Chapter 3 Rigid Bodies : Equivalent System of Forces

3.1 Introduction

principle of transmissibility

moment of a force about a point

moment of a force about an axis

body : combination of large number of particles

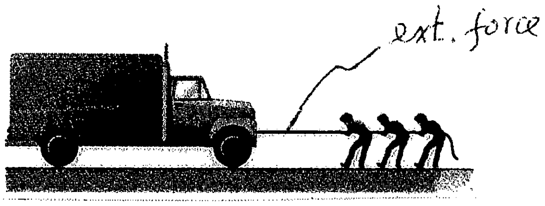
rigid body : does not deform under force acting on the body

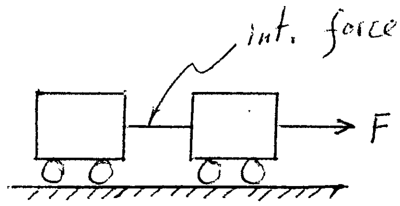
3.2 External and Internal Forces

forces acting on the body may be separated into two groups

1. external forces : represent the action of other bodies on the rigid body

2. internal forces : the forces which hold together the particles forming the rigid body

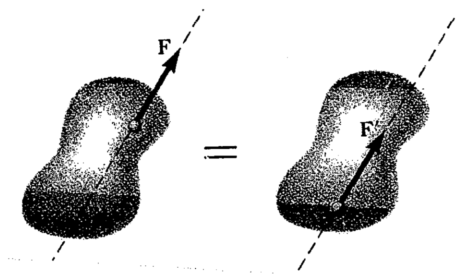


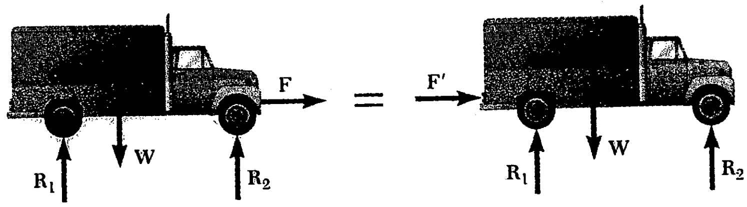


3.3 Principle of Transmissibility, Equivalent Forces

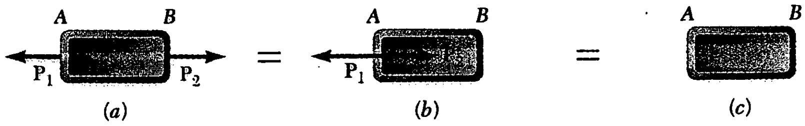
the condition of equilibrium or of motion of a rigid body will remain unchanged if a force $\boldsymbol{F}$ acting at a given point of the rigid body is replaced by a force $\boldsymbol{F'}$ of the same magnitude and same direction, but acting at a different point, provided that the two forces have the same line of action

the two forces $\boldsymbol{F}$ and $\boldsymbol{F'}$ have the same effect on the rigid body and are said to be equivalent





the principle of transmissibility and the concept of equivalent force have limitations

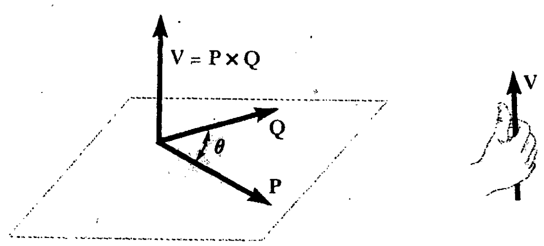


from the point of view of mechanics of rigid bodies, the system shown above are thus equivalent, but the internal forces and deformations are clearly different :   (a) tension,   (b) and (c) no stress

similarly for compression

thus, while the principle of transmissibility may be used freely to determine the conditions of motion or equilibrium of rigid bodies and the compute the external forces action on these bodies, it should be avoided, or at least used with care, in determining internal force and deformation

3.4 Vector Product of Two Vectors



$$V = P \times Q$$

1. $V \perp P$, $V \perp Q$
2. $V = PQ\sin\theta$
3. the direction of V obtained from the right hand rule

vector product also known as cross product

i. vector product are not commutative

$$P \times Q = -Q \times P$$

ii. distributive property can be applied

$$\boldsymbol{P} \times (\boldsymbol{Q}_1 + \boldsymbol{Q}_2) = \boldsymbol{P} \times \boldsymbol{Q}_1 + \boldsymbol{P} \times \boldsymbol{Q}_2$$

iii. associative property does not applied to vector product

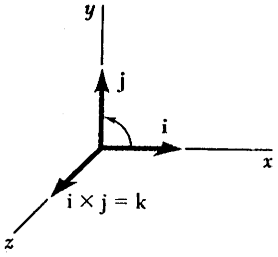
$$(\boldsymbol{P} \times \boldsymbol{Q}) \times \boldsymbol{S} \neq \boldsymbol{P} \times (\boldsymbol{Q} \times \boldsymbol{S})$$

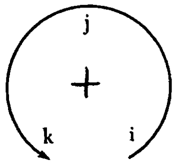
3.5 Vector Products Expressed in Terms of Rectangular Components

$\boldsymbol{i} \times \boldsymbol{i} = 0$

$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$

$\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$ etc.





$$V = \boldsymbol{P} \times \boldsymbol{Q} = (P_x \boldsymbol{i} + P_y \boldsymbol{j} + P_z \boldsymbol{k}) \times (Q_x \boldsymbol{i} + Q_y \boldsymbol{j} + Q_z \boldsymbol{k})$$

$$V_x \boldsymbol{i} + V_y \boldsymbol{j} + V_z \boldsymbol{k} = (P_y Q_z - P_z Q_y) \boldsymbol{i} + (P_z Q_x - P_x Q_z) \boldsymbol{j} + (P_x Q_y - P_y Q_x) \boldsymbol{k}$$

i.e. $V_x = P_y Q_z - P_z Q_y$ $\quad V_y = P_z Q_x - P_x Q_z$

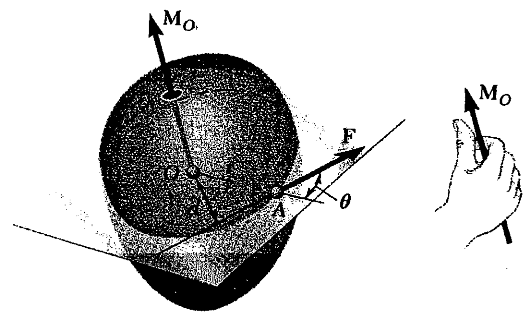
$V_z = P_x Q_y - P_y Q_x$

or

$$V = \boldsymbol{P} \times \boldsymbol{Q} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

