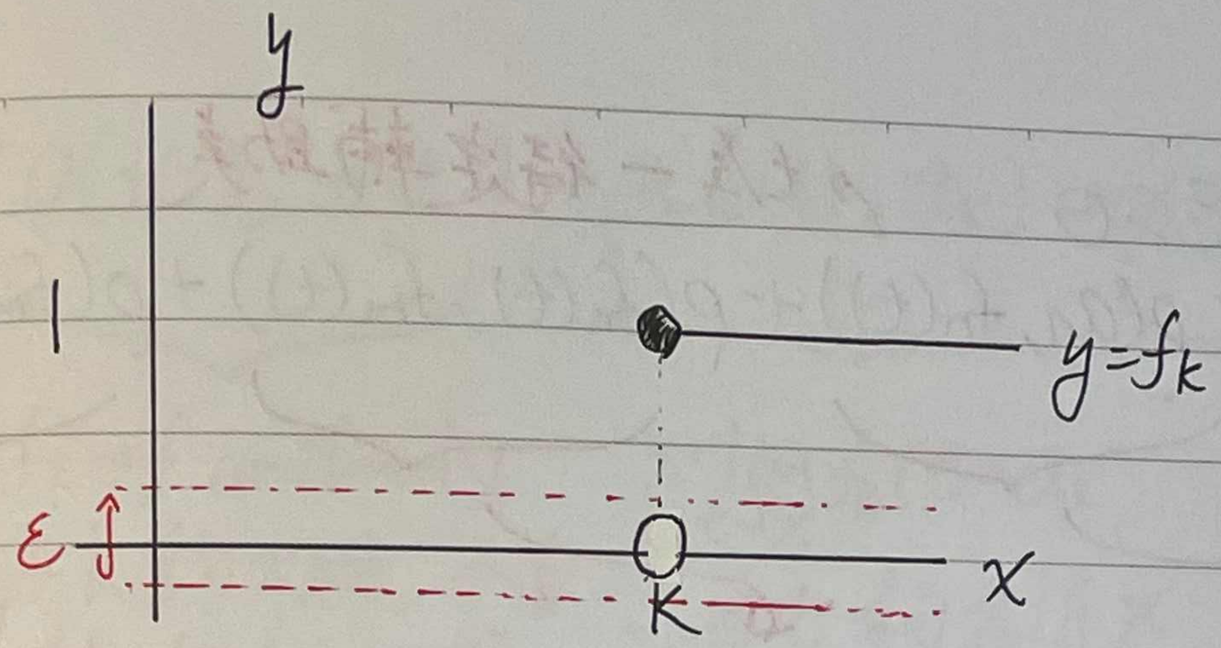
∵ For $\varepsilon=\frac{1}{2}$, $\forall k\in\mathbb{N}$,
$\left|f_k\left(\frac{1}{2k}\right)-0\right|=\frac{1}{2}\geq\frac{1}{2}$

∴ $f_k\to f$ not uniformly on A //

ex. $f(x)=\begin{cases}0, & \text{if } x<k\\ 1, & \text{if } x\geq k\end{cases}$

Does f_k converge? If so, does it uniformly?

pf: ① Let $x\in\mathbb{R}$, Then $\forall k>x$, $f_k(x)=0$
$\Rightarrow \lim_{k\to\infty} f_k(x)=0 \Rightarrow f_k\to 0$ on $\mathbb{R}$.



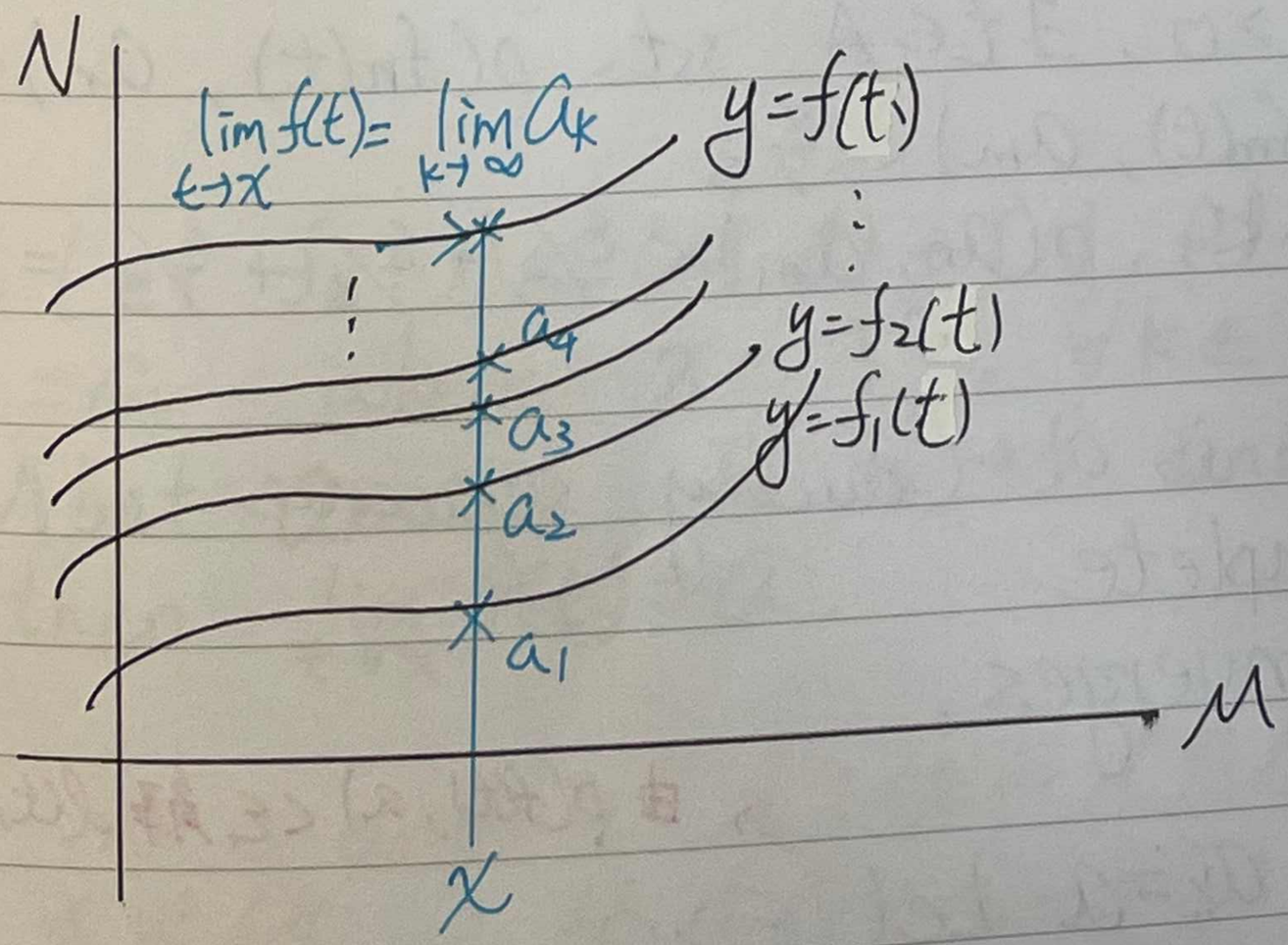
② ∵ For $\varepsilon=\frac{1}{2}$, $\forall k\in\mathbb{N}$, $|f_k(2k)-0|=1\geq\frac{1}{2}$

∴ $f_k\to f$ not uniformly on $\mathbb{R}$

⇒ 不連續的 seq. of functin 可收斂到 conti. functin ✌

Thm. Let $f_k: A\subset(M,d)\to(N,\rho)$ be functions with $\lim_{t\to x} f_k(t)=a_k$, where $x\in A'$, $\forall k\in\mathbb{N}$. Suppose that N is complete. If $f_k\to f$ uniformly on A, then a_k converges and $\lim_{t\to x} f(t)=\lim_{k\to\infty} a_k$.

(i.e. uniformly convergence preserves the limit at x).



pf. ① Let $\varepsilon>0$. ↑ t为一待定辅助点

$$\rho(a_n,a_m)\le \rho(a_n,f_n(t))+\rho(f_n(t),f_m(t))+\rho(f_m(t),a_m)$$

思考

在给定 n之下
因 $\lim_{t\to x} f_n(t)=a_n$
故当x夠靠近x，此项夠小

∵ f_k is uniformly convergence
∴当n,m夠大時，不管怎樣的t，此项夠小

在给定m下，
....

⇒故由第二项决定怎样的K夠大，再由给定的m与n来挑选t

∵ $f_k\to f$ uniformly on A
∴ For $\varepsilon/6$, $\exists K\in\mathbb{N}$ s.t. $\forall k>K$, $\rho(f_k(\tilde{x}),f(\tilde{x}))<\varepsilon/6$ $\forall \tilde{x}\in A$.

⇒ $\forall m,n>K$, $\rho(f_n(\tilde{x}),f_m(\tilde{x}))<\rho(f_n(\tilde{x}),f(\tilde{x}))+\rho(f(\tilde{x}),f_m(\tilde{x}))<\frac{\varepsilon}{6}+\varepsilon/6=\varepsilon/3$, $\forall \tilde{x}\in A$

∵ $\lim_{t\to x} f_n(t)=a_n$, and $\lim_{t\to x} f_m(t)=a_m$
∴ For $\varepsilon/3>0$, $\exists t\in A$ s.t. $\rho(f_n(t),a_n)<\varepsilon/3$ and $\rho(f_m(t),a_m)<\varepsilon/3$
⇒ $\forall m,n>K$, $\rho(a_n,a_m)<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon$

Therefore a_K is a Cauchy sequence in N.
∵ N is complete
∴ a_K converges.

② Let $\lim_{K\to\infty} a_K=a$. Let $\varepsilon>0$ ↑ 由 $\rho(f(t),a)<\varepsilon$ 解 $d(t,x)$

$\rho(f(t),a) \le \rho(f(t),f_k(t)) + \rho(f_k(t),a_k) + \rho(a_k,a)$

$\because f_k \to f$ uniformly
$\therefore$ 当k夠大時，不管怎樣的t，此項都會夠小。

$\because \lim_{t\to x} f_k(t) = a_k$
$\therefore$ 当t夠靠近x時，此項夠小

$\because a_k \to a$
$\therefore$ 当k夠大此項夠小

$\Rightarrow$ 故由第一項与第三項決定怎樣的k夠大，再由第二項去決定t如何靠近x

$\because f_k \to f$ uniformly on A and $a_k \to a$
$\therefore$ For $\varepsilon/3$, $\exists k \in \mathbb{N}$ s.t. $\forall k > K$ (取了式子K的max), $\rho(f(t),f_k(t)) < \varepsilon/3$, $\forall t \in A$ and $\rho(a_k,a) < \varepsilon/3$

$\because \lim_{t\to x} f_{2k}(t) = a_{2k}$
$\therefore$ For $\varepsilon/3$, $\exists \delta > 0$ s.t. $\forall t \in A$ with $d(t,x) < \delta$, $\rho(f_{2k}(t), a_{2k}) < \varepsilon/3$

$\Rightarrow \forall t \in A$ with $d(t,x) < \delta$, $\rho(f(t),a)$
$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

$\Rightarrow$ Therefore $\lim_{t\to x} f(t) = a$.

Q.E.D.

Corl. Let $f_k : A \subset (M,d) \to (N,\rho)$ with $\lim_{t\to x} f_k(t) = a_k$, where $x \in A'$, $\forall k \in \mathbb{N}$. Suppose that $a_k \to a$. If $f_k \to f$ uniformly on A, then $\lim_{t\to x} f(t) = a$.

(∵"N is complete"這了條件是用在讓a_k是Cauchy並收敛，∴替换成"$a_k \to a$"亦可)

Note: Moreover, If f_k is cont. on A, $\forall k \in \mathbb{N}$,
then $\lim_{t \to x} f_k(t) = f_k(x) \to f(x)$

$\because f_k \to f$ uniformly
$\therefore \lim_{t \to x} f(t) = f(x)$ by Thm (極限和對引之極限)
$\Rightarrow f$ is conti.
(如果有一連续的 seq. of function uniformly convergence, 則其 limit function 也 conti.)

⁂

Cor 2. Let $f_k : A \subset (M, d) \to (N, \rho)$ be conti functions. $\forall k \in \mathbb{N}$. If $f_k \to f$ uniformly, then f is conti. on A

↳ uniformly convergence preserve conti

ex. Let $f_n(x) = x^n$ on $[0, 1]$. Does f_n converge uniformly? What if $0 < x < 1$?

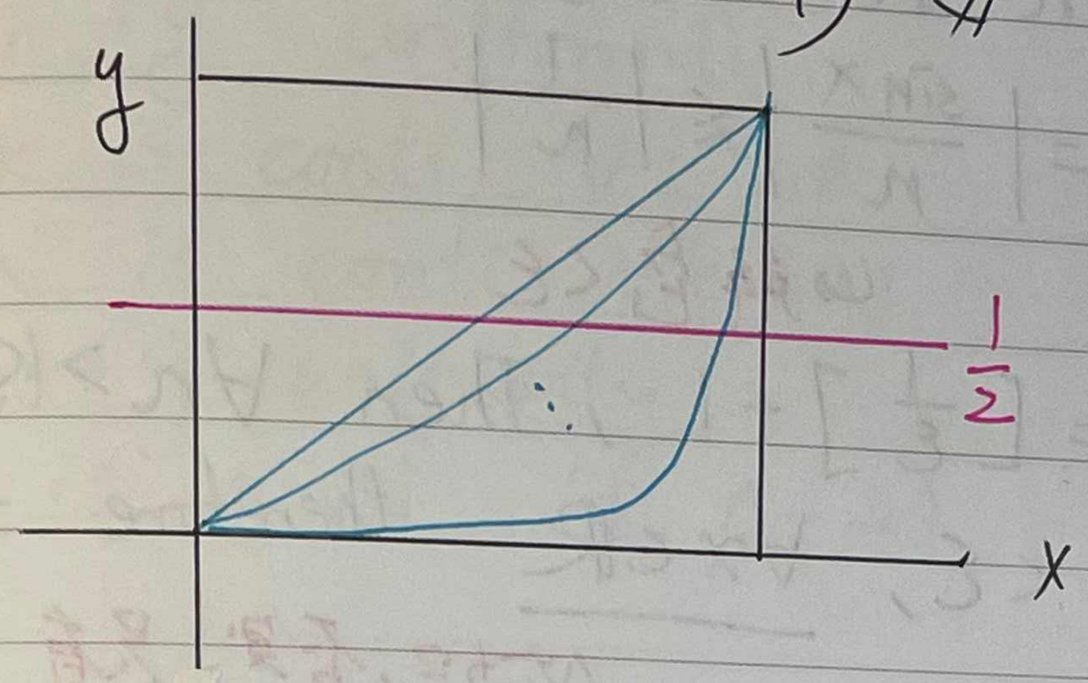
pf: ① $f_n(x) \to f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases}$

$\because f_n$ is conti. on $[0,1]$, $\forall n \in \mathbb{N}$
and f is not conti. on $[0,1]$
$\therefore f_n \to f$ not uniformly

② 考慮 limit function 的 ε-tube
→不管 n 有多大, 都有人会超过 $1/2$. 換言之,
$\because f_n$ 在 1 conti., 縱有人要很靠近 1,
以致於比 0 还靠近 1.

Let $\varepsilon = \frac{1}{2}$ $\because f_n$ is conti. at 1, $\forall n \in \mathbb{N}$
$\therefore \exists x_n$ closed enough to 1 s.t. $f(x_n) > \frac{1}{2}$

$\Rightarrow f_n \to f$ not uniformly $\#$



ex. Let $f_n(x) = \dfrac{x^n}{1+x^n}$ on $[0,2]$. Show that f_n converges but not uniformly.

pf. If $0 \le x < 1$ then $x^n \to 0$ as $n \to \infty$
$\Rightarrow f_n \to 0$ on $[0,1)$
If $x = 1$, $f_n(1) = \frac{1}{2} \to \frac{1}{2}$.
If $x \in (1,2]$, then $x^n \to \infty \Rightarrow f_n \to 1$.