3.6 Moment of a Force about a Point

defined the moment of $\boldsymbol{F}$ about O as the vector product of $\boldsymbol{r}$ and $\boldsymbol{F}$



$$\boldsymbol{M}_0 = \boldsymbol{r} \times \boldsymbol{F} \qquad \boldsymbol{r} = OA$$

$$M_0 = r F \sin\theta = F d$$

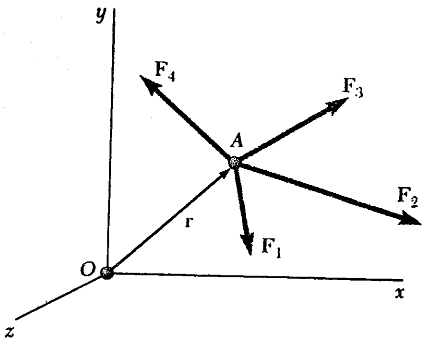
where d is the perpendicular distance from O to the line of action of the force $\boldsymbol{F}$

in two dimensional case

(a) $M_O = +Fd$ (b) $M_O = -Fd$

3.7 Varignon's Theorem

the moment about a given point $\boldsymbol{O}$ of a resultant of several concurrent forces is equal to the sum of the moments of various forces about the same point $\boldsymbol{O}$

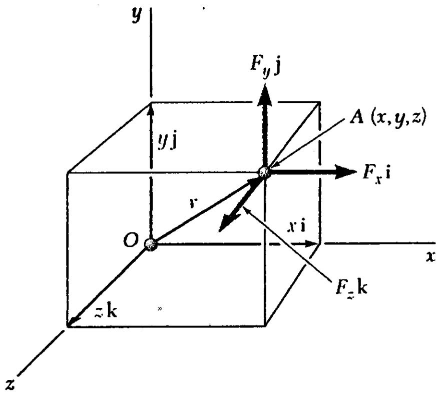


$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \cdots$$

$$\boldsymbol{M}_0 = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times (\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \cdots)$$

$$= \boldsymbol{r} \times \boldsymbol{F}_1 + \boldsymbol{r} \times \boldsymbol{F}_2 + \boldsymbol{r} \times \boldsymbol{F}_3 + \cdots$$

3.8 Rectangular Components of the moment of a Force



$$\boldsymbol{r} = x\,\boldsymbol{i} + y\boldsymbol{j} + z\,\boldsymbol{k}$$

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}$$

$$\boldsymbol{M}_0 = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

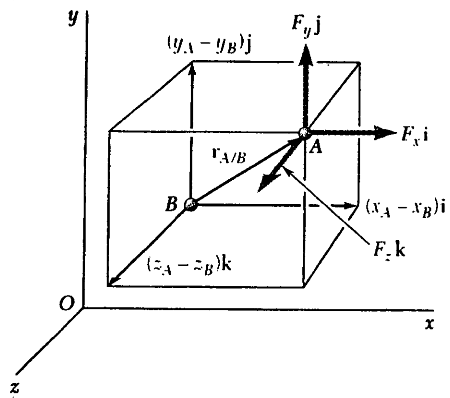
$$\boldsymbol{M}_0 = M_x\boldsymbol{i} + M_y\boldsymbol{j} + M_z\boldsymbol{k}$$

i.e. $M_x = yF_z - zF_y$ $\quad$ $M_y = zF_x - xF_z$

$M_z = xF_y - yF_x$

if we need to compute the moment $\boldsymbol{M}_B$ about an arbitrary at point B of a force $\boldsymbol{F}$ applied at A

$$\boldsymbol{M}_B = \boldsymbol{BA} \times \boldsymbol{F}$$



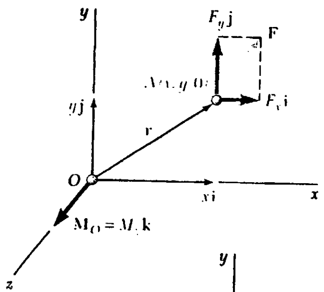
$$BA = \Delta x\, \boldsymbol{i} + \Delta y\, \boldsymbol{j} + \Delta z\, \boldsymbol{k}$$

$$x_{A/B} = x_A - x_B \qquad y_{A/B} = y_A - y_B$$

$$z_{A/B} = z_A - z_B$$

$$BA = \boldsymbol{r}_A - \boldsymbol{r}_B = \Delta \boldsymbol{r}$$

$$\therefore \quad \boldsymbol{M}_B = \Delta \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix}$$

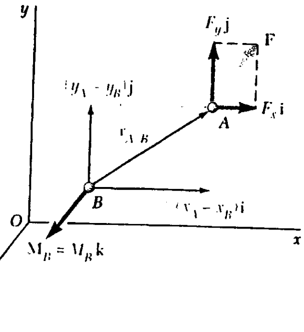


for two dimensional case, $F_z = 0, \; z = 0$

$$\boldsymbol{M}_0 = (x F_y - y F_x)\, \boldsymbol{k}$$

$$M_0 = x F_y - y F_x = M_z$$

$$M_x = M_y = 0$$



$$\text{or} \quad \boldsymbol{M}_B = [(x_A - x_B) F_y - (y_A - y_B) F_x]\, \boldsymbol{k}$$

$$M_B = M_z = (x_A - x_B) F_y - (y_A - y_B) F_x$$

$$M_x = M_y = 0$$

Sample Problem 3.1

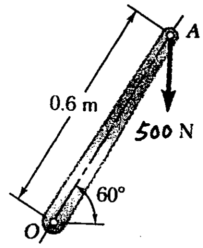
determine (a) M_0

(b) same M_0, find horizontal force F

(c) smallest force to get same M_0

(d) for 1200 N ↓, find d to get same M_0

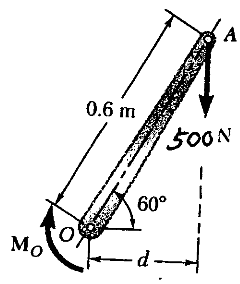
(e) is (b), (c), (d) is the force equivalent to the original force