Therefore $f_n \to f(x) = \begin{cases} 0, & \text{if } x \in [0,1) \\ 1/2, & \text{if } x = 1 \\ 1, & \text{if } x \in (1,2] \end{cases}$

$\because f_n$ is conti. $\forall n \in \mathbb{N}$ and f is not conti.
$\therefore f_n \to f$ not uniformly. $\#$

ex. Let $f_n : \mathbb{R} \to \mathbb{R}$ be function defined by $f_n(x) = \sin x / n$. Prove or disprove that f_n converges uniformly on $\mathbb{R}$.

pf. $f_n \to 0$ on $\mathbb{R}$. Let $\varepsilon > 0$,

$$\left| \frac{\sin x}{n} - 0 \right| = \left| \frac{\sin x}{n} \right| \le \left| \frac{1}{n} \right|$$

↳ 解它 $< \varepsilon$

Choose $K = \left[\frac{1}{\varepsilon} \right] + 1$, Then $\forall n > K$,

$\left| \frac{\sin x}{n} - 0 \right| < \varepsilon$, $\forall x \in \mathbb{R}$ therefore $f_n \to 0$

uniformly on $\mathbb{R}$ ≠ 必加x，否則另有在x收斂

Def. Let $g, g_k : A \subset (\mathcal{M}, d) \to (V, \|\cdot\|)$, a normed space, be a function, $\forall k \in \mathbb{N}$.

① We say that the series $\sum_{k=1}^{\infty} g_k$ converges to g if the seq. of partial sums $S_n = \sum_{k=1}^{n} g_k$ converges to g. Write $\sum_{k=1}^{\infty} g_k = g$

② We say that $\sum_{k=1}^{\infty} g_k$ converges uniformly to g if S_n converges uniformly to g.

③ We say that the series $\sum_{k=1}^{\infty} g_k$ converges, if $\exists$ a function $g : A \to V$ s.t. $\sum_{k=1}^{\infty} g_k = g$.

Thm. If $g_k : A \to (V, \|\cdot\|)$ is conti. and $\sum_{k=1}^{\infty} g_k = g$ uniformly, then g is conti. on A.

pf: Let $S_n = \sum_{k=1}^{n} g_k$.

$\therefore \sum_{k=1}^{\infty} g_k = g$ uniformly,

$\therefore S_n \to g$ uniformly.

$\because g_k$ is conti, $\forall k \in \mathbb{N}$

$\therefore S_n$ is conti. $\forall k \in \mathbb{N}$

$\because S_n \to g$ uniformly

$\therefore g$ is conti. (by Cor 2).

Q.E.D.

ex. Show that $\sin x = x - \frac{x^3}{3} + \frac{x^5}{x^5} + \cdots$

$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!}$ converges uniformly

on $[-r, r]$, $\forall r \in \mathbb{R}$

pf. Let $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k-1}$

$$|S_n(x) - \sin x| = \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k-1} \right|$$

$$\leq \sum_{k=n+1}^{\infty} \frac{r^{2k-1}}{(2k-1)!} \quad -(*)$$

$\Rightarrow$ 若 $\sum_{k=1}^{\infty} \frac{r^{2k-1}}{(2k-1)!}$ 收斂, 則 $(*)$ 亦收斂

$\because \frac{r^{2k+1}}{(2k+1)!} \cdot \frac{(2k-1)!}{r^{2k-1}} = \frac{r^2}{(2k+1)(2k)} \xrightarrow{\text{as } k \to \infty} 0 < 1$

$\therefore$ By ratio test, $\sum_{k=1}^{\infty} \frac{r^{2k}}{(2k-1)!}$ converges.

$\Rightarrow \sum_{k=n+1}^{\infty} \frac{r^{2k}}{(2k-1)!} \to 0$ as $n \to \infty$ (indep. of x)

$\Rightarrow S_n \to \sin x$ uniformly on $[-r, r]$

$$\Rightarrow \sin x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!}$$ uniformly on

$[-r, r]$ #

ex. Consider the series $\sum_{k=0}^{\infty} x^k$

① It converges to $\frac{1}{1-x}$ uniformly on

$[-a, a]$ for any $a \in [0, 1)$

② Show that it doesn't converge uniformly on $(-1, 1)$

pf. ① Let $S_n = \sum_{k=0}^{n} x^n$ Then $S_n = \frac{1-x^{n+1}}{1-x}$

$$\because \left| \frac{1-x^{n+1}}{1-x} - \frac{1}{1-x} \right| = \frac{|x|^{n+1}}{1-x} < \frac{a^{n+1}}{1-a} \to 0$$

($\uparrow$ indep. of x)

as $n \to \infty$

$\therefore S_n \to \frac{1}{1-x}$ uniformly on $[-a, a]$.

② 出問題的点在 $x=1$ 会爆掉

If $x > 1/2$, then

$$\frac{|x|^{n+1}}{1-x} \geq \frac{|x|^{n+1}}{1-\frac{1}{2}} = 2x^{n+1} \to 0 \text{ not}$$

uniformly on $[\frac{1}{2}, 1)$. (since $x^n \to 0$ not uniformly)

$$\Rightarrow \sum_{k=0}^{\infty} x^k \to \frac{1}{1-x}$$ is not uniformly on $(-1, 1)$.

§5.2 The Weierstrass M-test

Q: Let $f, f_k : A \subset (M,d) \to (N,\rho)$ be functions s.t. $f_k \to f$.

想問 convergence 是否会将函数 f_k 的性质遗传到 f 呢? 用数学的术语来说为 convergence 对函数的性质具有封闭 (preserve) 吗?

A: 函数有何性质呢?

① 最根本的为函数的极限: Let $x \in A'$. If $f_k \to f$ on A, does $\lim_{t\to x} f_k(t) \xrightarrow{?} \lim_{t\to x} f(t)$?

在討論此問題需要一些前提, 即 the seq. $\lim_{t\to x} f_k(t)$ in N 必須先收斂

ex. Let $f_k(x) = x^k$ on $[0,1]$ then $f_k \to f$ where

$$f(x) = \begin{cases} 0, & \text{if } x \in [0,1) \\ 1, & \text{if } x = 1 \end{cases}$$

在此例子中,

$\lim_{t\to x} f_k(t) = f_k(x) \to f(x) \neq \lim_{t\to x} f(x)$.

因此 convergence 对函数极限不具有封閉性.

Q: 那在 $f_k \to f$ uniformly on A 之下呢?

A: ① 在 §5.1, 当 N is complete, 若 $\lim_{t\to x} f_k(x)$ exists $\forall k \in \mathbb{N}$, 则 the seq. $\lim_{t\to x} f_k(t)$ 收斂

② In general case of N, 若 $\lim_{t\to x} f_k(t)$ 收斂 in N. 则 $\lim_{t\to x} f(t)$ exists and $\lim_{t\to x} f_k(t) \longrightarrow \lim_{t\to x} f(t)$

(此二句話可用一句話來表示: $\lim_{t\to x} f(t) = \lim_{k\to\infty} \lim_{t\to x} f_k(t)$)

§5.1 中的 a_k

② 函数的連續性：

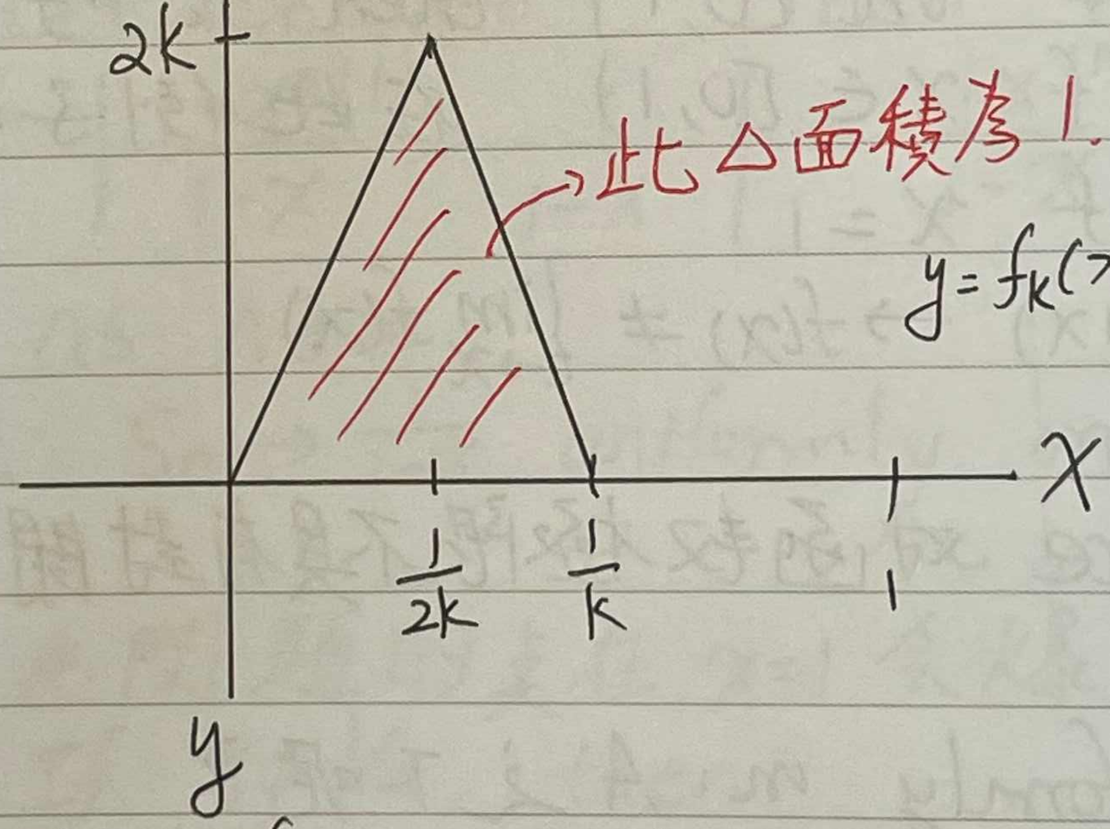
由① 中的例子可知 convergence 对函数的連续性不具有封閉性。但 uniformly convergence 会对函数的連续性具有封閉性

Note：若 $(M,d)=(N,\rho)=(\mathbb{R}^n,\|\cdot\|)$，我們还可以討論函数的可積性和可微性

③ 函数的可積性：

可積函数不一定是連續函数，因此問題也可能成立．看一下例子：

Let $f_k:[0,1]\to\mathbb{R}$ be a function defined by



Then $f_k\to 0$ but not uniformly on $[0,1]$

(why? $\because f_k(\frac{1}{2k})=2k\to\infty$ as $k\to\infty$ $\therefore |f_k(\frac{1}{2k})-0|\geq 2$)

$\int_0^1 f_k(x)\,dx=1 \not\to \int_0^1 0\,dx=0$

对 convergence 对定積分不具有封閉性，故在 uniform convergence 之下去討論此問題．

(將在 §5.3 討論)

④ 函数的可微性

故要在 uniform convergence 之下去討論此問題

Q: 如何決定一个 seq. of functions $f_k \to f$ uniformly in A 呢?

Thm. Let $f, f_k : A \subset (M,d) \to (N,\rho)$ be functions. Suppose N is complete. Then f_k converges uniformly on A if $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall m,n > K$, $\rho(f_m(x), f_n(x)) < \varepsilon$ for all $x \in A$.

pf: $\Rightarrow$) Let $\varepsilon > 0$, $\rho(f_m(x), f_n(x)) \le \rho(f_m(x), f(x)) + \rho(f(x), f_n(x))$.

$\because f_k \to f$ uniformly on A

$\therefore$ For $\varepsilon/2$, $\exists K \in \mathbb{N}$ s.t. $\rho(f_k(x), f(x)) < \varepsilon/2$, $\forall x \in A$

$\Rightarrow \forall m,n > K$, $\rho(f_m(x), f_n(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$\Leftarrow$) $\because \forall x \in A$, $f_k(x)$ is a Cauchy seq. in N and N is complete

$\therefore \exists y_x \in N$ s.t. $f_k(x) \to y_x$

Define a function $f: A \to N$ by $f(x) = y_x$.

Then $f_n \to f$. Let $\varepsilon > 0$, For $0 < \varepsilon' < \varepsilon$, $\exists K \in \mathbb{N}$ s.t. $\forall n,m > K$, $\rho(f_n(x), f_m(x)) < \varepsilon'$, $\forall x \in A$.

$\because (f_n(x), f_m(x)) \xrightarrow{m \to \infty} (f_n(x), f(x))$ and ρ is conti. on $N \times N$

$\therefore \rho(f_n(x), f_m(x)) \xrightarrow{m\to\infty} \rho(f_n(x), f(x))$

$\Rightarrow \rho(f_n(x), f(x)) \le \varepsilon' < \varepsilon. \quad \forall x \in A.$

Therefore $f_k \to f$ uniformly on A.

Q.E.D.

Cor. Let $(V, \|\cdot\|)$ be a normed space (complete).

Let $g_k : A \subset (M,d) \to (V, \|\cdot\|)$ be functions, $\forall k \in \mathbb{N}$

Then $\sum_{k=1}^{\infty} g_k$ converges uniformly on A

$\iff \forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t. $\forall m > n > K$,

$\| g_{n+1}(x) + \cdots + g_m(x) \| < \varepsilon, \forall x \in A.$

Cor. (Weistrass M-test)

Suppose that $(V, \|\cdot\|)$ is complete normed space and $A \subset (M,d) \to V$ is a function, $\forall k \in \mathbb{N}$.

If for each k, $\exists M_k \ge 0$ s.t. $\|g_k(x)\| \le M_k$, $\forall x \in A$ and $\sum_{k=1}^{\infty} M_k$ converges, then $\sum_{k=1}^{\infty} g_k$ converges uniformly.

pf: Let $\varepsilon > 0$. Let $m > n$. $\| g_{n+1}(x) + \cdots + g_m(x) \|$

$\le \|g_{n+1}(x)\| + \cdots + \|g_m(x)\|$

$\le M_{n+1} + \cdots + M_m, \quad \forall x \in A.$

$\therefore \sum_{k=1}^{\infty} M_k$ converges.

$\therefore$ For $\varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall m > n > K$,

$M_{n+1} + \cdots + M_m < \varepsilon$

$\Rightarrow \forall m > n > K, \| g_{n+1}(x) + \cdots + g_m(x) \| < \varepsilon, \forall x \in A.$

By Cauchy criteria,

$\sum_{k=1}^{\infty} g_k$ is uniformly convergence on A.

Q.E.D.

ex. Show that the series $\sum_{n=1}^{\infty} \frac{(\sin nx)^2}{n^2}$ converges uniformly on $\mathbb{R}$.

pf: $\left|\frac{(\sin nx)^2}{n^2}\right| \le \frac{1}{n^2}$, $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}$

$\because 2 > 1$

$\therefore$ By p-test, $\sum \frac{1}{n^2}$ converges.

By Weierstrass M-test, $\sum \frac{(\sin nx)^2}{n^2}$ converges on $\mathbb{R}$ #

ex2. Prove that $f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$ is cont. on $\mathbb{R}$.

pf: $\left(\frac{x^n}{n!}\right)^2 \to \infty$ as $x \to \infty$. Let $x \in [-r, r]$, for some $r > 0$. Then $\left(\frac{x^n}{n!}\right)^2 \le \frac{r^{2n}}{(n!)^2}$, $\forall x \in [-r, r]$. $\forall n \in \mathbb{N}$.

$\because \frac{r^{2(n+1)}}{((n+1)!)^2} \cdot \frac{(n!)^2}{r^{2n}} = \frac{r^2}{(n+1)} \to 0 < 1$.

$\therefore$ By ratio test, $\sum \frac{r^{2n}}{n!}$ converges.