(a) $M_0 = F d = 500 \times 600 \cos 60^\circ$

$= 500 \times 300 = 150{,}000$ N-mm

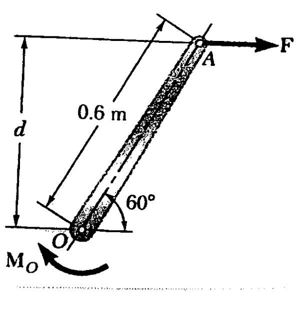
$= 150$ N-m

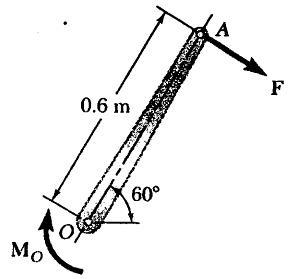


(b) $M_0 = F d = F \times 600 \sin 60^\circ$

$150 = F \times 0.5196$

$F = 288.68$ N



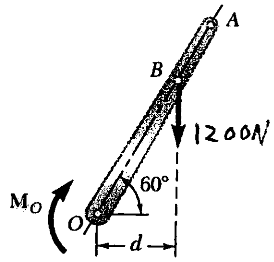


(c) $M_0 = F\,d$ $\qquad F_{min} \rightarrow d_{max}$

i.e. $F \perp OA$ $\qquad d_{max} = OA = 0.6$ m

$150 = F_{min} \times 0.6$

$F_{min} = 250$ N $\qquad \searrow 30^\circ$



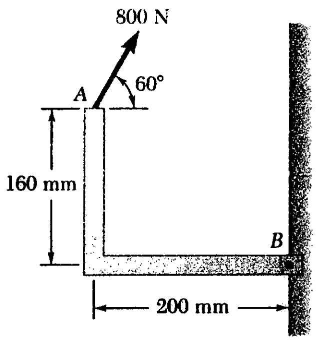
(d) $M_0 = F\,d$

$150 = 1200\,d$ $\qquad d = 0.125$ m

$OB = d / \cos 60^\circ = 0.25$ m $= 250$ mm

(e) none of (b), (c), (d) is equivalent, although M_0 is equal in each case, but F is not equal

Sample Problem 3.2



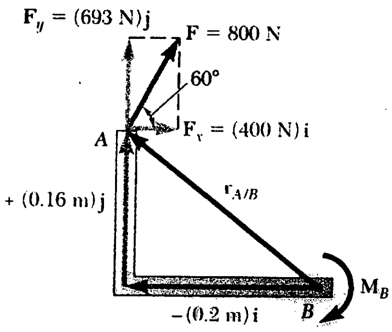
determine $\boldsymbol{M}_B$

$\boldsymbol{M}_B = \boldsymbol{r}_{A/B} \times \boldsymbol{F}$

$\boldsymbol{r}_{A/B} = BA = -0.2\,\boldsymbol{i} + 0.16\,\boldsymbol{j}$ (m)

$\boldsymbol{F} = 800 \cos 60^\circ\,\boldsymbol{i} + 800 \sin 60^\circ\,\boldsymbol{j}$

$= 400\,\boldsymbol{i} + 693\,\boldsymbol{j}$ (N)

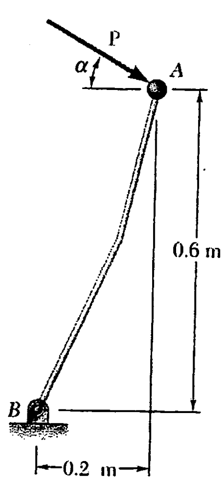


$\boldsymbol{M}_B = (-0.2\,\boldsymbol{i} + 0.16\,\boldsymbol{j}) \times (400\,\boldsymbol{i} + 693\,\boldsymbol{j})$

$= -138.6\,\boldsymbol{k} - 64\,\boldsymbol{k} = -202.6\,\boldsymbol{k}$ (N-m)

$= 202.6$ N-m $(\curvearrowright)$

Sample Problem 3.3



$P = 40$ N $\qquad a = 25^\circ$

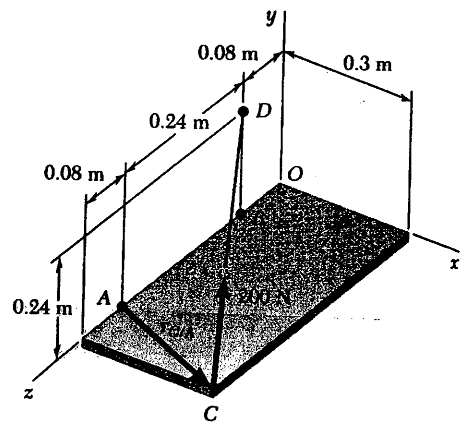
$P_x = 40 \cos 25^\circ = 36.252$ N

$P_y = 40 \sin 25^\circ = 16.905$ N

$\because\ M_B = -x\,P_y - y\,P_x$

$= -0.2 \times 16.905 - 0.6 \times 36.252 = -25.1$ N-m

Sample Problem 3.4



T_{CD} = 200 N

find $\boldsymbol{M}_A$ of $\boldsymbol{T}_{CD}$ at C

$\boldsymbol{M}_A = \boldsymbol{r}_{C/A} \times \boldsymbol{T}_{CD}$

$A(0, 0, 0.32)$ $\quad$ $C(0.3, 0, 0.4)$

$D(0, 0.24, 0.08)$ $\quad$ unit : m

$\boldsymbol{r}_{C/A} = \boldsymbol{AC} = 0.3\,\boldsymbol{i} + 0.08\,\boldsymbol{k}$ (m)

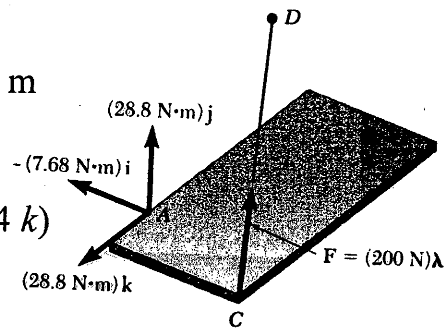
$\boldsymbol{CD} = -0.3\,\boldsymbol{i} + 0.24\,\boldsymbol{j} - 0.32\,\boldsymbol{k}$ (m)

$CD = (0.3^2 + 0.24^2 + 0.32^2)^{1/2} = 0.5$ m

$\lambda_{CD} = \boldsymbol{CD} / CD = -0.6\,\boldsymbol{i} + 0.48\,j - 0.64\,k$

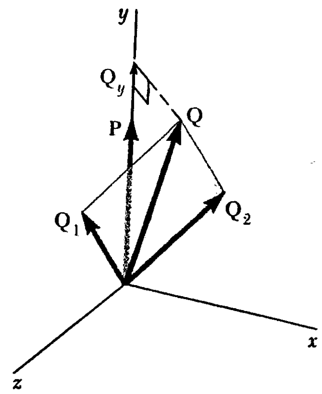
$\boldsymbol{T}_{CD} = T_{CD}\,\lambda_{CD} = 200\,(-0.6\,\boldsymbol{i} + 0.48\,j - 0.64\,k)$

$= -120\,\boldsymbol{i} + 96\,j - 128\,k$ (N)

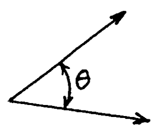


$$\boldsymbol{M}_A = \boldsymbol{r}_{C/A} \times \boldsymbol{T}_{CD} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix} = -7.68\,\boldsymbol{i} + 28.8\,j + 28.8\,k \text{ (N-m)}$$

3.9 Scalar Product of Two Vector (dot product)



$\boldsymbol{P} \cdot \boldsymbol{Q} = P\,Q \cos\theta$ (scalar)



commutative property $\quad \boldsymbol{P} \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{P}$

distributive property $\quad \boldsymbol{P} \cdot (\boldsymbol{Q}_1 + \boldsymbol{Q}_2) = \boldsymbol{P} \cdot \boldsymbol{Q}_1 + \boldsymbol{P} \cdot \boldsymbol{Q}_2$

associative property cannot be applied, $\because$ $(\boldsymbol{P} \cdot \boldsymbol{Q}) \cdot \boldsymbol{S}$ has no meaning

dot product of unit vectors (= 0 or 1)

$\boldsymbol{i} \cdot \boldsymbol{i} = 1 \times 1 \cos 0^\circ = 1 \qquad \boldsymbol{j} \cdot \boldsymbol{j} = 1 \qquad \boldsymbol{k} \cdot \boldsymbol{k} = 1$

$\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{i} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{j} = 0$

$\boldsymbol{P} \cdot \boldsymbol{Q} = (P_x\,\boldsymbol{i} + P_y\,\boldsymbol{j} + P_z\,\boldsymbol{k}) \cdot (Q_x\,\boldsymbol{i} + Q_y\,\boldsymbol{j} + Q_z\,\boldsymbol{k})$

$$= P_x Q_x + P_y Q_y + P_z Q_z$$

if $\quad \boldsymbol{P} = \boldsymbol{Q} \qquad \boldsymbol{P} \cdot \boldsymbol{P} = P_x^2 + P_y^2 + P_z^2 = P^2$

applications :

1. angle formed by two given vectors

$$\boldsymbol{P} = P_x \boldsymbol{i} + P_y \boldsymbol{j} + P_z \boldsymbol{k}$$

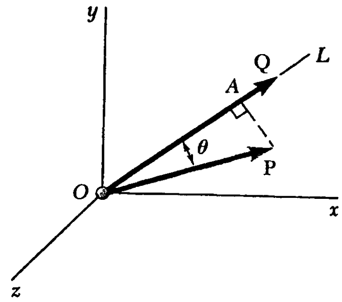
$$\boldsymbol{Q} = Q_x \boldsymbol{i} + Q_y \boldsymbol{j} + Q_z \boldsymbol{k}$$

$$\boldsymbol{P} \cdot \boldsymbol{Q} = P\,Q \cos\theta = P_x Q_x + P_y Q_y + P_z Q_z$$

$$\cos\theta = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{P\,Q}$$

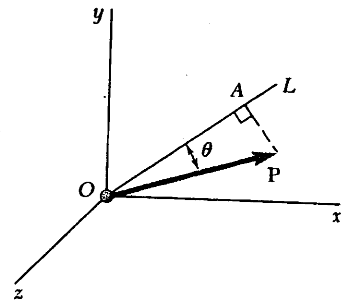
where $P = (P_x^2 + P_y^2 + P_z^2)^{1/2} \qquad Q = (Q_x^2 + Q_y^2 + Q_z^2)^{1/2}$

2. projection of a vector on a given axis



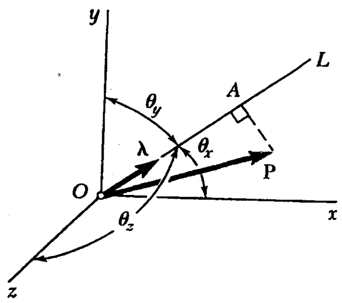
$$P_{OL} \equiv OA = P\cos\theta$$

$$\boldsymbol{P} \cdot \boldsymbol{Q} = P\,Q\cos\theta = P_{OL}\,Q$$



$$P_{OL} = \frac{\boldsymbol{P} \cdot \boldsymbol{Q}}{Q} = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{Q}$$

now, if we choose $\boldsymbol{Q}$ is an unit vector



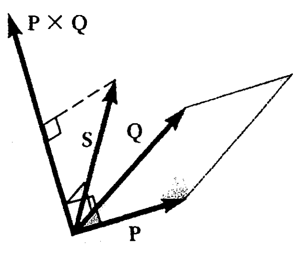
$$\boldsymbol{Q} = \boldsymbol{\lambda}_{OL} \qquad Q = 1$$

$$P_{OL} = \boldsymbol{P} \cdot \boldsymbol{\lambda}_{OL}$$

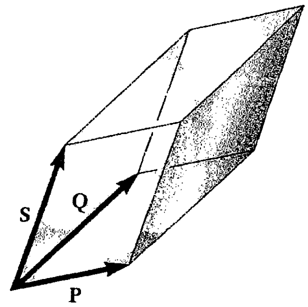
$$\boldsymbol{\lambda}_{OL} = \cos\theta_x\,\boldsymbol{i} + \cos\theta_y\,\boldsymbol{j} + \cos\theta_z\,\boldsymbol{k}$$

$$\therefore \quad P_{OL} = P_x\cos\theta_x + P_y\cos\theta_y + P_z\cos\theta_z$$

3.10 Mixed Triple Product of Three Vectors



$\boldsymbol{S} \cdot (\boldsymbol{P} \times \boldsymbol{Q})$	scalar triple product
$\boldsymbol{S} \times (\boldsymbol{P} \times \boldsymbol{Q})$	vector triple product (used in dynamics)

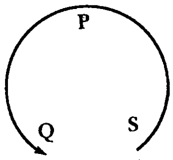


$U = \boldsymbol{S} \cdot (\boldsymbol{P} \times \boldsymbol{Q}) = \boldsymbol{S} \cdot \boldsymbol{V}$

U = the volume of the parallelogram having the vector $\boldsymbol{S}, \boldsymbol{P}, \boldsymbol{Q}$ for sides

if $\boldsymbol{S}, \boldsymbol{P}, \boldsymbol{Q}$ are coplanar, the scalar triple product will be zero

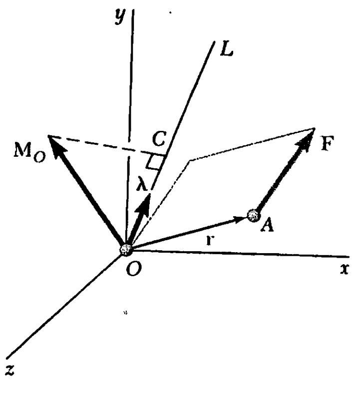
$$\boldsymbol{S} \cdot (\boldsymbol{P} \times \boldsymbol{Q}) = \boldsymbol{P} \cdot (\boldsymbol{Q} \times \boldsymbol{S}) = \boldsymbol{Q} \cdot (\boldsymbol{S} \times \boldsymbol{P})$$
$$= -\boldsymbol{S} \cdot (\boldsymbol{Q} \times \boldsymbol{P}) = -\boldsymbol{Q} \cdot (\boldsymbol{P} \times \boldsymbol{S}) = -\boldsymbol{P} \cdot (\boldsymbol{S} \times \boldsymbol{Q})$$



$$\boldsymbol{S} \cdot (\boldsymbol{P} \times \boldsymbol{Q}) = \boldsymbol{S} \cdot \boldsymbol{V} = S_x V_x + S_y V_y + S_z V_z$$
$$= S_x (P_y Q_z - P_z Q_y) + S_y (P_z Q_x - P_x Q_z) + S_z (P_x Q_y - P_y Q_x)$$

$$\boldsymbol{S} \cdot (\boldsymbol{P} \times \boldsymbol{Q}) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