By Weierstrass M-test, $\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$ converges uniformly on $[-r, r]$

$\therefore \left(\frac{x^n}{n!}\right)^2$ is cont. on $[-r,r]$ $\forall n \in \mathbb{N}$.

uniformly conv. ⇐ $\therefore$ f is cont. on $[-r,r]$. Let $x \in \mathbb{R}$

Then $\exists \tilde{r} > 0$ s.t. $x \in [-\tilde{r}, \tilde{r}]$

$\therefore$ f is cont. on $[-\tilde{r}, \tilde{r}]$

$\therefore$ f is cont. at x

$\Rightarrow$ f is cont. on $\mathbb{R}$. #

ex. Suppose that $a_0, a_1, \dots$ is a bdd seq. of real numbers. Show that the series $\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ converges to a cont. function

pf: Let $|a_k| \le M$, $\forall k = 0, 1, \dots$, and $r > 0$.

$$\left|\frac{a_k}{k!} x^k\right| \le \frac{M}{k!} r^k, \quad \forall x \in [-r, r]$$

$$\because \frac{M}{(k+1)!} r^{k+1} \cdot \frac{k!}{M r^k} = \frac{r}{k+1} \to 0 < 1$$

$\therefore$ By Ratio test, $\sum_{k=0}^{\infty} \frac{a_k}{k!} r^k$ converges

$\Rightarrow$ By Weiestrass M test, $\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ uniformly converges on $[-r, r]$.

$\because \frac{a_k}{k!} x^k$ is cont. on $[-r,r]$ $\forall k$

$\therefore \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ is cont. on $[-r,r]$

$\because r$ is arbitrary

$\therefore \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ is cont. on $\mathbb{R}$ #

MON TUE WED THU FRI SAT SUN | DATE: / / | NO.: | SUBJECT:

△高等微積分整理

高微：用遠近关係、研究集合，用d去描述（空間感

⇒ (interior pt, accumulation pt ⟩ metric topology

↓ bdy point), compact, connected

↳ 分佈情形

↳ 點集合的相对关係

→ "附近"的概念 → 用 ε-δ 建立 limit 的概念

⇕

用 nbd 建立的 limit 的概念

↓

cont.（一种特别的极限）

→ 研究 "f cont" 对 metric topology 有沒有封閉性

↓

sequence of functions:

"$f_n \to f$" 对函数的特質是否具有封閉性？(5.1-5.3)

→ 考虑「所有函数」？(functim space)(5.4)

general topology: ① 用"空間感"研究集合 topology

→ 高微裡做的事能否重現？

② 空間歸化

（不牽涉到"数"与"運算"）

数的

formly

§5.3 Integration and differentiation of series

Thm. Suppose that f_k are integrable over $[a,b]$, $\forall k\in\mathbb{N}$. If $f_k\to f$ uniformly, then f is integrable over $[a,b]$. and

$$\int_a^b f_k(x)dx \to \int_a^b f(x)dx \quad \left(\text{or } \lim_{k\to\infty}\int_a^b f_k(x)dx = \int_a^b f(x)dx\right)$$

pf: ① f is bdd

$\because f_k\to f$ uniformly cont. on $[a,b]$.

$\therefore$ For $\varepsilon=1$, $\exists K\in\mathbb{N}$ s.t. $\forall k>K$,

$|f_k(x)-f(x)|<1$, $\forall x\in[a,b]$

$\Rightarrow |f_{2K}(x)-f(x)|<1$, $\forall x\in[a,b]$

$\Rightarrow |f(x)|<|f_{2K}(x)|+1$, $\forall x\in[a,b]$

$\because f_{2K}$ is bdd on $[a,b]$

$\therefore f$ is bounded on $[a,b]$

② f is integrable over $[a,b]$:

Let $\varepsilon>0$, $\because f_k\to f$ uniformly on $[a,b]$

$\therefore \forall \frac{\varepsilon}{2}>0$, $\exists K\in\mathbb{N}$ s.t. $\forall k>K$,

$|f_k(x)-f(x)|\le\frac{\varepsilon}{4(b-a)}$ $\forall x\in[a,b]$.

其实写 $>K$ 也可以 试的像"$<\varepsilon$"跟"$\le\varepsilon$"等价

$\Rightarrow \forall k>K$, $f_k(x)-\frac{\varepsilon}{4(b-a)}\le f(x)<f_k(x)+\frac{\varepsilon}{4(b-a)}$, $\forall x\in[a,b]$.

Let $P=\{a=x_0<x_1<\cdots<x_n=b\}$ be a partition of $[a,b]$. $\forall k>K$,

$\because f$ is bdd $\therefore$ 可取 sup

$$\sup_{[x_{i-1},x_i]} f(x) \le \sup_{[x_{i-1},x_i]}\left(f_k(x)+\frac{\varepsilon}{4(b-a)}\right) = \sup_{[x_{i-1},x_i]} f_k(x)+\frac{\varepsilon}{4(b-a)}$$

$\forall\ i=1, 2, \dots, n$ and

$\inf_{[x_{i-1}, x_i]} f(x) \geq \inf_{[x_{i-1}, x_i]} f_k(x) - \underbrace{\frac{\varepsilon}{2}}$, $\forall\ i=1, 2, \dots, n$

$\Rightarrow \forall k > K$, $\sup_{[x_{i-1}, x_i]} f(x) - \inf_{[x_{i-1}, x_i]} f(x) \leq \sup_{[x_{i-1}, x_i]} f_k(x) -$

$\inf_{[x_{i-1}, x_i]} f_k(x) + \frac{\varepsilon}{2(b-a)}$, $\forall\ i=1, 2, \dots, n$

$\Rightarrow \forall k > K$, $U(f, P) - L(f, P) \leq U(f_k, P) - L(f_k, P) + \frac{\varepsilon}{2}$.

$\times \Delta x$

$\because f_{2k}$ is integrable over $[a, b]$

$\therefore$ For $\frac{\varepsilon}{2}$, $\exists$ a partition $\check{P}$ of $[a, b]$ s.t.

$U(f_{2k}, \check{P}) - L(f_{2k}, \check{P}) < \frac{\varepsilon}{2}$

$\Rightarrow U(f, \check{P}) - L(f, \check{P}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Therefore f is integrable over $[a, b]$.

③ $\int f_k \to \int f$:

Let $\varepsilon > 0$, we want

$\left| \int_a^b f_k(x)\,dx - \int_a^b f(x)\,dx \right| \leq \left| \int_a^b f_k(x) - f(x)\,dx \right|$

$\leq \int_a^b \underbrace{|f_k(x) - f(x)|}_{< \frac{\varepsilon}{b-a}}\,dx < \varepsilon$

$\because f_k \to f$ uniformly in $[a, b]$

$\therefore$ For $\frac{\varepsilon}{b-a}$, $\exists K \in \mathbb{N}$ s.t. $k > K$, $|f_k(x) - f(x)|$

$< \frac{\varepsilon}{b-a}$ $\forall x \in [a, b]$

$\Rightarrow \forall k' > K,\ \left| \int_a^b f_k(x)\,dx - \int_a^b f(x)\,dx \right| \le \int_a^b \frac{\varepsilon}{b-a}\,dx = \varepsilon$

$\Rightarrow$ Therefore $\int_a^b f_k(x)\,dx \to \int_a^b f(x)\,dx$.

Q.E.D.

Cor. Suppose that $g_k : [a,b] \to \mathbb{R}$ is integrable $\forall k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} g_k$ converges uniformly over $[a,b]$. Then $\sum_{k=1}^{\infty} g_k$ is integrable over $[a,b]$, and $\sum_{k=1}^{\infty} \int_a^b g_k(x)\,dx = \int_a^b \sum_{k=1}^{\infty} g_k(x)\,dx$

ex. Let $f_k(x) = \frac{\sin(k^2 x)}{k}$, $\forall x \in \mathbb{R}$, $\forall k \in \mathbb{N}$. Then

$|f_k(x)| \le \frac{1}{k} \to 0$, $\forall x \in \mathbb{R}$, $\forall k \in \mathbb{N}$

$\Rightarrow f_k \to 0$ uniformly on $\mathbb{R}$.

$f_k'(x) = k\cos(k^2 x)$ $\forall k \in \mathbb{N}$

$\because f_k'(0) = k \to \infty$ as $k \to \infty$

$\therefore f_k'$ doesn't converge on $\mathbb{R}$

ex. Let $f_k(x) = \frac{kx^2}{1+kx^2}$ on $[-1,1]$, $\forall k \in \mathbb{N}$

Then f_k is diff. on $[-1,1]$, $\forall k \in \mathbb{N}$.

$f_k \to f$, where $f = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{otherwise} \end{cases}$

$\because f$ is not diff.

$\therefore f$ is not diff.

$\because f_k\left(\frac{1}{\sqrt{k}}\right) = \frac{1}{2}$, $\forall k \in \mathbb{N}$

$\therefore \left| f_k\left(\frac{1}{\sqrt{k}}\right) - 0 \right| \geq \frac{1}{2}$

$\Rightarrow f_k \to f$ not uniformly on $[-1,1]$

$f_k'(x) = \frac{2kx}{(1+kx^2)^2} \to 0$. → 微分会收敛，但 f' 不存在！

$\because f_k'\left(\frac{1}{k}\right) = \frac{2}{1+\frac{2}{k}+\frac{1}{k^2}} \overset{\text{代 } k=1}{\geq} \frac{2}{1+2+1} = \frac{1}{2}$, $\forall k \in \mathbb{N}$.

$\therefore f_k' \to 0$ not uniformly.

Remark. 此問題要成立，則此問題的前提必須比「f_k' is conti. and f_k' 是 pointwise 收斂的」還強才行.

($f_k' \to f'$?)

Q: If $f_k : I \to \mathbb{R}$ are diff. functions that converges uniformly to f. Does $f_k' \to f'$?

A: 第一个問題：$f_k \to f$ 是否会使得 f_k' 收斂？

→由此例知答案為否，故討論此問題需要一个前提條件，即 f_k' 收斂

第二个問題：若 f_k' 收斂，可保證 f 可微否？

由上例可知不能保證 f 可微.

Assume that $f_k' \to g$ uniformly converges on I.

Let $x \in I$. 去計算 $\lim_{t \to x} \frac{f(t)-f(x)}{t-x}$

此收斂 uniform 嗎？

能否找到一个 seq. of function 收斂至此？

$\because f_k \to f$

$\therefore \frac{f_k(t)-f_k(x)}{t-x} \overset{k \to \infty}{\longrightarrow} \frac{f(t)-f(x)}{t-x}$, $\forall t \in I \setminus \{x\}$

$\because f_k$ is diff. at x $\forall k \in \mathbb{N}$

$\therefore \lim_{t\to x} \dfrac{f_k(t)-f_k(x)}{t-x} = f_k'(x) \xrightarrow{k\to\infty} g(x)$ by 假設

Let $\varepsilon>0$, $\left|\dfrac{f_n(t)-f_n(x)}{t-x} - \dfrac{f_m(t)-f_m(x)}{t-x}\right|$

$= \left|\dfrac{[f_n(t)-f_m(t)]-[f_n(x)-f_m(x)]}{t-x}\right|$

$= |f_n'(c_t)-f_m'(c_t)|$, where c_t is between t and x.

↓ Applying MVT to (f_n-f_m)

$\because f_k' \to g$ uniformly

$\therefore$ For ε, $\exists K\in\mathbb{N}$ s.t. $n,m>K$, $|f_n'(t)-f_m'(t)| < \varepsilon$, $\forall t\in I$.

$\Rightarrow \forall\, n,m>K, \left|\dfrac{f_n(t)-f_m(x)}{t-x}\right| < \varepsilon$

Therefore $\dfrac{f_k(t)-f_k(x)}{t-x}$ is a uniformly Cauchy sequence in $I\setminus\{x\}$

$\because \mathbb{R}$ is complete

$\therefore \dfrac{f_k(t)-f_k(x)}{t-x}$ converges uniformly on $I\setminus\{x\}$

$\because \lim_{t\to x} \dfrac{f_k(t)-f_k(x)}{t-x} = f_k'(x) \to g(x)$

$\therefore \lim_{t\to x} \dfrac{f(t)-f(x)}{t-x} = f(x)$

$\Rightarrow f$ is diff. at x and $f'(x)$ and $g(x)$.

Thm. Let $f_k: I \to \mathbb{R}$ be differentiable function s.t. $f_k \to f$ uniformly. If f_k' converges uniformly to g, then $f' = g$.

→由於 $f_k \to f \not\Rightarrow f_k' \to g$，所以先好到為前提之一

→ How about $\Leftarrow$?

MVT: $f_k(t) = f_k(x) + f_k'(c_t^k)(x-t)$, $\forall t \in I, \forall k \in \mathbb{N}$.

If f_k' is cont. $\forall k \in I$, then first fundamental thm. of integral calculus,

$$f_k(t) = f_k(x) + \int_x^t f_k'(u)\,du$$

Q: 若 $f_k' \to g$ uniformly, 可否得到 $f_k \to f$ uniformly?

A: Let $\varepsilon > 0$, $|f_n(t) - f_m(t)| \le |f_n(x) - f_m(x)| + |f_n'(c_t^n) - f_m'(c_t^m)|\,|t-x|$

$|f_n(x) - f_m(x)|$ < 區間長度. ($\le$?, 但可加「在此收斂」之條件.

$|f_n'(c_t^n) - f_m'(c_t^m)|$ 不同矣→不知收斂至!

(用3.2之MVT)

⇒不該各apply一次MVT

應對 "$f_n - f_m$" apply

→改為 $|f_n(t) - f_m(t)| = |(f_n(t) - f_m(t)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|$

$= |f_n'(c_t) - f_m'(c_t)|\,|t - x_0| + |f_n(x_0) - f_m(x_0)|$

Apply MVT for $f_n - f_m$

$|f_n'(c_t) - f_m'(c_t)|$: $\because f_k' \to g$ uniformly $\therefore$ 当 n, m 夠大時, 不管怎樣的 c_t, 此項都夠靠近

$|t - x_0|$: bdd

$|f_n(x_0) - f_m(x_0)|$: Assume 夠靠近

Assume that I is a bounded interval with length $\leq L$. Assume that $\exists x_0 \in I$ s.t. $f_k(x_0)$ converges

$\because f_k(x_0)$ converges

$\therefore$ For $\frac{\varepsilon}{2}$, $\exists K \in \mathbb{N}$ s.t. $\forall n,m > K$, $|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$

and $|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2L}$, $\forall t \in I$ → 要給t用

$\Rightarrow \forall m,n > K$, $|f_n(t) - f_m(t)| < \frac{\varepsilon}{2L} \cdot L + \frac{\varepsilon}{2} = \varepsilon$

Therefore f_k is a uniform Cauchy sequence

$\because \mathbb{R}$ is complete

$\therefore f_k$ converges uniformly on I.

Thm. Let $f_k : I \to \mathbb{R}$ be a seq. of diff. functions where I is a bdd interval and $f_k(x_0)$ converges for some $x_0 \in I$. Suppose that f_k' converges uniformly to a function $g : I \to \mathbb{R}$, then f_k converges uniformly to a diff. function f on I s.t. $f' = g$.

Cor. If $g_k : I \to \mathbb{R}$ be a sequence of diff. function defined on a bdd interval I s.t. $\sum_{k=1}^{\infty} g(x_0)$ converges for some $x_0 \in I$.

If $\sum_{k=1}^{\infty} g_k'$ converges uniformly, then $\sum_{k=1}^{\infty} g_k$ converges uniformly on I s.t. $\left(\sum_{k=1}^{\infty} g_k\right)' = \sum_{k=1}^{\infty} g_k'$

ex. Suppose that $a_0, a_1, \dots, a_k$ is a bdd seq. in $\mathbb{R}$. Show that the series $\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$

converges to a diff. function on $\mathbb{R}$. s.t. $f'(x) = \sum_{k=0}^{\infty} \frac{a_{k+1}}{k!} x^{k+1}$

pf. Let $r>0$, and $|a_k| \le M$, $\forall k \in \mathbb{N}$.

$$\left| \left(\frac{a_k}{k!} x^k \right)' \right| \le \frac{M}{(k-1)!} r^{k-1}, \quad \forall x \in [-r, r], \forall k \in \mathbb{N}.$$

$$\because \frac{Mr^k}{k!} \cdot \frac{(k-1)!}{Mr^{k-1}} = \frac{r}{k} \xrightarrow[\text{as } k \to \infty]{} 0 < 1$$

$\therefore$ By Ratio test, $\sum_{k=1}^{\infty} \frac{Mr^{k-1}}{(k-1)!}$ converges.

By Weiestrass M-test, $\sum_{k=1}^{\infty} \frac{a_k}{(k-1)!} x^{k-1}$ converges uniformly on $[-r, r]$.