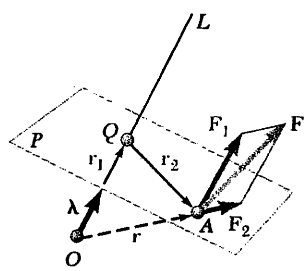
3.11 Moment of a Force about a Given Axis



$$\boldsymbol{M}_0 = \boldsymbol{r} \times \boldsymbol{F}$$

define the moment M_{OL} of $\boldsymbol{F}$ about OL as the projection OC of the moment $\boldsymbol{M}_0$ on the axis OL

$$\therefore \quad M_{OL} = \lambda \cdot \boldsymbol{M}_O = \lambda \cdot (\boldsymbol{r} \times \boldsymbol{F})$$
$$= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$



resolve $\boldsymbol{F}$ into $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ in which $\boldsymbol{F}_1$ // OL and $\boldsymbol{F}_2$ on the plane $\perp$ OL

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2$$

and $\boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{r}_2$ where $\boldsymbol{r}_1 \,//\, \lambda$

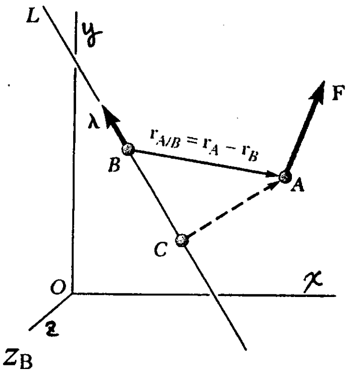
$$M_{OL} = \lambda \cdot (\boldsymbol{r} \times \boldsymbol{F}) = \lambda \cdot [(\boldsymbol{r}_1 + \boldsymbol{r}_2) \times (\boldsymbol{F}_1 + \boldsymbol{F}_2)]$$
$$= \lambda \cdot (\boldsymbol{r}_1 \times \boldsymbol{F}_1) + \lambda \cdot (\boldsymbol{r}_1 \times \boldsymbol{F}_2) + \lambda \cdot (\boldsymbol{r}_2 \times \boldsymbol{F}_1) + \lambda \cdot (\boldsymbol{r}_2 \times \boldsymbol{F}_2)$$
$$= \lambda \cdot (\boldsymbol{r}_2 \times \boldsymbol{F}_2) = |\boldsymbol{r}_2 \times \boldsymbol{F}_2|$$

the moment M_{OL} of $\boldsymbol{F}$ about OL measures the tendency of the force $\boldsymbol{F}$ to impart to the rigid body a motion or rotation about the fixed axis OL

$$\therefore \quad \boldsymbol{M} \cdot \boldsymbol{i} = M_x = y F_z - z F_y$$
$$\boldsymbol{M} \cdot \boldsymbol{j} = M_y = z F_x - x F_z \qquad \text{moment about the coord. axes}$$
$$\boldsymbol{M} \cdot \boldsymbol{k} = M_z = x F_y - y F_x$$

the moment components M_x, M_y and M_z of $\boldsymbol{F}$ about the coordinate axes measure the tendency of $\boldsymbol{F}$ to impart to the rigid body a motion of rotation about the x, y and z axes, respectively

the moment of F applied at A about an axis which does not pass through O is obtained by choosing an arbitrary point B on the axis



$$M_{BL} = \lambda \cdot \boldsymbol{M}_B = \lambda \cdot (\Delta \boldsymbol{r} \times \boldsymbol{F})$$
$$\Delta \boldsymbol{r} = \boldsymbol{r}_A - \boldsymbol{r}_B$$

$$M_{BL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ \Delta x & \Delta y & \Delta z \\ F_x & F_y & F_z \end{vmatrix}$$

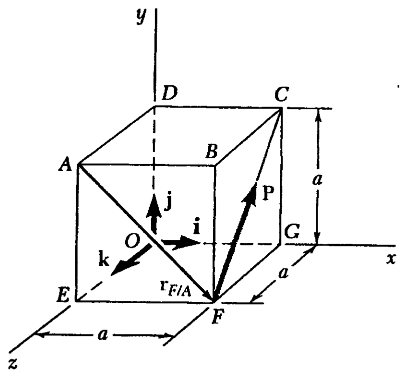
$$\Delta x = x_A - x_B \qquad \Delta y = y_A - y_B \qquad \Delta z = z_A - z_B$$

the moment of F about an axis is independent of the choice of the point on the axis

choose another point C

$$M_{CL} = \lambda \cdot [(\boldsymbol{r}_A - \boldsymbol{r}_C) \times \boldsymbol{F}]$$
$$\boldsymbol{r}_A - \boldsymbol{r}_C = (\boldsymbol{r}_A - \boldsymbol{r}_B) + (\boldsymbol{r}_B - \boldsymbol{r}_C)$$
$$M_{CL} = \lambda \cdot [(\boldsymbol{r}_A - \boldsymbol{r}_B) \times \boldsymbol{F}] + \lambda \cdot [(\boldsymbol{r}_B - \boldsymbol{r}_C) \times \boldsymbol{F}]$$
$$= \lambda \cdot (\Delta \boldsymbol{r} \times \boldsymbol{F}) = M_{BL}$$

Sample Problem 3.5



a cube of side a, determine

(a) $\boldsymbol{M}_A$ (b) M_{AB}

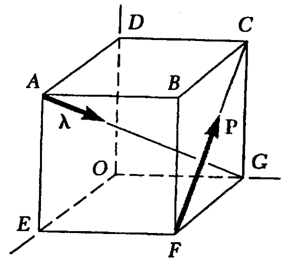
(c) M_{AG} (d) d from AG to FC

$\boldsymbol{P} = P\,\lambda_{FC}$ $F(a, 0, a)$ $C(a, a, 0)$

$\lambda_{FC} = \boldsymbol{FC} / FC = (\boldsymbol{j} - \boldsymbol{k}) / \sqrt{2}$

(a) $\boldsymbol{M}_A = \boldsymbol{AF} \times \boldsymbol{P} = (a\boldsymbol{j} - a\boldsymbol{k}) \times P(\boldsymbol{j} - \boldsymbol{k}) / \sqrt{2}$

$= a P(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) / \sqrt{2}$



(b) $M_{AB} = \lambda_{AB} \cdot \boldsymbol{M}_A$ $\lambda_{AB} = \boldsymbol{i}$

$= \boldsymbol{i} \cdot a P(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) / \sqrt{2} = a P / \sqrt{2}$

(c) $M_{AG} = \lambda_{AG} \cdot \boldsymbol{M}_A$

$A(0, a, a)$ $G(a, 0, 0)$ $\boldsymbol{AG} = a(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$ $AG = a\sqrt{3}$