$\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ converges uniformly to f on $[-r, r]$

s.t. $f'(x) = \sum_{k=1}^{\infty} \frac{a_k}{(k-1)!} x^{k-1} = \sum_{k=0}^{\infty} \frac{a_{k+1}}{k!} x^k$

$\because r$ is arbitrary

$\therefore f$ is diff. on $\mathbb{R}$ s.t. $f'(x) = \sum_{k=0}^{\infty} \frac{a_{k+1}}{k!} x^k$. //

§5.5 The Space of Continuous Functions

Def. Let $f: A \subset (M,d) \to (N, \|\cdot\|)$ be a function. (→norm (才能來考慮))
We write $C(A,N) = \{$ functions $f: A \to N \mid$
f is conti. $\}$.
Then $C(A,N)$ is a vector space over $\mathbb{R}$.
(Note: 在不混淆之下，僅以 C 代表 $C(A,N)$)

Def. $C_b(A,N) = \{f \in C(A,N) \mid f$ is bdd$\}$ (Then $C_b \subset C$).

Remark. $\overset{\text{well}}{=}$ A is compact, then $C_b = C$

Def. Let $f \in C_b$, we define the norm of f, denoted by $\|f\| = \sup_{x \in A} \|f(x)\|$

Prop. Let $f \in C_b$, then $\|f\| \overset{(<)}{\leq} L \iff \|f(x)\| \overset{(<)}{\leq} L$
$\forall x \in A$.
→ 改成 "<" 時，"≤" 要 C_b ompact 才会对!

Rmk. If N is only a metric space, say (N,ρ). We can still define C and C_b. In this case, we define $d(f,g) = \sup_{x \in A} \rho(f(x), g(x))$ for $f, g \in C_b$.
→ 要 bdd above 才能取 sup!
(Note: $\because f, g$ are bdd on A
$\therefore \exists\, p \in N$ s.t. $f(x), g(x) \in D(p,r)$ $\forall x \in A$, for some $r > 0$.
$\rho(f(x), g(x)) \leq \rho(f(x), p) + \rho(p, g(x)) < 2r$
$\Rightarrow \{\rho(f(x), g(x)) \in \mathbb{R} \mid x \in A\}$ is bounded above
↳ metric space 裡無法比大小，一定要 Double A
收的 set 才有可能找 sup

$\Rightarrow \sup_{x\in A} \rho(f(x), g(x))$ exists.

Thm. ① If (M, d) and (N, ρ) are metric spaces. then so is $C_b(A, N)$

② If $(N, \|\cdot\|)$ is a normed space, so is $C_b(A, N)$.

pf: ① Let $f, g \in C_b(A, N)$.

要以所有x作文考量

$\because \rho(f(x), g(x)) \geq 0, \quad \forall x \in A$

$\therefore \sup_{x\in A} \rho(f(x), g(x)) \geq 0$

$\Rightarrow d(f, g) \geq 0$. ⟵ $\because \geq 0$

Moreover, $d(f, g) = 0 \Leftrightarrow \sup_{x\in A} \rho(f(x), g(x)) = 0$

$\Leftrightarrow \rho(f(x), g(x)) = 0, \quad \forall x \in A$.

$\Leftrightarrow f(x) = g(x), \quad \forall x \in A \Leftrightarrow f = g$.

$\because \rho$ is a metric

Clearly $d(f, g) = d(g, f)$ for $\rho(f(x), g(x)) = \rho(g(x), f(x))$.

Let $f, g, h \in C_b(A, N)$. $d(f, g) = \sup_{x\in A} \rho(f(x), g(x))$

$\because \rho(f(x), g(x)) \leq \rho(f(x), h(x)) + \rho(h(x), g(x)), \quad \forall x \in A$.

$\leq \sup_{x\in A} (\rho(f(x), h(x)) + \sup_{x\in A} \rho(h(x), g(x)), \quad \forall x \in A$.

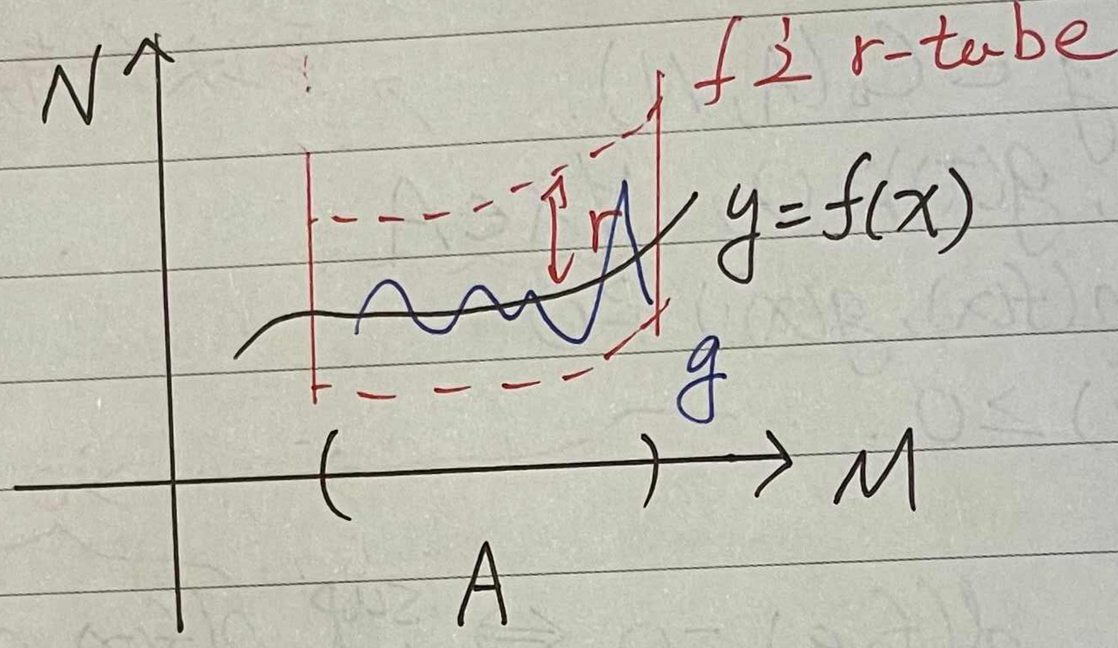
$= d(f, h) + d(h, g), \quad \forall x \in A$.

$\therefore \sup_{x\in A} \rho(f(x), g(x)) \leq d(f, h) + d(h, g)$

$\Rightarrow d(f, g) \leq d(f, h) + d(h, g)$. Therefore d is a metric on $C_b(A, N)$.

② 同理 Q.E.D.

Q: What does $D(f,r)$ look like?
A: If $d(f,g)<r$, then $\rho(f(x),g(x))<r$, $\forall x\in A$



Q: 在 $C_b(A,N)$, 怎樣的集合会是 open? closed? compact? connected?

a normed space

Thm. Let $f_k, f\in C_b(A,N)$. Then $f_k\to f$ in $C_b(A,N)$
$\Leftrightarrow f_k\to f$ uniformly on A.

pf. $f_k\to f$ uniformly on $A \Leftrightarrow \forall \varepsilon>0, \exists K>0$ s.t.
$\forall k>K, \|f_k(x)-f(x)\|<\varepsilon, \forall x\in A$.
$\Leftrightarrow \forall k>K, \|f_k-f\|\le\varepsilon$
$\Leftrightarrow f_k\to f$ in $C_b(A,N)$.

Q.E.D.

Thm. If (N,ρ) (c.f. $(N,\|\cdot\|)$) is complete, then $(C_b(A,N),d)$ (c.f. $(C_b(A,N),\|\cdot\|)$) is complete

pf. Let f_k be a Cauchy sequence in C_b.

Then f_k is a uniform Cauchy seq. in A.
$\because N$ is complete
$\therefore$ By Cauchy Criterion, $f_k \to f$ uniformly on A
$\Rightarrow f_k \to f$ in C_b.

Q.E.D.

Remark. While both $C_b(A, N)$ and $\mathbb{R}^n$ are normed spaces, they are quite different. For instance, $(C_b, \|\cdot\|)$ is not an inner product space. We will show that C_b is not finite dimensional vector space.

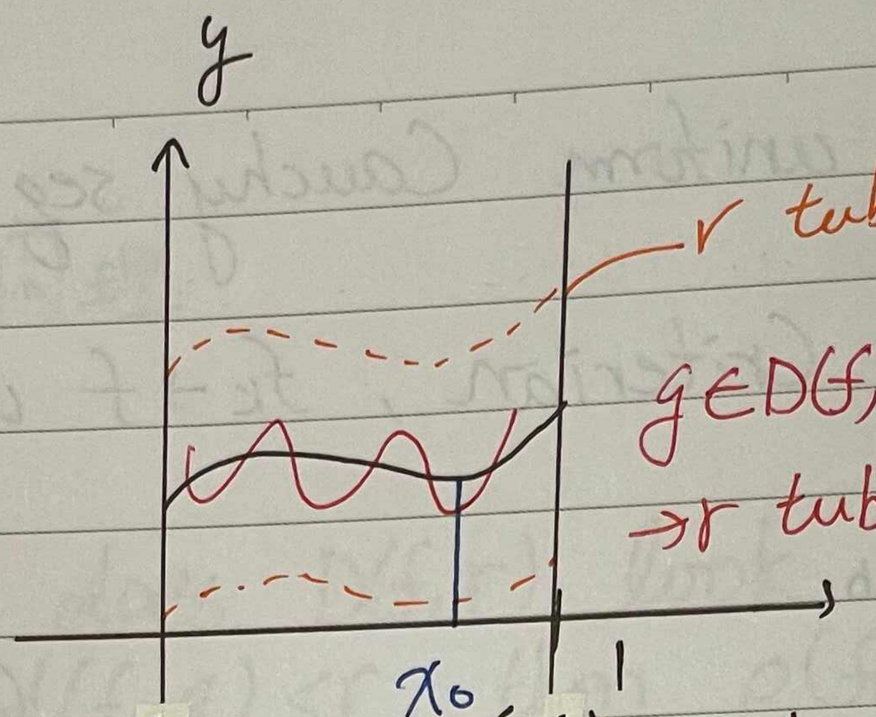
(N: a normed space)

Def. ① A complete normed space is called a Banach space
② A complete inner product space is called a Hilbert space

ex. Let $E = \{f \in C([0,1], \mathbb{R}) \mid f(x) > 0 \ \forall x \in [0,1]\}$
Show that ① E is open in $C([0,1], \mathbb{R})$ ② Find $cl(E)$

pf: ① $\because [0,1]$ is compact
$\therefore C([0,1], \mathbb{R}) = C_b([0,1], \mathbb{R})$
Let $f \in E$. $\because f$ is conti. on $[0,1]$
$\therefore \exists\ x_0 \in [0,1]$ s.t. $f(x_0) = \inf_{x \in [0,1]} f(x)$.

Let $g \in D(f, \frac{f(x_0)}{2})$ Then $\|g - f\| < \frac{f(x_0)}{2}$
$\Rightarrow |g(x) - f(x)| < \frac{f(x_0)}{2}$, $\forall x \in [0,1]$.



r tube of f

$g \in D(f,r) \subset E$

$D(f,r)$中的 g 要如何取能保證 $g>0$?

→ r tube 要如何取能保證 $g>0$?

$$\Rightarrow f(x)-\frac{f(x_0)}{2} < g(x) < f(x)+\frac{f(x_0)}{2}, \quad \forall x \in [0,1]$$

$$\because f(x)-\frac{f(x_0)}{2} \geq f(x_0)-\frac{f(x_0)}{2} = \frac{f(x_0)}{2} > 0, \quad \forall x \in [0,1]$$

$$\therefore g(x) > 0, \quad \forall x \in [0,1].$$

$\Rightarrow g \in E \Rightarrow D(f, f(x_0)/2) \subset E$

$\Rightarrow E$ is open.

→ 用 open 好似不大方便!

② Let $F = \{ f \in C([0,1], \mathbb{R}) \mid f \geq 0 \text{ on } [0,1] \}$.

Then $E \subset F$. Let f_k be a seq. in F that converges to f. Then $f_k \to f$ uniformly on $[0,1]$.

($\because f_k$ is conti. on $[0,1]$, $\forall k \in \mathbb{N}$

$\therefore f$ is cont. on $[0,1]$)

$\because \forall x \in [0,1]$, $f_k(x) \geq 0$, $\forall k \in \mathbb{N}$

$\therefore \forall x \in [0,1]$, $f(x) \geq 0$.

Hence $f \in F \Rightarrow F$ is closed.

Let $f \in F$. Let $f_k = f + \frac{1}{k}$, $\forall k \in \mathbb{N}$. Then $f_k \in \bar{E}$, $\forall k \in \mathbb{N}$, and $f_k \to f$ in $C([0,1], \mathbb{R})$

$\Rightarrow f \in cl(E) \Rightarrow F \subset cl(E)$

Therefore $F = cl(E)$

※

ex. Let $f_n \in C_b(M, \mathbb{R})$, $\forall n \in \mathbb{N}$ s.t. $\|f_{n+1} - f_n\| \leq r_n$ where $\sum r_n$ converges. Prove that f_n converges in M.

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§ 5.6 Arzela-Ascoli Thm.

In function space, there are three basic Thm. the Arzela Ascoli thm, contraction mapping Thm and the stone-Weiestrass thm.

Q: 怎樣的集合在 $C_b(A,N)$ 会是 compact.

Def. Let $B \subset C(A,N)$. We say that B is an equicontinuous set of functions, if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, y \in A$ with $d(x,y) < \delta$, $\rho(f(x), f(y)) < \varepsilon$, $\forall f \in B$.

→所有 f 都有一致的 uniformly conti. 之速度

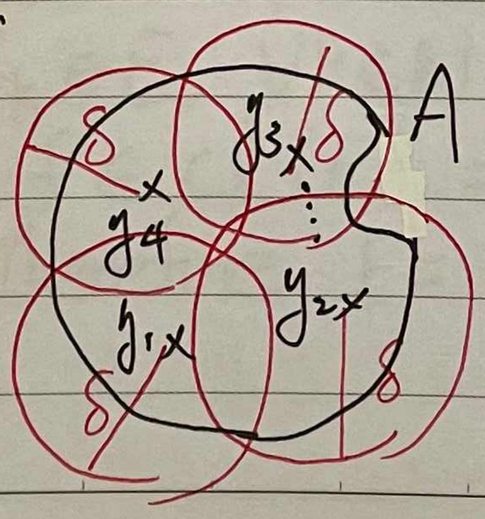
Def. Let $B \subset C(A,N)$

① Define $B_x \overset{\subset N}{=} \{f(x) \mid f \in B\}$ where $x \in A$ ← 逐点被認知的

② We say that B_x is pointwise compact, if B_x is compact in N. $\forall x \in A$.

Prop. Let $A \subset (M,d)$ be compact. Then for any $\delta > 0$, there is a finite set $S_\delta = \{y_1, \dots, y_n\} \subset A$ s.t. for each $x \in A$, $\exists\, y_i \in S_\delta$ s.t. $x \in D(y_i, \delta)$ (i.e. each $x \in A$ is within δ-distance of some $y_i \in S_\delta$

take δ



Double A

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Prop. A. Let $A \subset M$ be compact and N complete. Let $B \subset C(A,N)$. If B is equicontinuous and pointwise compact, then every seq. in B has a convergence subsequence. (in $C(A,N)$)

pf: Notice that $\because A$ is compact $\therefore C(A,N) = C_b(A,N)$. Let $S = \bigcup_{n=1}^{\infty} S_{\frac{1}{n}}$ where $S_{\frac{1}{n}}$ is defined in the previous prop.

$\because S_{\frac{1}{n}}$ is a finite set, $\forall n \in \mathbb{N}$

$\therefore S$ is countable.

Hence we can let $S = \{x_1, x_2, \dots\}$. Evaluate all functions of B at the point x_1. Then $f_k(x_1)$ is a seq. in Bx_1

$\because B$ is pointwise compact

$\therefore Bx_1$ is compact.

$\Rightarrow f_k(x_1)$ has a convergent subsequence $f_{11}(x_1), f_{12}(x_1), \dots$

Now evaluate all the functions f_{1k} at the pt x_2. Similarly, the seq. $f_{1k}(x_2)$ in Bx_2 has a convergent subseq. $f_{21}(x_2), f_{22}(x_2), \dots$ (此時 f_{2k} 在 x_1, x_2 皆收斂). Continuing this process, the seq. $f_{k-1n}(x_{k-1})$ has a convergent subseq. $f_{k1}(x_k), f_{k2}(x_k) \dots \forall k \in \mathbb{N}$.