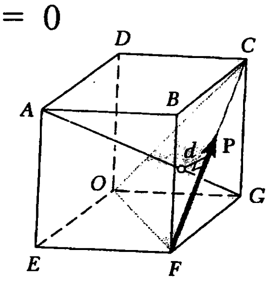
$\lambda_{AG} = (\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) / \sqrt{3}$

$M_{AG} = [(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) / \sqrt{3}] \cdot [a P(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) / \sqrt{2}] = -a P / \sqrt{6}$

or $M_{AG} = \lambda_{AG} \cdot (\boldsymbol{AF} \times \boldsymbol{P})$

$$= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ a & -a & 0 \\ 0 & P/\sqrt{\ } & -P/\sqrt{\ } \end{vmatrix} = -a P / \sqrt{6}$$

(d) $\because \boldsymbol{P} \cdot \lambda_{AG} = [P(\boldsymbol{j} - \boldsymbol{k}) / \sqrt{2}] \cdot [(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) / \sqrt{3}] = 0$



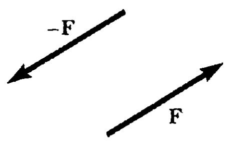
$\therefore$ $\boldsymbol{P} \perp \lambda_{AG}$

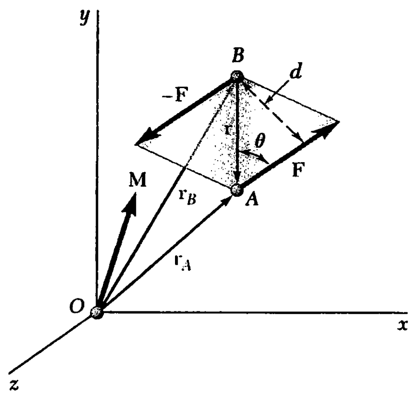
$\therefore$ $M_{AG} = |P\,d|$ $a P / \sqrt{6} = P\,d$

thus $d = a / \sqrt{6}$

3.12 Moment of a Couple

two forces $\boldsymbol{F}$ and $-\boldsymbol{F}$ having the same magnitude, parallel line of action, opposite direction are said to form a couple

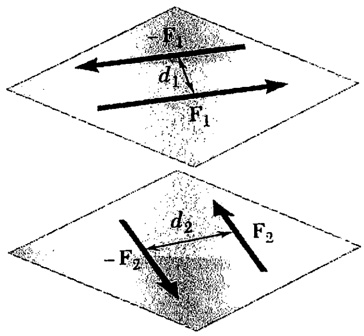




$$M_0 = r_A \times F + r_B \times (-F)$$
$$= (r_A - r_B) \times F = r \times F$$

where r is the vector jointing the points of application of the two forces, M_0 is called the moment of couple

$$M_0 = r F \sin\theta = F d$$

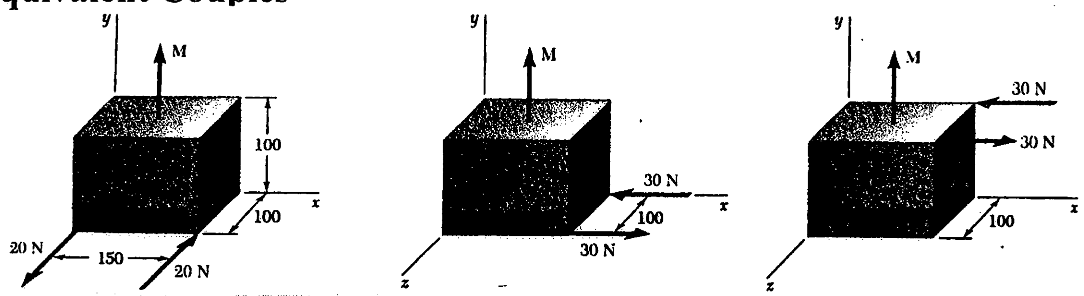


r is independent of the origin O, $\therefore$ the moment M of a couple is a free vector, i.e. which may be applied at any point

if $\quad F_1 d_1 = F_2 d_2$

then the moment of couple is said to be equal

2.13 Equivalent Couples



three couples have the same effect on the box

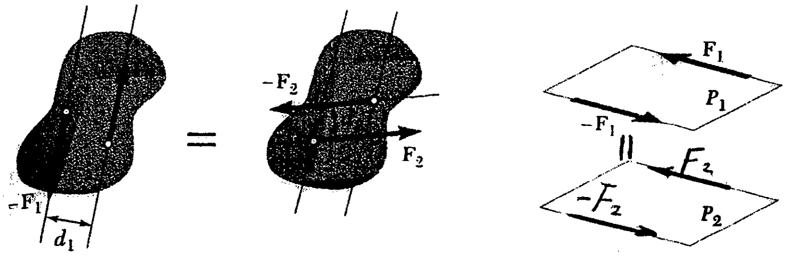
two system of forces are equivalent if we can transform one of them into other by means of one or several of the following operations :

1. replacing two forces acting on the same particle by their resultant
2. resolving a force into two components
3. canceling two equal and opposite forces acting on the same particle
4. attaching to the same particle two equal and opposite forces
5. moving a force along its line of action

two couples having the same moment $\boldsymbol{M}$ are equivalent

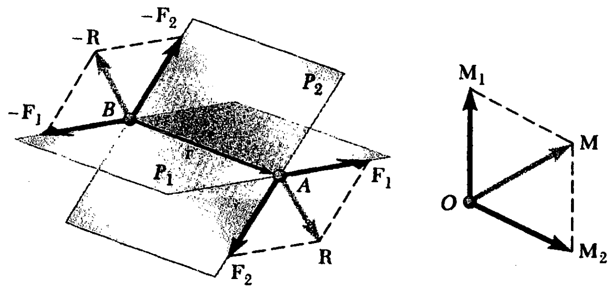
1. two couples contained in the same plane

2. two couples contained in the parallel planes



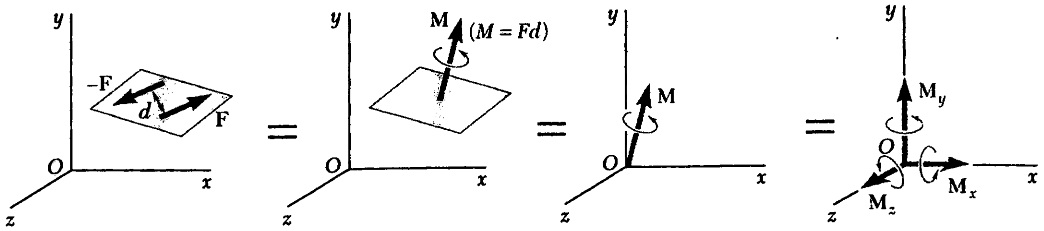
3.14 Addition of Couples

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{R} = \boldsymbol{r} \times (\boldsymbol{F}_1 + \boldsymbol{F}_2)$$
$$= \boldsymbol{r} \times \boldsymbol{F}_1 + \boldsymbol{r} \times \boldsymbol{F}_2$$
$$= \boldsymbol{M}_1 + \boldsymbol{M}_2$$