By diagonal process, we have

f_{11} f_{12} f_{13} ...
f_{21} f_{22} f_{23}
f_{31} f_{32} f_{33} ⋱
⋮
f_{k1} f_{k2} f_{k3} ⋱
⋮

of f_k that converges on

不知極限，改證其為Cauchy

We have subsequence $f_{11}, f_{22}, f_{33}, \dots$ converges on the set S. For simplicity, let $f_{kk} = g_k$. Let $x \in A$. Let $\varepsilon > 0$,

$$\rho(g_n(x), g_m(x)) \leq \rho(g_n(x), g_n(y)) + \rho(g_n(y), g_m(y)) + \rho(g_m(y), g_m(x))$$

要利用S中y來作估計

$\rho(g_n(x), g_n(y))$：∵ B is equicontinuous ∴不管怎樣的g_n，只要x距y夠靠近，此項夠小。

$\rho(g_n(y), g_m(y))$：∵$g_k(y)$收斂 ∴只要m與n夠大，此項夠小。

$\rho(g_m(y), g_m(x))$：同第一項。

(⇒故由第一項的x決定y的選取，再由所得$g_k(y)$去決定m, n的選取)

∵ B is equicontinuous

∴ For $\varepsilon/3$, $\exists \delta > 0$ s.t. $\forall x, y \in A$ with $\rho(f(x), f(y)) < \varepsilon/3$, $\forall f \in B$.

$= \{y_1, \dots, y_e\}$

Let $n_0 \in \mathbb{N}$ s.t. $\frac{1}{n_0} < \delta$. Then $\exists y_i \in S_{\frac{1}{n}}$ s.t. $x \in D(y_i, \frac{1}{n}) \Rightarrow d(x, y_i) < \frac{1}{n_0} < \delta$.

$\Rightarrow \rho(g_n(x), g_n(y_i)) < \varepsilon/3$ and $\rho_m(g_m(x), g_m(y_i)) < \varepsilon/3$

∵ $g_k(y)$ converges

∴ For $\varepsilon/3$, $\exists K_i \in \mathbb{N}$ s.t. $\forall m, n > K_i$, $\rho(g_n(y_i), g_m(y_i)) < \varepsilon/3$.

$\Rightarrow \forall\, n, m \geq \max\{k_1, \dots, k_\ell\}$, $\rho(g_n(x), g_m(x)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$, $\forall x \in A$

$\Rightarrow g_k$ is a uniform Cauchy sequence.

$\because \mathcal{N}$ is complete

$\therefore g_k$ uniformly converges on A.

$\Rightarrow g_k$ converges in $C(A, \mathcal{N})$

Q.E.D.

∵在$\mathbb{R}^n$中bdd set一定可包含在 compact set中

Cor. Let $A \subset M$ be compact. If $B \subset C(A, \mathbb{R}^n)$ is equiconti. and pointwise bounded, then every sequence in B has a convergent subseq.

pf: ∵ Every bdd set in $\mathbb{R}^n$ lies in a compact set

Q.E.D.

Note: Prop. A之敘述已很接近 "sequentially compact", 只差在收斂是否在 $C_b(A, \mathcal{N})$ 之中. "sequentially compact" 在 metric space 中與 compact 等价.

Thm. (Arzela-Ascoli Thm) → 加上 "B is closed".

Let $A \subset (M, d)$ be compact and $\mathcal{N}$ is complete. Let $B \subset C(A, \mathcal{N})$. Then B is compact $\Leftrightarrow$ B is closed, equicontinuous and pointwise compact.

② ①

pf: ($\Leftarrow$) ∵ By Prop A, every seq. in B has a convergence subsequence.

∵ B is closed

$\therefore$ The subseq. converges in B
$\Rightarrow$ B is sequentially compact
$\Rightarrow$ B is compact (by B-W Thm)

($\Leftarrow$) ① $\because$ B is compact
$\therefore$ B is closed
Let $x \in A$. Let $f_k(x)$ be a seq. in B_x. Then f_k is a seq. in B.
$\because$ B is compact.
$\therefore$ $\exists$ a convergent subseq. f_{k_n} of f_k converges to $f \in B$.
$\Rightarrow$ $f_{k_n} \to f$ uniformly on A
$\Rightarrow$ $f_{k_n}(x) \to f(x)$

$\because$ $f(x) \in B_x$
$\therefore$ B_x is compact.
$\Rightarrow$ B is pointwise compact.

② Suppose B is not equicontinuous. Then $\exists \varepsilon_0 > 0$. $\forall n \in \mathbb{N}$. $\exists$ $x_n, y_n \in A$ and $f_n \in B$ s.t. $d(x_n, y_n) < 1/n$ and $\rho(f_n(x_n), f_n(y_n)) \geq \varepsilon_0$.
(compact, compact)
$\therefore$ $\exists$ a subseq. (依舊記為) s.t. $x_n \to x$, $y_n \to y$, $f_n \to f$ (f conti.)

$$\rho(f_n(x_n), f_n(y_n)) \leq \rho(f_n(x_n), f(x_n)) + \rho(f(x_n), f(y_n)) + \rho(f_n(y_n), f(y_n))$$

亂寫的但方法 $\because$ $f_n \to f$ and f is conti. we can find some n s.t. all three terms $< \frac{\varepsilon}{3}$

差不多就是由 n 去控制之.

$\Rightarrow \rho(f_n(x_n), f_n(y_n)) < \varepsilon$ $\not\rightarrow$

$\Rightarrow$ B is equiconti.

Q.E.D.

Cor. Let $A \subset (M,d)$ be compact, and let $B \subset C(A, \mathbb{R}^n)$. Then B is compact $\Leftrightarrow$ B is closed, equicont. and pointwise bdd.

Q: 什麼時候要用 diagonal process 呢?

ex. Let $f_n : [0,1] \to \mathbb{R}$ be conti. and $|f_n(x)| \le 100$ on $[0,1]$, $\forall n \in \mathbb{N}$. and f_n' are uniformly bdd. Prove that f_n has a convergent subsequence.

pf: $\because |f_n(x)| \le 100$, $\forall x \in [0,1]$. $\forall n \in \mathbb{R}$

$\therefore \{f_n\}$ is pointwise bdd.

$\because f_n'$ is uniformly bounded on $(0,1)$

$\therefore \exists M \ge 0$ s.t. $|f_n'(x)| \le M$, $\forall x \in (0,1)$ and $\forall n \in \mathbb{N}$.

By M.V.T. $|f_n(x) - f_n(y)| = |f_n'(c_{x,y}^{(n)})||x-y|$

$\le M|x-y|$. $\forall x, y \in [0,1]$, $\forall n \in \mathbb{N}$.

uniformly conti. equi-conti.

So $\{f_n\}$ is equi-conti.

$\because [0,1]$ is compact

$\therefore$ By prop. A. f_n has a convergent subseq. #

ex. Is the result of previous example valid if we omit the condition that $|f_n(x)| \le 100$.

pf: No! For example, $f_n(x) = n$, $f_n' \equiv 0$, $\forall n \in \mathbb{N}$

Note: $BCC(A,N)$ 在 C 採用不同 norm (空間感不同) 時，可能会由 open 变不 open，compact 变不 compact 等。故 Arzela-Ascoli Thm 其实很狭隘，只限制在 $\sup|f(x)|$ 這个 norm (L^∞) 上.
同理，Heine-Borel 也只考虑 $(\mathbb{R}^n, \|\cdot\|)$，换其它空間又不对了。任意度量下改为 complete + totally bounded.

ex. Let $I: C([0,1], \mathbb{R}) \to \mathbb{R}$ be defined by $P(f) = \int_0^1 f(x)\,dx$. Prove that I is cont. on $C([0,1], \mathbb{R})$.

pf. $\because [0,1]$ is compact
$\therefore C([0,1], \mathbb{R}) = C_b([0,1], \mathbb{R})$

思考 | 如何證連续?
1. ε-δ
2. seq. $\to I(f_k) \to I(f)$?
3. open 拉回为 open
4. closed " closed

Let $f_k \to f$ in $C([0,1], \mathbb{R})$. Then $f_k \to f$ uniformly on $[0,1]$. $\Rightarrow \int_0^1 f_k(x)\,dx \to \int_0^1 f(x)\,dx$

$\Rightarrow I(f_k) \to I(f)$. Therefore I is conti. #

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§5.7 The contraction mapping principle and

Thm. The contraction mapping principle.
Let (M,d) be a complete metric space and $\Phi: M \to M$ be a mapping. Assume $\exists\, k \in \mathbb{R}$ s.t. $0 \le k < 1$ s.t. $d(\Phi(x), \Phi(y)) \le k \cdot d(x,y)$, $\forall\, x, y \in M$. Then $\exists!$ fixed point for Φ i.e. $\exists$ a unique point $x_* \in M$ s.t. $\Phi(x_*) = x_*$.
In fact, if x_0 is any point in M, define $x_1 = \Phi(x_0)$, $x_2 = \Phi(x_1)$, ..., $x_{n+1} = \Phi(x_0)$, ⋯, then $\lim_{n\to\infty} x_n = x_*$

Remark. 我們可以假設有一个力(force)作用於M這个空間，使得M中的点会在M中移动(假設此力使终作用在那裡). 今令Φ這个Mapping描述M中的点如何在M中移动. 由定理的假設，$0 \le k < 1$，可知Φ会收缩点之間的距离. 因此M中的任意 region D 会被Φ"移动"到一个較小的 region Φ(D)，而後Φ(D)又被移动到一个更小的 region Φ(Φ(D))。定理說D最後会缩成一个点。該点為Φ的唯一定点。

pf: (1) ① Existence of a fixed point of Φ: (show the lim exists and ② is a fixed pt.)
Let $x_0 \in M$ and $x_1, x_2, x_3, \dots, x_n, \dots$ be as in the Thm. Let $\varepsilon > 0$, $d(x_{n+p}, x_n) \le d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + d(x_{n+1}, x_n)$.
By triangle ineq., $d(x_{m+1}, x_m) = d(\Phi(x_m), \Phi(x_{m-1})) \le k \cdot d(x_m, x_{m-1}) \le \cdots \le k^m d(x_1, x_0)$
(定数)

Double A

its applications

$< \varepsilon$ 解 n.

$\therefore d(x_{n+p}, x_n) \leq (k^{n+p-1} + k^{n+p-2} + \cdots + k^n)\, d(x_1, x_0)$

$\because |k|<1, \therefore \sum_{n=0}^{\infty} k^n$ converges $\Rightarrow$ For $\frac{\varepsilon}{d(x_1,x_0)}$,

$\exists K \in \mathbb{N}$ s.t. $\forall n > K$ and $p > 0$,

$|k^{n+p-1} + \cdots + k^n| < \frac{\varepsilon}{d(x_1, x_0)}$

$\because k > 0 \Rightarrow \forall n > K$ and $p > 0$, $d(x_{n+p}, x_n) < \frac{\varepsilon}{d(x_1, x_0)}$.

$d(x_1, x_0) = \varepsilon$.

Therefore x_n is a Cauchy seq. in M.

$\because M$ is complete

$\therefore \exists\, x_* \in M$ s.t. $x_n \to x_*$

② $\because \underline{\Phi}$ is Lipschitz

$\therefore \underline{\Phi}$ is cont. on M

$\Rightarrow \underline{\Phi}(x_n) \to \underline{\Phi}(x_*)$

$\Rightarrow x_{n+1} \to \underline{\Phi}(x_*)$

$\Rightarrow \underline{\Phi}(x_*) = x_*$

$\Rightarrow x_*$ is a fixed point of $\underline{\Phi}$.

(2). Uniqueness of fixed point of $\underline{\Phi}$:

Let $\underline{\Phi}(x) = x$ and $\underline{\Phi}(y) = y$. then

$d(\underline{\Phi}(x), \underline{\Phi}(y)) \leq k\, d(x, y)$. $\Rightarrow d(x, y) \leq k\, d(x, y)$

$\Rightarrow (1-k)\, d(x, y) \leq 0$

$\because 1 - k > 0$

$\therefore d(x, y) \leq 0 \Rightarrow d(x, y) = 0 \Rightarrow x = y$

Q.E.D.

Double A

Prop. The norm $\|\cdot\|$ in a normed space $(N, \|\cdot\|)$ is Lipschitz

假設在 $\mathbb{R}$ 上有一个速度場，可以將 $\mathbb{R}$ 想像成一條河流。因此 $\mathbb{R}$ 上每一点都有它的水流速度。事实上也不一定要討論整个 $\mathbb{R}$，也可以討論 $\mathbb{R}$ 中的 open set，or $\mathbb{R}^n$ or $\mathbb{R}^n$ 中的 open set. 假設這個速度場会隨時間的改变。用 $f: \mathbb{R}^2 \to \mathbb{R}$ 來描述 $\mathbb{R}$ 上隨時間改变而改变的速度場，即 $f(t,x)$ 表示在 t 時間 $\mathbb{R}$ 上 x 之位置那一点的速度。今在 t_0 時間，有一質点（或一根羽毛 or 一个乒乓球）在 $\mathbb{R}$ 中的 x_0 位置上，受到速度場 f 的影响。此質点会在 $\mathbb{R}$ 中跑动。用 $x(t)$ 描述此質点在 t 時間時於 $\mathbb{R}$ 中的位置，則 $x(t_0) = x$ and

$$\frac{dx}{dt} = f(t,x).$$

Q: 在給定 $f(t,x)$ 之下，若給定一開始的時間与位置 $x(t_0)=x_0$，如何找到 $x(t)$ 呢？即如何解微分方程

$$\begin{cases} \frac{dx}{dt} = f(x,t) \\ x(t_0) = x_0 \end{cases}$$

若 f is cont. $\Longleftarrow$ on a nbh. of (t_0, x_0), by 2nd fundamental thm of calculus.

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds$$

積分方程

令 $x = \varphi(t)$, then $\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, \underline{\varphi(s)})\, ds$

↑ input a cont. functi

output a function

為

將此裝置設Ψ，放入材料為φ，我們要找的積分方程的解就是找Ψ的 fixed point了。即$\Psi(\varphi)=\varphi$。想要用 contraction mapping principle 來幫找Ψ的 fixed pt，此時問題就變成要去建構 input data φ的所在空間 M使得 contraction mapping principle works!

$\because \frac{d\varphi}{dt} = \underbrace{f(t,\varphi(t))}_{\text{cont.}}$ $\therefore \varphi$ is a C^1 function

$\because \varphi(t)$ 要代入 f 的第2個變數中

再看 $\varphi(t)$ 中 t 變數是 f 的第一個變數，故 φ 的定義域必須在 f 的第一個變數的定義域中。且 $\varphi(t)$ 必須落在 f 的第二個變數的定義域。

→ 劃對 f 之定義域，但有2變數，圓形

今令 f is cont. on $D((t_0,x_0),r)$ for some $r>0$

並不好用 → 改為長方形

故於 D 中選一夠小之閉長方形。Let $[t_0-\delta, t_0+\delta]\times[x_0-a, x_0+a] \subset D((t_0,x_0),r)$. Then φ is defined on $[t_0-\delta, t_0+\delta]$. $|\varphi(t)-x_0|\le a$.

↳ δ, a 雖不唯一，但是是給定的值.

Let $M=\{$ functions $\varphi: [t_0-\delta, t_0+\delta]\to\mathbb{R} \mid \varphi$ is C^1, $\varphi(t_0)=x_0$, $|\varphi(t_0)-x_0|\le a\}$ ∵ Ψ必須將 M 送到 M

∴ 必須檢驗 $\Psi(\varphi)\in M$? $\forall \varphi\in M$.