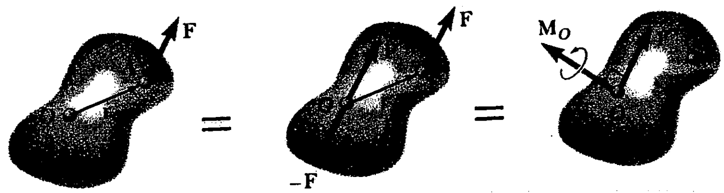
3.15 Couples may be Represented by Vectors

the vectors representing a couple is called a couple vector



3.16 Resolution of a Given Force into a Force at O and a Couple

any force acting on a rigid body may be moved to an arbitrary point O, provided that a couple is added, of moment equal to the moment of $\boldsymbol{F}$ about O



i.e. $\quad \boldsymbol{M}_0 = \boldsymbol{r} \times \boldsymbol{F} \qquad M_0 = F\,d$

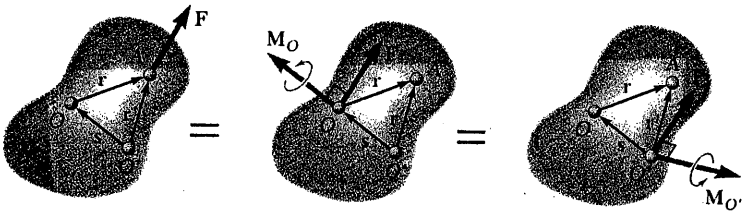
and the combination obtained is referred to as a force-couple system

if the force $\boldsymbol{F}$ had been moved from A to a different point O', then

$$\boldsymbol{M}_{0'} = \boldsymbol{r}' \times \boldsymbol{F} \qquad \text{[from (a) to (c)]}$$

$$= (\boldsymbol{r} + \boldsymbol{s}) \times \boldsymbol{F} = \boldsymbol{r} \times \boldsymbol{F} + \boldsymbol{s} \times \boldsymbol{F}$$

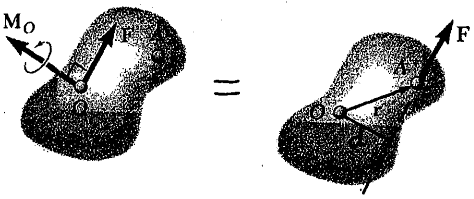
$$= \boldsymbol{M}_0 + \boldsymbol{s} \times \boldsymbol{F} \qquad \text{[from (b) to (c)]}$$



(a) (b) (c)

replacing a force-couple system into a single force

$$d = M_0 / F$$

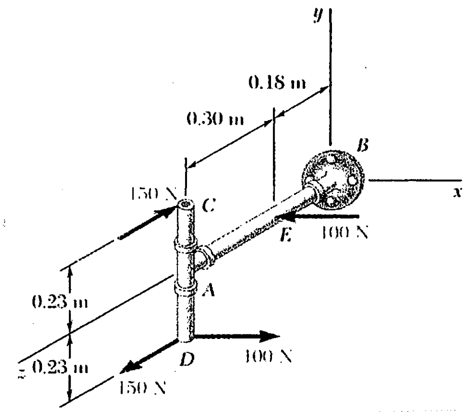


in this case, $\boldsymbol{F}$ must be $\perp$ $\boldsymbol{M}_0$

Sample Problem 3.6

determine $\boldsymbol{M}$ of the couples

$$\boldsymbol{DE} = 0.23\boldsymbol{j} - 0.3\boldsymbol{k}$$



$\boldsymbol{M}_1$ for the 100 N force

$$\boldsymbol{M}_1 = \boldsymbol{DE} \times (-100\boldsymbol{i})$$

$$= (0.23\boldsymbol{j} - 0.3\boldsymbol{k}) \times (-100\boldsymbol{i})$$

$$= 30\boldsymbol{j} + 23\boldsymbol{k} \qquad \text{(N-m)}$$

$\boldsymbol{M}_2$ for the 150 N force

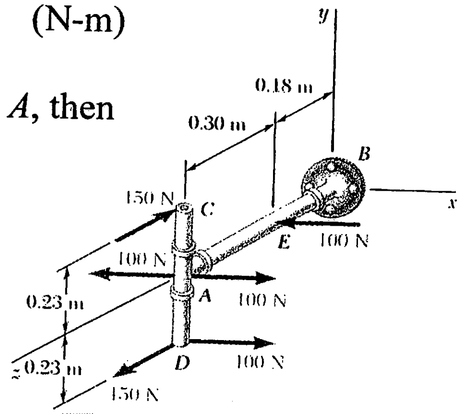
$$\boldsymbol{M}_2 = \boldsymbol{DC} \times (-30\boldsymbol{k})$$

$$= 0.46\boldsymbol{j} \times (-150\boldsymbol{i})$$

$$= -69\boldsymbol{i} \qquad \text{(N-m)}$$

$$\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2 = -69\,\boldsymbol{i} + 30\,\boldsymbol{j} + 23\,\boldsymbol{k} \quad \text{(N-m)}$$

alternate solution, add + 100 N and - 100 N at A, then



$$M_x = -150 \times 0.46 = -69 \text{ N-m}$$

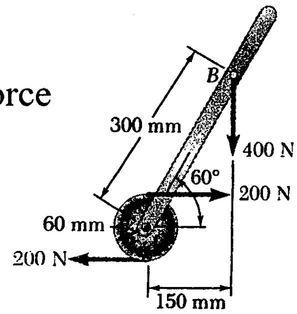
$$M_y = 100 \times 0.3 = 30 \text{ N-m}$$

$$M_z = 100 \times 0.23 = 23 \text{ N-m}$$

then $\quad \boldsymbol{M} = -69\,\boldsymbol{i} + 30\,\boldsymbol{j} + 23\,\boldsymbol{k}$ (N-m)