$\Psi(\varphi)(t_0) = \varphi(t_0) = x_0$ → t 可以 $< t_0$

$$|\Psi(\varphi)(t)-x_0| = \left|\int_{t_0}^{t} f(s,\varphi(s))\,ds\right|$$

$$\le \left|\int_{t_0}^{t} \underbrace{|f(s,\varphi(s))|}_{\substack{f \text{ is cont. on } [t_0-\delta, t_0+\delta]\times[x_0-a] \\ \text{compact} \downarrow \\ \times[x_0+a]}}\,ds\right|$$

Double A

∵ f is conti. on the compact set $[t_0-\delta, t_0+\delta]\times[x_0-a, x_0+a]$

∴ $\exists L>0$ s.t. $|f(t,x)|\le L$, $\forall (t,x)\in[t_0-\delta, t_0+\delta]\times[x_0-a, x_0+a]$

$\Rightarrow$ 原式 $\le \left|\int_{t_0}^{t} L\,ds\right| \le L|t-t_0| \le L\delta$

Choose $L\delta \doteq a$.

∵ $(\Phi(\varphi))'(t) = f(t,\varphi(t))$ is cont.

∴ $\Phi(\varphi)$ is C^1.

$\Rightarrow \Phi(\varphi)\in M$, $\forall \varphi\in M$. Therefore $\Phi: M\to M$.

Notice that $M\subset C_b([t_0-\delta, t_0+\delta], \mathbb{R})$ and $C_b([t_0-\delta, t_0+\delta])$ is complete. To show M is closed, let $\varphi_n\in M$ that converges to φ.

∵ $\varphi_n\to\varphi$ in $C_b([t_0-\delta, t_0+\delta], \mathbb{R})$

∴ $\varphi_n\to\varphi$ uniformly on $[t_0-\delta, t_0+\delta]$.

∵ φ_n is cont., $\forall n\in\mathbb{N}$ and $\varphi_n(t_0)=x_0$

∴ φ is cont. and $\varphi(t_0)=x_0$.

∵ $\varphi_n(t)-x_0\to\varphi(t)-x_0$ and $|t|$ is cont. on $\mathbb{R}$

∴ $|\varphi_n(t)-x_0|\to|\varphi(t)-x_0|$.

∵ $|\varphi_n(t)-x_0|\le L\delta$, $\forall n\in\mathbb{N}$.

∴ $|\varphi(t)-x_0|\le L\delta$ → 無法證 C^1！original contraction principle work, 需改 M 之定義（放寬）

$\Rightarrow$ 修正：Let $M=\{$ cont. functions $\varphi: [t_0-\delta, t_0+\delta]\to\mathbb{R}: \varphi(t_0)=x_0$ and $|\varphi(t)-x_0|\le L\delta\}$

Then $\varphi\in M \Rightarrow M$ is complete

(雖然 M 變大，但也變簡單了)

To check $d(\Psi(\varphi_1), \Psi(\varphi_2))$ 與 $d(\varphi_1, \varphi_2)$ 之關係、

$$\| \Psi(\varphi_1) - \Psi(\varphi_2) \| = \sup_{[t_0-\delta, t_0+\delta]} |\Psi(\varphi_1)(t) - \Psi(\varphi_2)(t)|$$

$$\Rightarrow |\Psi(\varphi_1)(t) - \Psi(\varphi_2)(t)| \leq \left| \int_{t_0}^{t} |f(s, \varphi(s)) - f(s, \varphi_2(s))| \right| ds$$

↓ 與 $|\varphi_1(s) - \varphi_2(s)|$ 比較

→ 無法改變 f 之條件，只好 (想爆..) 的假設 $|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|$, $\forall (t, x_1), (t, x_2)$ in a n.b.d. of (t_0, x_0) (uniformly Lipchitz)

$$\leq K \left| \int_{t_0}^{t} |\varphi_1(s) - \varphi_2(s)| \right| ds$$

$$\leq K \left| \int_{t_0}^{t} \|\varphi_1 - \varphi_2\| \, ds \right|$$

$$\leq K \|\varphi_1 - \varphi_2\| |t - t_0|$$

$$\leq K\delta \, d(\varphi_1, \varphi_2)$$

$\Rightarrow d(\Phi(\varphi_1), \Phi(\varphi_2)) \leq K\delta \, d(\varphi_1, \varphi_2)$ Choose $\delta > 0$ s.t. $K\delta < 1$. Then Φ is a contraction.

By contraction mapping principle, $\exists! \tilde{\varphi} \in M$ s.t. $\Phi(\tilde{\varphi}) = \tilde{\varphi} \Rightarrow \tilde{\varphi}$ is C^1.

(Note: $\varphi_0 \equiv x_0$, $\varphi_1 = \Phi(\varphi_0)$, ..., $\varphi_n = \Phi(\varphi_{n-1})$, ..., $\varphi_n \to \tilde{\varphi}$)

Thm. Let f be a cont. function on a n.b.d. of $(t_0, x_0) \in \mathbb{R}^2$. Assume that $|f(t,x_1)-f(t,x_2)| \leq K|x_1-x_2|$, for all (t,x_1) and (t,x_2) in this n.b.d. of (t_0,x_0). Then $\exists \delta>0$ s.t.
$\frac{dx}{dt}=f(t,x)$ and $x(t_0)=x_0$ has a unique C^1 solution $x=\varphi(t)$ with $\varphi(t_0)=x_0$ on the interval $(t_0-\delta, t_0+\delta)$.

雖然[ ]是对的，但習慣取開

Def. The integral equation of the form

$$f(x)=\lambda\int_a^b K(x,y)f(y)\,dy+\varphi(x)$$

are called Fredholm equations, where λ is a given constant and K and φ are given cont. functions, for all $x\in[a,b]$ and $y\in[a,b]$.

Q: Fredholm equations 有解否？如何解？
A: Let $M=C([a,b],\mathbb{R})$ and

$$\Phi(f)(x)=\lambda\int_a^b K(x,y)f(y)\,dy+\varphi(x),\quad \forall f\in M \quad -(*)$$

Let $M=C([a,b],\mathbb{R})$. $\because [a,b]$ is compact
$\therefore M=C_b([a,b],\mathbb{R})$. $\because \mathbb{R}$ is complete $\therefore M$ is complete. Define $\Phi(f)(x)$ as $(*)$, $\forall f\in M$.
→ (1) $\Phi(f)$ conti.? 是的話 Φ 就是 $M\to M$
Let $x_0\in[a,b]$. Let $\varepsilon>0$,

Double A

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y也会跟着变　　含各项小於 $\frac{\varepsilon}{2}$

→用 uni. cont.，夠靠近就夠小、

$$|\Phi(f)(x)-\Phi(f)(x_0)| \le \lambda\int_a^b |K(x,y)-K(x_0,y)||f(y)|dy + |\varphi(x)-\varphi(x_0)|$$

$|f(y)| \le \|f\|$

$\because$ K is cont. on a compact set $[a,b]\times[a,b]$

$\therefore$ K is uniformly cont. on $[a,b]\times[a,b]$

$\Rightarrow$ For $\frac{\varepsilon}{2|\lambda|\cdot\|f\|(b-a)}$, $\exists\,\delta_1>0$ s.t. $\forall (x_1,y_1),(x_2,y_2)\in[a,b]\times[a,b]$ with $\|(x_1,y_1)-(x_2,y_2)\|<\delta$,

$$|K(x_1,y_1)-K(x_2,y_2)| < \frac{\varepsilon}{2|\lambda|\|f\|(b-a)}$$

$\because$ φ is cont. at x_0

$\therefore$ For $\frac{\varepsilon}{2}$, $\exists\,\delta_2>0$ s.t. $\forall |x-x_0|<\delta_2$,

$|\varphi(x)-\varphi(x_0)| < \varepsilon/2$.

So $\forall |x-x_0|<\min\{\delta_1,\delta_2\}$, $|\Phi(f)(x)-\Phi(f)(x_0)|$ $< \varepsilon/2+\varepsilon/2=\varepsilon$. $\Rightarrow \Phi(f)$ is cont. at x_0 (x_0 为任意点) $\Rightarrow \Phi(f)$ is cont. on $[a,b]$ $\Rightarrow \Phi(f)\in M$, i.e. $\Phi: M\to M$

(2) Φ 为 contraction?

$$|\Phi(f_1)(x)-\Phi(f_2)(x)| \le |\lambda|\int_a^b \underbrace{|K(x,y)|}_{\text{bdd by } M}\underbrace{|f_1(y)-f_2(y)|}_{\le\|f_1-f_2\|}dx$$

$$\le |\lambda||M|\,\|f_1-f_2\|\int_a^b 1\,dy$$

$$= |\lambda||M|(b-a)\|f_1-f_2\|$$

$$\Rightarrow d(\Phi(f_1),\Phi(f_2)) \le |\lambda|M(b-a)\,d(f_1,f_2)$$

Double A

If $|\lambda| M(b-a) < 1$, then Φ is a contraction.
By contraction mapping principle, Φ has a unique solution.

Let $f_0 \equiv 0$ 不断代, $f_1 = \Phi(f_0)$, $f_2 = \Phi(f_1)$, ...
可得 fixed point

Thm. If $|\lambda| M(b-a) < 1$, then Fredholm eq. has a unique continuous solution.

*現把 Φ 改成 x，則 證明修改後之條件為 (的 b)
$d(\Phi(f_1), \Phi(f_2)) \le |\lambda| M \|f_1 - f_2\| \sup_{[a,b]} (x-a)$.

Def. The integral equations of the form (K, f, ϕ, ... 之條件與 Fredholm 一樣.)
$$f(x) = \lambda \int_a^x K(x,y) f(y)\, dy + \varphi(x)$$
are called Voltera equation

$$|\Phi^2(f_1)(x) - \Phi^2(f_2)(x)| \le |\lambda| \int_a^x M\left(|\lambda| M \|f_1 - f_2\| (y-a)\right) dy$$
(Φ^2: 代2次)
$$\le |\lambda|^2 M^2 \|f_1 - f_2\| \int_a^x (y-a)\, dy$$
$$= |\lambda|^2 M^2 \|f_1 - f_2\| \cdot \frac{1}{2}(x-a)^2$$

$$\Rightarrow |\Phi^n(f_1)(x) - \Phi^n(f_2)(x)\| \le |\lambda|^n M^n \|f_1 - f_2\| \frac{1}{n!}(x-a)^n$$

$$\Rightarrow d\left(\Phi^n(f_1), \Phi^n(f_2)\right) \le |\lambda|^n M^n \|f_1 - f_2\| \sup_{[a,b]} \frac{(x-a)^n}{n!}$$
$$= \frac{|\lambda|^n M^n (b-a)^n}{n!} d(f_1, f_2)$$
↳ 如何說明 → 0? 可用 series 證

$\therefore \sum_{n=0}^{\infty} \frac{\lambda^n \mu^n (b-a)^n}{n!}$ converges by ratio test

$\therefore \lim_{n\to\infty} \frac{|\lambda^n \mu^n (b-a)^n}{n!} \to 0$

$\Rightarrow \exists\, N \in \mathbb{N}$ s.t. $1 > \frac{|\lambda|^n \mu^N (b-a)^N}{N!}$

$\Rightarrow \underline{\Phi}^N$ is a contraction

By contraction mapping principle, $\underline{\Phi}^N$ has a unique fixed point.

Q: Does $\underline{\Phi}$ has a unique fixed point?
A: Let $\underline{\Phi}^N(x_*) = x_*$. Then $\underline{\Phi}^{N+1}(x_*) = \underline{\Phi}(x_*)$
$\Rightarrow \underline{\Phi}(x_*)$ is a fixed point of $\underline{\Phi}^N$
$\because \underline{\Phi}^N$ has a unique fixed point
$\therefore \underline{\Phi}(x_*) = x_*$
$\Rightarrow x_*$ is a fixed point of $\underline{\Phi}$
$\because$ Any fixed point of $\underline{\Phi}$ is a fixed of $\underline{\Phi}^N$
$\therefore$ By uniqueness of fixed point of $\underline{\Phi}^N$
$\Rightarrow \underline{\Phi}$ has a unique fixed point.

Thm: Let $\mathcal{M}$ be a complete metric space, $\underline{\Phi}: \mathcal{M} \to \mathcal{M}$ be a mapping. If for some integer N, $\underline{\Phi}^N = \underline{\Phi} \circ \cdots \circ \underline{\Phi}$ is a contraction, then $\underline{\Phi}$ has a unique fixed point.

Thm. The Volterra equation has a unique continuous solution.

→对任意之 λ, a, φ 原以为把 b 改成 x 会变很难，但反而得到必定有连续之解的结論

→平移没有定关

ex. Let $M=\mathbb{R}$, and $\Phi(x)=x+1$. $|\Phi(x)-\Phi(y)|$ $=|x-y|$, but Φ has no fixed point.
($\because k$ 要 <1)

ex. Show that the method of successive approximations applied to the differential equation $f'(x)=f(x)$ with $f(0)=1$ leads to formula for e^x

pf. The differential equation of $f'(x)=f(x)$ with $f(0)=1$ is equivalent to the integral equation.

$$f(x)=1+\int_0^x f(y)\,dy$$

Volterra equation with $\lambda=1$, $K(x,y)=1$ and $\varphi(x)=1$. Define $\Phi(f)=1+\int_a^x f(y)\,dy$.

$\Phi(0)(x)=1$. $\Phi^2(0)(x)=1+x$. $\Phi^3(0)(x)=1+\int_0^x(1+y)\,dy$ $=1+x+x^2/2$. ..... $\Phi^n(0)=1+x+x^2/2+\cdots+x^{n-1}/(n-1)!$

$\Rightarrow \sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$

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§5.8 The Stone-Weierstrass Thm.

稠密

Def. Let A be a subspace of a metric space M. A is said to be dense in M, if $cl(A) = M$.

Q: The set $\mathcal{P}$ of all polynomials in $C([a,b], \mathbb{R})$ 是否 open? 是否 closed?

Thm. (Weierstrass Thm.)
Let $f: [0,1] \to \mathbb{R}$ be a cont. function. Then for any $\varepsilon > 0$, $\exists$ a polynomial P s.t. $\|f - P\| < \varepsilon$.
In fact, the seq. of Bernstein polynomials

$$P_n(x) = \sum_{k=0}^{n} C_k^n f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$ converges

uniformly to f on $[0,1]$.

Note: $P_n(0) = f(0)$ and $P_n(1) = f(1)$, $\forall n \in \mathbb{N}$.

Remark. ① 此定理成立的理由可用机率論解釋. 請自閱

② f is analytic [Weierstrass Thm. / Taylor Thm.

→ Taylar 收斂速度快很多

pf: $(x+y)^n = \sum_{k=0}^{n} C_k^n x^k y^{n-k}$ —①

對 x 微 → $nx(x+y)^{n-1} = \sum_{k=0}^{n} k C_k^n x^k y^{n-k} = \sum_{k=0}^{n} k C_k^n x^k y^{n-k}$ —②

再 x 乘 x

Double A

$$n(n-1)x^2(x+y)^{n-2}=\sum_{k=2}^{n}\overset{C_k^n}{\vee}k(k-1)x^k y^{n-k}=\sum_{k=0}^{n}\overset{C_k^n}{\vee}k(k-1)x^k y^{n-k} \quad -③$$

Let $r^k=C_k^n x^k(1-x)^{n-k}$, Then $r(x)\geq 0$ on $[0,1]$.

Replacing $y=1-x$ in ①, ②, ③,

r_k是某种概率 $\leftarrow \sum_{k=0}^{n} r_k(x)=1$, $\sum_{k=0}^{n} k r_k(x)=nx$, $\sum_{k=0}^{n} \overbrace{k(k-1)}^{k^2-k} r_k(x)$

$=n(n-1)x^2 \Rightarrow \sum_{k=0}^{n} k^2 r_k(x)=n(n-1)x^2+nx$

Moreover, $\sum_{k=0}^{n}(k-nx)^2 r_k(x)=[n(n-1)x^2+nx]-2nx\cdot nx+n^2x^2$

$=nx-nx^2$

$=nx(1-x)$.

Let $\varepsilon>0$, $\because f$ is cont. on $[0,1]$, $\therefore f$ is bdd and uniformly cont. on $[0,1]$.

$\Rightarrow \exists M>0$ s.t. $|f(x)|\leq M$ on $[0,1]$ and $\exists \delta>0$ s.t. $\forall x,y\in[0,1]$ with $|x-y|<\delta$, $|f(x)-f(y)|<\varepsilon$.

$$|f(x)-P_n(x)|=\left|f(x)-\sum_{k=0}^{n} f\left(\frac{k}{n}\right) r_k(x)\right|, \quad f\text{乘}\Sigma r_k(x)\ (\Sigma r_k=1)$$

$$\leq \sum_{k=0}^{n}\left|f(x)-f\left(\frac{k}{n}\right)\right| r_k(x)$$

把n取很大 这样任意x都有夹逼近它

$$=\sum_{\substack{k \text{ with} \\ |x-\frac{k}{n}|<\delta}} \text{“} \quad + \sum_{\substack{k \text{ with} \\ |x-\frac{k}{n}|\geq\delta}} \text{“}$$

$$\leq \frac{\varepsilon}{2}\sum_{\substack{k \text{ with} \\ |x-\frac{k}{n}|<\delta}} r_k(x) + 2M \sum_{\substack{k \text{ with} \\ |x-\frac{k}{n}|\geq\delta}} r_k$$

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$* \ |x-\frac{k}{n}| \geq \delta \Rightarrow \frac{|k-nx|}{n\delta} \geq 1$

$$\leq \frac{\varepsilon}{2} \sum_{\substack{k \text{ with} \\ |x-\frac{k}{n}|<\delta}} r_k(x) + 2M \sum_{\substack{k \text{ with} \\ |x-\frac{k}{n}|\geq\delta}} \frac{(k-nx)^2}{n^2\delta^2} r_k(x)$$

$$\leq \frac{\varepsilon}{2} \sum_{k=0}^{n} r_k(x) + \frac{2M}{n^2\delta^2} \sum_{k=0}^{n} (k-nx)^2 r_k(x)$$

把各自之 k 補齊了

$$= \frac{\varepsilon}{2} + \frac{2M}{n^2\delta^2} nx(1-x)$$

$$= \frac{\varepsilon}{2} + \frac{2Mx(1-x)}{\delta^2 n}, \quad x(1-x) \leq 1$$

$$\leq \frac{\varepsilon}{2} + \frac{2M}{\delta^2 n}$$

$-n^2x^2$

$\because \frac{1}{n} \to 0 \quad \therefore$ For $\frac{\delta^2\varepsilon}{4M}$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$\left|\frac{1}{n}\right| < \frac{\delta^2\varepsilon}{4M} \Rightarrow \forall n > N, \ |f(x) - P_n(x)| < \frac{\varepsilon}{2} + \frac{2M}{\delta^2} \cdot \frac{\delta^2\varepsilon}{4M}$

$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \underline{\forall x \in [0,1]}.$

Therefore $P_n \to f$ uniformly on $[0,1]$.

Q.E.D.

Cor. Weierstrass Thm is true if we change $[0,1]$ to $[a,b]$.

pf: Let $\Phi_1 : [0,1] \to [a,b]$ be a function defined by $\Phi_1(x) = (b-a)x + a$. Then Φ_1 is cont. and 1-1 and onto.

Let $\Phi_2(x) : [a,b] \to [0,1]$ be a function defined by $\Phi_2(x) = (x-a)/(b-a)$. Then

Double A

Φ_2 ... is cont. and 1-1 and onto. Moreover $\Phi_1 \circ \Phi_2(x) = x$ and $\Phi_2 \circ \Phi_1(x) = x$

(思考 | $[a,b] \xrightarrow{g} \mathbb{R}$, $\Phi_1 \uparrow \downarrow \Phi_2$, $[0,1] \nearrow g \circ \Phi_1$)

Then $g \circ \Phi_1 \in C([0,1], \mathbb{R})$. by Weiestrass Thm, $\exists$ polynomial P_n s.t. $P_n \to g \circ \Phi_1$ uniformly on $[0,1]$.

$\Rightarrow |P_n(x) - g \circ \Phi_1(x)| \to 0$, $\forall x \in [0,1]$

$\Rightarrow |P_n(\Phi_2(x)) - (g \circ \Phi_1)(\Phi_2(x))| \to 0$, $\forall x \in [a,b]$

$\Rightarrow \left| P_n\left(\frac{x-a}{b-a}\right) - g(x) \right| \to 0$, $\forall x \in [a,b]$.

$\because P_n\left(\frac{x-a}{b-a}\right)$ is a polynomial of x, $\forall n \in \mathbb{N}$

$\therefore$ We are done

Q.E.D.

Thm. (Stone-Weiestrass Thm.)

Let M be a metric space, $A \subset M$ a compact set. and $\mathcal{B} \subset C(A, \mathbb{R})$ satisfy the following:

(i) $\mathcal{B}$ is an algebra; (i.e. if $f, g \in \mathcal{B}$, $\alpha \in \mathbb{R}$, then $f+g \in \mathcal{B}$, $fg \in \mathcal{B}$, $\alpha f \in \mathcal{B}$) ↗有四則運算

(ii) The constant 1-function lies in $\mathcal{B}$ → 常數函數 $\in \mathcal{B}$

(iii) $\mathcal{B}$ seperates points; that is, for $x, y \in A$, $x \neq y$, there is an $f \in \mathcal{B}$ s.t. $f(x) \neq f(y)$

Then $\mathcal{B}$ is dense in $C(A, \mathbb{R})$

Remark. $\mathcal{P}$ is an such $\mathcal{B}$.

pf: Let $f, g \in \overline{\mathcal{B}} (= cl(\mathcal{B}))$. Then $\exists\ f_n, g_n \in \mathcal{B}$.
s.t. $f_n \to f$, $g_n \to g$
$\Rightarrow$ by 極限四則運算, $f_n + g_n \to f+g$, $\alpha f_n \to \alpha f$
and $f_n g_n \to fg$.
$\because f_n + g_n, \alpha f_n, f_n g_n \in \mathcal{B}$
$\therefore f+g, \alpha f, fg \in \overline{\mathcal{B}}$
$\Rightarrow \overline{\mathcal{B}}$ is an algebra

Denote $(f \vee g)(x) = \max(f(x), g(x))$ and
$(f \wedge g)(x) = \min(f(x), g(x))$. Then

$$\begin{cases} f \vee g = \dfrac{f+g}{2} + \dfrac{|f-g|}{2} \\ f \wedge g = \dfrac{f+g}{2} - \dfrac{|f-g|}{2} \end{cases} \text{ and}$$

Let $f \in \overline{\mathcal{B}}$. $\because f \in C(A, \mathbb{R})$ and A is compact
$\therefore \exists\ M > 0$ s.t. $|f(x)| \le M$, $\forall x \in A$.
$\because |t|$ is cont. on $[-M, M]$.
$\therefore \forall n \in \mathbb{N}$, $\exists$ a polynomial P_n s.t. $||t| - P_n(t)|$
$< 1/n$, $\forall t \in [-M, M]$ by Weiestrass Thm. (uniform ←)
$\Rightarrow ||f(x)| - P_n(f(x))| < 1/n$, $\forall x \in A$.
↳ 此為 f 之多項式 $\therefore P_n \circ f \in \overline{\mathcal{B}}$

$\Rightarrow P_n \circ f \to |f|$ uniformly on A
$\Rightarrow \boxed{P_n \circ f} \to |f|$ in $C(A, \mathbb{R})$
$\Rightarrow |f| \in cl(\mathcal{B}) = \mathcal{B}$
Hence if $f, g \in \overline{\mathcal{B}}$, then $f \vee g$, $f \wedge g \in \overline{\mathcal{B}}$

$\in \overline{\mathcal{B}}$ 改取 $\in \mathcal{B}$

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hint: $\overline{cl}(A) = A \cup A'$

→ 欲證 $\overline{\mathcal{B}} = C(A,\mathbb{R})$?

$\overline{cl}(A) = \{x \in M \mid D(x,r) \cap A \neq \emptyset, \forall r > 0\}$ ✓

Let $h \in C(A,\mathbb{R})$. First notice that $\forall x_1, x_2 \in A$ with $x_1 \neq x_2$, by (iii) $\exists g \in \mathcal{B}$ s.t. $g(x_1) \neq g(x_2)$.

Let $\alpha = \dfrac{h(x_1) - h(x_2)}{g(x_1) - g(x_2)}$ and $\beta = \dfrac{g(x_1)h(x_2) - g(x_2)h(x_1)}{g_1(x) - g(x_2)}$

and let $f_{x_1,x_2} = \alpha g + \beta$ Then $f_{x_1,x_2} \in \mathcal{B}$ s.t. $f_{x_1,x_2}(x_1) = h(x_1)$ and $f_{x_1,x_2}(x_2) = h(x_2)$. Notice $\forall x_1 \in A$, $\exists f_{x_1} \in \mathcal{B}$ s.t. $f_{x_1}(x_1) = h(x_1)$.

↳ 想證 $D(h,r) \cap \overline{\mathcal{B}} \neq \emptyset$

Let $r > 0$. Let $x \in A$. For $y \in A$, $\exists f_{yx} \in \mathcal{B}$ s.t. $f_{yx}(y) = h(y) \Rightarrow (f_{yx} - h)(y) = 0$.

$\because f_{yx} - h$ is cont. at y

$\therefore$ For this r, $\exists$ a n.b.d. $U(y)$ of y s.t. $(f_{yx} - h)(z) > -r$, $\forall z \in U(y)$.

$\Rightarrow f_{yx}(z) > h(z) - r$, $\forall z \in U(y)$.

$\because A$ is compact

$\therefore \exists\ y_1, \dots, y_n \in A$ s.t. $\{U(y_1), \dots, U(y_n)\}$ covers A.

$f_x \in \overline{\mathcal{B}}$

Let $f_x = f_{y_1 x} \vee \cdots \vee f_{y_n x}$. Then $f_x(z) > h(z) - r$, $\forall z \in A$. ✓

$\because f_{y_i x}(x) = h(x)$, $\forall y_i$

$\therefore f_x(x) = h(x)$.

$\because f_x - h$ is cont.

$\therefore \exists$ a nbd $V(x)$ s.t. $f_x(z) < h(z) + r$, $\forall z \in V(x)$

$\because A$ is compact

$\therefore \exists x_1, \dots, x_m \in A$ s.t. $\{V(x_1), \dots, V(x_m)\}$ covers A.

Let $f = f_{x_1} \wedge \cdots \wedge f_{x_m}$. Then $f \in \overline{\mathcal{B}}$ and $f(z) < h(z) + r$, $\forall z \in A$.

$\because f(z) > h(z) - r$, $\forall z \in A$

$\therefore |f(z) - h(z)| < r$, $\forall z \in A$

$\because A$ is compact

$\therefore \sup_{x \in A} |f(x) - h(x)| < r$

$\Rightarrow \|f - h\| < r$

$\Rightarrow f \in D(h, r)$

$\Rightarrow D(h, r) \cap \overline{\mathcal{B}} \neq \emptyset$

$\because r$ is arbitrary

$\therefore h \in cl(\overline{\mathcal{B}}) = \overline{\mathcal{B}}$

$\Rightarrow C(A, \mathbb{R}) \subset \overline{\mathcal{B}}$

$\Rightarrow C(A, \mathbb{R}) = \overline{\mathcal{B}}$

$\Rightarrow \mathcal{B}$ is dense in $C(A, \mathbb{R})$

Q.E.D.

ex. Show that $\exists$ a polynomial P s.t. $|P(x) - \sin x| < 1/100$, $\forall x \in [0, 2\pi]$

pf: $\because \sin x$ is cont. on $[0, 2\pi]$

$\therefore$ By Weierstrass Thm, $\exists$ a polynomial P s.t. $|P(x) - \sin x| < 1/100$, $\forall x \in [0, 2\pi]$.

ex. Let P_n be uniformly convergent seq. of polynomials on $[0, 1]$ and $\lim_{n \to \infty} P_n(x)$. Must f be differentiable?

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-sol- 否，可找一連续但不可微之函数，由多项式逼近之。※

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§5.9 The Dirichlet and Abel tests

When Weierstrass M-test fails to determine the uniform convergence of a series, there are two other tests to apply to: the Dirichlet test and Abel test.

Lemma. (Abel's partial sum formula)
Let a_k and b_k be two seq. of real numbers. Let $S_n = \sum_{k=1}^{n} a_k$ then

$$\sum_{k=1}^{n}(a_k b_k) = S_n b_{n+1} - \sum_{k=1}^{n}\left[S_k(b_{k+1}-b_k)\right]$$

$$= S_n b_1 + \sum_{k=1}^{n}\left[(S_n - S_k)(b_{k+1}-b_k)\right]$$

pf. $\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n}\left[(S_k - S_{k-1})b_k\right]$, where $S_0 = 0$.

$$= \sum_{k=1}^{n}(S_k b_k) - \sum_{k=1}^{n}(S_{k-1} b_k)$$

$$= \sum_{k=1}^{n}(S_k b_k) - \sum_{k=0}^{n-1}(S_k b_{k+1})$$

$$= \sum_{k=1}^{n}(S_k b_k) - \left[\sum_{k=1}^{n}(S_k b_{k+1}) - S_n b_{n+1}\right]$$

$$= \sum_{k=1}^{n}\left[S_k(b_k - b_{k+1})\right] + S_n b_{n+1}$$

$$= S_n b_{n+1} - \sum_{k=1}^{n}\left[S_k(b_{k+1}-b_k)\right]$$

$$= S_n\left[\sum_{k=1}^{n}(b_{k+1}-b_k) + b_1\right] - \sum_{k=1}^{n} S_k(b_{k+1}-b_k)$$

Double A

$$= \sum_{k=1}^{n} \left[(S_n - S_k)(b_{k+1} - b_k) \right] + S_n b_1$$

Q.E.D.

Cor. For $m < n$,

$$\sum_{k=m+1}^{n} (a_k b_k) = S_n b_{n+1} - S_m b_{n+1} - \sum_{k=m+1}^{n} \left[S_k (b_{k+1} - b_k) \right]$$

$$= (S_n - S_m) b_{m+1} + \sum_{k=m+1}^{n} \left[(S_n - S_k)(b_{k+1} - b_k) \right]$$

pf: $\sum_{k=m+1}^{n} (a_k b_k) = S_n b_{n+1} - S_m b_{m+1} - \sum_{k=m+1}^{n} \left[S_k (b_{k+1} - b_k) \right]$

$$= S_n \sum_{k=m+1}^{n} \left[(b_{k+1} - b_k) + b_{m+1} \right] - S_m b_{m+1}$$

$$- \sum_{k=m+1}^{n} \left[S_k (b_{k+1} - b_k) \right]$$

$$= \sum_{k=m+1}^{n} \left[(S_n - S_k)(b_{k+1} - b_k) \right] + (S_n - S_m) b_{m+1}$$

Q.E.D.

Thm. (Abel test)

Let $A \subset \mathbb{R}^m$. Let $\sum_{k=1}^{\infty} f_k$ converges uniformly on A, where $f_k : A \to \mathbb{R}$. If $\varphi_k : A \to \mathbb{R}$ is a bounded, monotone sequence of functions, then $\sum_{k=1}^{\infty} \varphi_k f_k$ converges uniformly

→ 不知收斂到哪
⇒ 改證 uniformly Cauchy

Note: ① By bounded seq. we mean $\exists M > 0$ s.t. $|\varphi_k(x)| \le M$, $\forall^{\circ} x \in A$, and $\forall k \in \mathbb{N}$.

② Monotone means increasing or decreasing: $\varphi_k \le \varphi_{k+1}$ on A, $\forall k \in \mathbb{N}$ or $\varphi_k \ge \varphi_{k+1}$ on A. $\forall k \in \mathbb{N}$.

pf: We assume φ_k is a increasing seq. on A. Otherwise consider $(-\varphi_k)$.

Let $S_n = \sum_{k=1}^{n} f_k$ and $r_n(x) = \sum_{k=1}^{n} (\varphi_k f_k)$

Let $\varepsilon > 0$, let $m < n$,

$$r_n(x) - r_m(x) = (S_n(x) - S_m(x)) + \sum_{k=m+1}^{n} \left[(S_n(x) - S_k(x)) \right.$$
$$\left. (\varphi_{k+1}(x) - \varphi_k(x)) \right]$$

$$\Rightarrow |r_n(x) - r_m(x)| \le M |S_n(x) - S_m(x)| + \sum_{k=m+1}^{n} \left[|S_n(x) - S_k(x)| \right.$$
$$\left. (\varphi_{k+1}(x) - \varphi_k(x)) \right]$$

$\because \sum_{k=1}^{\infty} f_k$ is uniformly converges on A

$\therefore$ For $\varepsilon/3M$, $\exists K \in \mathbb{N}$ s.t. $\forall n > m > K$,

$|S_n(x) - S_m(x)| < \varepsilon/3M$, $\forall x \in A$.

$$\Rightarrow |r_n(x) - r_m(x)| \le M \cdot \frac{\varepsilon}{3M} + \sum_{k=m+1}^{n} \frac{\varepsilon}{3M} (\varphi_{k+1}(x) - \varphi_k(x))$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3M} \sum_{k=m+1}^{n} (\varphi_{k+1}(x) - \varphi_k(x))$$

$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3M} (\varphi_{n+1}(x) - \varphi_{m+1}(x))$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3M} \cdot 2M$$

$$= \varepsilon. \quad \forall x \in A.$$

Therefore r_k is a uniform Cauchy sequence on A.

$\because \mathbb{R}$ is complete

$\therefore r_k$ is uniformly convergent on A.

$\Rightarrow \sum_{k=1}^{\infty}(\varphi_k f_k)$ converges uniformly on A.

Q.E.D.

Thm. (Dirichlet test)

Let $A \subset \mathbb{R}^m$. Let $\varphi_k, f_k : A \to \mathbb{R}$ be functions $\forall k \in \mathbb{N}$. Let $\sum_{k=1}^{\infty} f_k$ has the bounded sequence of partial sums. Let φ_k be non-negative decreasing of functions that uniformly converges to 0 on A. Then $\sum_{k=1}^{\infty}\varphi_k f_k$ converges uniformly on A.

pf: Let $S_n = \sum_{k=1}^{n} f_k$ and $r_k = \sum_{k=1}^{n}\varphi_k f_k$, and $M > 0$ s.t. $|S_n(x)| \le M$, $\forall x \in A$ and $n \in \mathbb{N}$.

Let $\varepsilon > 0$, $\because \varphi_k \to 0$ uniformly on A

$\therefore$ For $\frac{\varepsilon}{4M}$, $\exists K \in \mathbb{N}$ s.t. $\forall k > K$, $\varphi_k(x) < \frac{\varepsilon}{4M}$, $\forall x \in A$.

Let $m < n$, $\because r_n(x) - r_m(x) = S_n(x)\varphi_{n+1}(x) - S_m(x)\varphi_{m+1}$

$$- \sum_{k=m+1}^{n}\left[S_k(x)(\varphi_{k+1}(x) - \varphi_k(x))\right]$$

$$\therefore |r_n(x) - r_m(x)| \le M \cdot \frac{\varepsilon}{4M} + M \cdot \frac{\varepsilon}{4M} + M\left|\sum_{k=m+1}^{n}(\varphi_k(x) - \varphi_{k+1}(x)\right.$$

$\forall n > m > K$

$$= \frac{\varepsilon}{2M} + M\left(\varphi_{m+1}(x) - \varphi_{n+1}(x)\right)$$

$$< \frac{\varepsilon}{2M} + M \cdot \left(\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}\right)$$

$$= \varepsilon, \ \forall x \in A.$$

Therefore $\sum_{k=1}^{\infty} \varphi_k f_k$ is uniformly Cauchy on A.

$\because \mathbb{R}$ is complete

$\therefore \sum_{k=1}^{\infty} \varphi_k f_k$ converges uniformly on A.

Q.E.D.

Note: 此二定理皆在 Weiestrass M-test 不適用下才考慮的，皆是將原來的每一个函数拆解成二个相乘的函数來看，其中一个看成 series，另外一个看成 seq，且有一个是 uniformly convergence 和另一个为 uniformly bdd.

ex. Show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ converges uniformly on $[\delta, 2\pi-\delta]$

-sol-. 先決定誰当 series，誰当 seq.

~~$\sum \frac{1}{n}$~~ 只好取 $\frac{1}{n}$ 作 seq.

① $\because 2\sin(kx)\sin\frac{x}{2} = \cos\left[(k-\frac{1}{2})x\right] - \cos\left[(k+\frac{1}{2})x\right]$

$$\therefore \sum_{k=1}^{n}\left[\sin(kx)\sin\left(\frac{x}{2}\right)\right]=\frac{1}{2}\sum_{k=1}^{n}\left[\cos\left[\left(k-\frac{1}{2}\right)x\right]-\cos\left(\left(k+\frac{1}{2}\right)x\right]\right.$$

$$=\frac{1}{2}\left(\cos\frac{x}{2}-\cos\left[\left(n+\frac{1}{2}\right)x\right]\right)$$

$$<\frac{1}{2}(1+1)$$

$$=1$$

$$\Rightarrow \sin\frac{x}{2}\sum_{k=1}^{n}\sin(kx)\leq 1$$

$\because x\in[\delta, 2\pi-\delta]$

$\therefore \sin\frac{x}{2}>0.$

$\because \sin\frac{x}{2}$ is cont. on $[\delta, 2\pi-\delta]$.

$\therefore \exists L>0$ s.t. $\sin\frac{x}{2}\geq L$ on $[\delta, 2\pi-\delta]$

$$\Rightarrow \sum_{k=1}^{n}\sin(kx)\leq\frac{1}{\sin\frac{x}{2}}\leq\frac{1}{L},\ \underline{\forall n\in\mathbb{N}}\ (\text{bdd}),\ \underline{\forall x\in[\delta, 2\pi-\delta]}\ (\text{uniform})$$

$$\Rightarrow \sum_{k=1}^{\infty}\sin kx \text{ is bdd on } [\delta, 2\pi-\delta].$$

② $\because \frac{1}{n}\geq 0$ and $\frac{1}{n}\to 0$ uniformly on $[\delta, 2\pi-\delta]$

$\therefore$ By Dirichlet test, $\sum_{k=1}^{\infty}\frac{\sin(nx)}{n}$ uniformly converges on $[\delta, 2\pi-\delta]$ ※

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ex. Show that $\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}e^{-nx}\right]$ converges uniformly on $[0,\infty)$.

-sol-

$$\sum \frac{(-1)^n}{n} \quad e^{-nx}$$

記憶：級数收斂比較自然，比 seq. 收斂好一点

(A)bel (D)irichlet

好 坏

$\because |e^{-nx}| \le 1$, $\forall x \in [0,\infty)$, $\forall n \in \mathbb{N}$.

$\therefore \{e^{-nx}\}$ is uniformly bounded on A

$\because e^{-nx}$ is decreasing, $\forall n \in \mathbb{N}$. $\forall x \in A$.

seq. $\downarrow$ decreasing

$\therefore e^{-nx}$ is monotone sequence on A.

$\because \sum \frac{(-1)^n}{n}$ uniformly convergence on A

$\therefore$ By Abel test, $\sum\left(\frac{(-1)^n}{n}e^{-nx}\right)$ uniformly converges on A. ※

Double A

Ch6 Differentiable mappings

Euclidean space $(\mathbb{R}^n, \|\cdot\|)$ will be viewed as a normed space

Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then $\exists m$ functions $f_1, \dots, f_m : A \to \mathbb{R}$ s.t. $f = (f_1, \dots, f_m)$

§6-1

Thm: ① Let $x_0 \in A'$ and $L = (L_1, \dots, L_m) \in \mathbb{R}^m$ (分量函数)

Then $\lim_{x \to x_0} f(x) = L \iff \lim_{x \to x_0} f_j(x) = L_j, \ \forall j = 1, \dots, m$

② f is cont. at $x_0 \iff f_j$ is cont. at x_0 $\forall j = 1, \dots, m$

Q: How to define the differentiability of f at x_0?

Def. Let (原点) $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function $0 \in A'$. We say that f is $o(x)$, read little oh of x and written $f(x) = o(x)$ if $\lim_{x \to 0} \frac{\|f(x)\|}{\|x\|} = 0$

Prop. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function and $0 \in A'$. Then:

① If $f(x) \in o(x)$, then $\lim_{x \to 0} f(x) = 0$

② Let $f(x) = (f_1, f_2, \dots, f_m)$. $f = (f_1, \dots, f_m)$

$$f(x)=\mathcal{O}(x) \iff f_j(x)=\mathcal{O}(x),\ \forall j=1,\dots,n.$$ 寫成一個向量

pf. ②: $f(x)=\mathcal{O}(x) \iff \lim_{x\to 0}\frac{\|f(x)\|}{\|x\|}=0 \iff$ ★ $\lim_{x\to 0}\left\|\frac{f(x)}{\|x\|}\right\|=0$

$\iff \lim_{x\to 0}\frac{f(x)}{\|x\|}=0 \iff \lim_{x\to 0}\frac{f_j(x)}{\|x\|}=0 \iff \lim_{x\to 0}\frac{|f_j(x)|}{\|x\|}=0$

$\iff f_j(x)=\mathcal{O}(x),\ \forall j=1,\dots,m$

Q.E.D.

Thm. ($\mathcal{O}(x)$ 的四則運算)

Let $f,g: A\subset\mathbb{R}^n\to\mathbb{R}^m$ be functions and $\mathcal{O}\in A'$. Suppose that $f(x)=\mathcal{O}(x)$ and $g(x)=\mathcal{O}(x)$. Then

① $f(x)+g(x)=\mathcal{O}(x)$

② $\alpha f(x)=\mathcal{O}(x),\ \forall \alpha\in\mathbb{R}$

③ $\langle f(x),g(x)\rangle=\mathcal{O}(x)$ (內積)

pf. ③ Let $f=(f_1,\dots,f_m)$ and $g=(g_1,\dots,g_m)$. Then

$$f\cdot g=\sum_{j=1}^{m} f_j g_j$$

$\because f(x)=\mathcal{O}(x)$ and $g(x)=\mathcal{O}(x)$

$\therefore f_j(x)=\mathcal{O}(x)$ and $g_j(x)=\mathcal{O}(x),\ \forall j=1,\dots,m$

$\Rightarrow (f\cdot g)(x)=\mathcal{O}(x)$

Q.E.D.

→ Go back to differentiability problem. Recall two def.

Def. ① Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a function. We say that f is diff. at x if $\exists$ a number in $\mathbb{R}$, denoted by $f'(x)$ s.t. (差值的解定)

$f(x+h) - f(x) = f'(x)h + o(h)$ ⇒ 函數差值表成變數

↳ Let $t = x+h$, then $f(t) = f(x) + f'(x)(t-x) + o(h) \Rightarrow f(t) \doteq f(x) + f'(x)(t-x)$ → a va and 存在

② Let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ be a function. We say that f is diff at x if $\exists$ a vector in $\mathbb{R}^n$, denoted by $\nabla f(x)$ s.t. $f(x+h) - f(x)$

$= \nabla f(x) \cdot h + o(h)$

↳ Let $t = x+h$, then $f(t) \doteq f(x) + \nabla f(x) \cdot h$

③ In general, let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function. Consider $f(x+h) - f(x) = \square\, h$

$f(x+h) - f(x) \in \mathbb{R}^m$, $\square$ 定 (與 h 無關), $h \in \mathbb{R}^n$

⇒ 欲取定 something 為一個固定的 linear function

$T: \mathbb{R}^n \to \mathbb{R}^m$ s.t. $f(x+h) - f(x) = T(h) + o(h)$

(若用標準基底表 $\mathbb{R}^n$, linear function T 為固定的 $m \times n$ 矩陣!)

現在檢驗取 T 為成新定義，則在 $m=n=1$ 時，

T 為 1×1 的矩陣為一個數，故 $T(h)$ = 此數與 h 相乘

若 $m=1$, 則 T 為 $1 \times n$ 的矩陣，為 $\mathbb{R}^n$ 中的一個定向量且 $T(h)$ = 此向量與 h 作內積.

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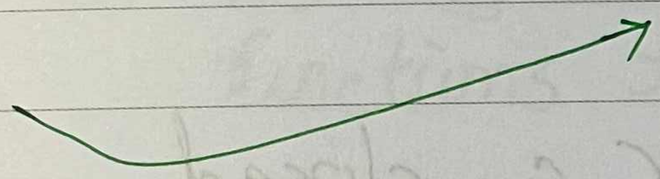
→ a polynomial of variable t of degree 1.
and $y = f(x) + f'(x)(t-x)$
为函数 $y = f(t)$ 的图形在 $(x, f(x))$ 之切线.

polynomial of n variable $t_1, \dots, t_n$ of degree 1. 且 $z = f(x) + \nabla f(x)(t-x)$ 为函数 $z = f(t)$ 的图形在 $(x, f(x))$ 处的切平面



→ Let $t = x+h$, $f(t) \doteq f(x) + T(t-x)$

$$(a_{ij}) \begin{pmatrix} t_1 - x_1 \\ \vdots \\ t_n - x_n \end{pmatrix} = \begin{pmatrix} \sum a_{ij}(t_i - x_i) \\ \vdots \end{pmatrix}$$

Def. A mapping $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be diff. at x if there is a linear function, denoted by $Df(x): \mathbb{R}^n \to \mathbb{R}^m$ s.t.
$f(x+h) - f(x) = Df(x)(h) + o(h)$
Note: ① $Df(x)$ is called the derivative of f at x

② $Df(x)(h)$ 为 $Df(x)$ 取值在 h 上

③ f is diff. at $x \iff \lim_{h \to 0} \frac{\| f(x+h) - f(x) - Df(x)(h) \|}{\| h \|} = 0$

剩余项是 $o(h)$

Convention: $Df(x)(h) = Df(x) \cdot h$

Q: 直觀上我們期待 $f(t) \approx f(x) + Df(x)(t-x)$ 為 f 在 x 附近的最好的 linear approximation. 故期待当 $Df(x)$ 存在時, $Df(x)$ 应該要唯一, 但 $Df(x)$ 会唯一嗎?

A: Let $A = \{x\} \subset \mathbb{R}^n$. Then $Df(x)(t-x) = Df(x)(0)$ $= 0$, $f(t) - f(x) = 0 + 0 = 0 + o(h)$

$\Rightarrow$ 独立矣, $Df(x)$ 沒有唯一的取法, 任意 linear function 都可以

Thm. Let A be an open set or a closed disk in $\mathbb{R}^n$. If $A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x, then $Df(x)$ is uniquely determined by f.

pf. (只證 A being open!)

Let T_1 and T_2 be two linear functions s.t.

$$\begin{cases} f(x+h) - f(x) = T_1(h) + o(h) \\ f(x+h) - f(x) = T_2(h) + o(h) \end{cases}$$

$$\Rightarrow T_1(h) - T_2(h) = o(h)$$

$$\Rightarrow \lim_{h \to 0} \frac{\|T_1(h) - T_2(h)\|}{\|h\|} = 0$$

Let $h \to 0$ along x_i-axis. Then $\lim_{h_i \to 0} \| T_1\left(\frac{h}{\|h\|}\right) - T_2\left(\frac{h}{\|h\|}\right) \| = 0$.

$\Rightarrow \lim_{h_i \to 0} \| T_1(e_i) - T_2(e_i) \| = 0$, where $e = (0, \dots, 0, \overset{i^{th}}{1}, 0, \dots, 0)$

定数

$\Rightarrow \|T_1(e_i) - T_2(e_i)\| = 0$

$\Rightarrow T_1(e_i) = T_2(e_i)$, $\forall i = 1, \dots, n$

$\because \{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$

$\therefore T_1 = T_2$

$\Rightarrow Df(x)$ is unique.

Q.E.D.

A中所有f可微的点所成集合

Note. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function, where A is open or a closed disk. Then $Df: \widetilde{A} \subset A \to L(\mathbb{R}^n, \mathbb{R}^m) = \{$ all linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m\}$

Def. We say $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ is diff. on A if f is diff. at every point of A.

Thm. (微分的四則運算)

Let $f, g: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be functions. If f and g are diff. at x, then

① $f+g$ is diff. at x and $D(f+g)(x) = Df(x) + Dg(x)$

② αf is diff at x and $D(\alpha f)(x) = \alpha Df(x)$

pf. ① Exercise

② $\because \alpha f(x+h) - \alpha f(x) = \alpha(f(x+h) - f(x))$

$= \alpha(Df(x)(h) + o(h))$ ($\because f$ is diff.)

$= (\alpha Df(x))(h) + o(h)$

$\therefore \alpha f$ is diff at x and $D(\alpha f)(x) = \alpha Df(x)$

Q.E.D.