Sample Problem 3.7

replace the force and couple by a single equivalent force

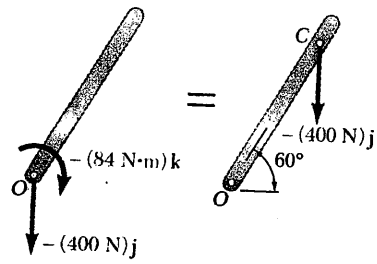


due to the force, the moment about O is

$$M_0 = 400 \times d = 400 \times 0.3 \cos 60^{\circ}$$

$$= 60 \text{ N-m} \quad (\circlearrowright)$$

due to the couple



$$M_0 = 200 \times 0.12 = 24 \text{ N-m} \quad (\circlearrowright)$$

then the total moment at O is

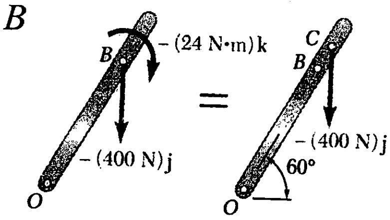
$$M_0 = 84 \text{ N-m} \quad (\circlearrowright)$$

the distance of the application of the equivalent force is

$$d = M_0 / F = 84 / 400 = 0.21 \text{ m}$$

and $\quad OC = 0.21 / \cos 60^{\circ} = 0.42$ m

alternative method, move the (- 24 N-m) couple to B



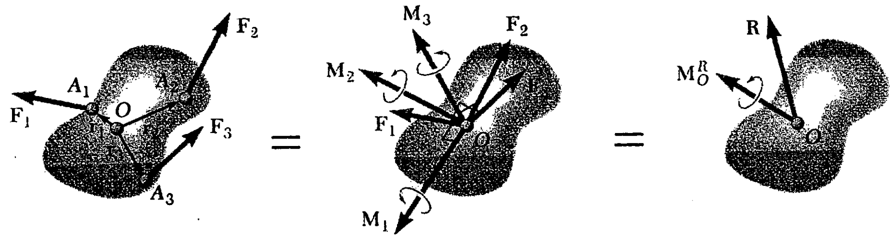
$$d = M / F = 24 / 400 = 0.06 \text{ m}$$

$$BC = 0.06 / \cos 60^{\circ} = 0.12 \text{ m}$$

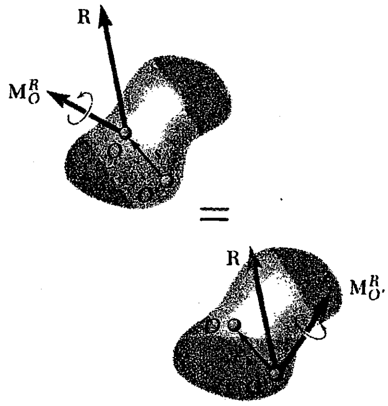
$$OC = OB + BC = 0.3 + 0.12 = 0.42 \text{ m}$$

3.17 Reduction of a System of Force to One Force and One Couple

consider a system of forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ $\cdots$ acting at the rigid body at the points A_1, A_2, A_3 $\cdots$, the forces can be removed to a given point O



then the equivalent force-couple system is defined



$$\boldsymbol{R} = \Sigma \boldsymbol{F} \qquad \boldsymbol{M}_0^{\mathrm{R}} = \Sigma \boldsymbol{M}_0 = \Sigma (\boldsymbol{r} \times \boldsymbol{F})$$

$\boldsymbol{M}_0^{\mathrm{R}}$ is called the moment resultant of the system at O

it can also to a force and a couple at another point O'

$$\boldsymbol{R} = \Sigma \boldsymbol{F} \qquad \boldsymbol{M}_{0'}^{\mathrm{R}} = \boldsymbol{M}_0^{\mathrm{R}} + \boldsymbol{s} \times \boldsymbol{R}$$

for each force, the position and force components is

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\boldsymbol{j} + z\,\boldsymbol{k}$$
$$\boldsymbol{F} = F_x\,\boldsymbol{i} + F_y\boldsymbol{j} + F_z\,\boldsymbol{k}$$

and the resultant force and moment can be written as

$$\boldsymbol{R} = R_x\,\boldsymbol{i} + R_y\boldsymbol{j} + R_z\,\boldsymbol{k}$$
$$\boldsymbol{M}_0^{\mathrm{R}} = M_x^{\mathrm{R}}\,\boldsymbol{i} + M_y^{\mathrm{R}}\boldsymbol{j} + M_z^{\mathrm{R}}\,\boldsymbol{k}$$

R_x, R_y, R_z : the sums of x, y, z components of the given force

M_x^{R}, M_y^{R}, M_z^{R} : the sums of the moments of the given force about the x, y, z axes

3.18 Equivalent Systems of Forces

two systems of forces are equivalent, therefore, if they may reduced to the same force-couple system at given at point O of a rigid body

i.e. they are equivalent if, and only if, the sums of forces and the sums of moments about a given point O are, respectively, equal

$\Sigma \boldsymbol{F} = \Sigma \boldsymbol{F'} \quad \rightarrow \quad \Sigma F_x = \Sigma F_{x'} \quad \Sigma F_y = \Sigma F_{y'} \quad \Sigma F_z = \Sigma F_{z'}$

$\Sigma \boldsymbol{M}_0 = \Sigma \boldsymbol{M}_{0'} \rightarrow \quad \Sigma M_x = \Sigma M_{x'} \quad \Sigma M_y = \Sigma M_{y'} \quad \Sigma M_z = \Sigma M_{z'}$

3.19 Equipollent Systems of Vectors

two systems of vectors satisfy the equations in last section, the two system are said to be equipollent

if two systems of forces acting on a rigid body are equipollent, they are also equivalent

two systems of forces acting on different particles may be equipollent but not equivalent

3.20 Further Reduction of a System of Forces

for a force-couple system, it can be reduced as $\boldsymbol{R}$ and $\boldsymbol{M}_0^R$ at O

when $\boldsymbol{R} = 0$ => single couple vector $\boldsymbol{M}_0^R$

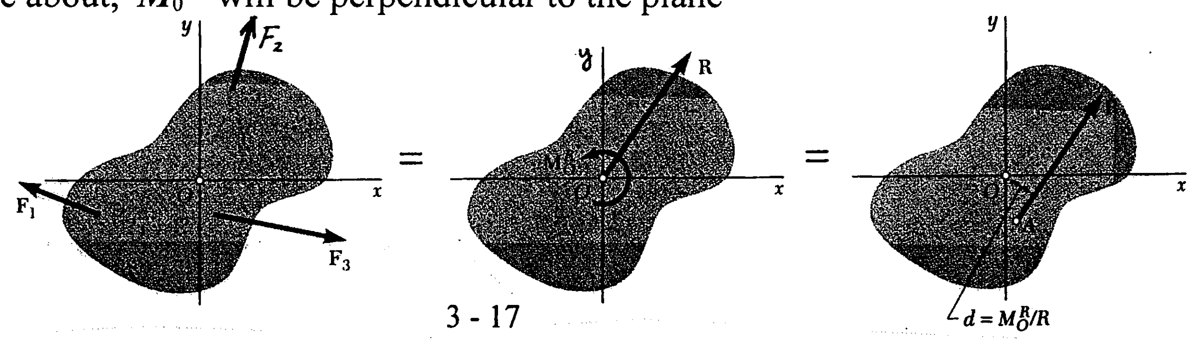
we now investigate the conditions under which a given system of forces may be reduced to a single force

two systems of forces which may be reduced to a single force, or resultant, are therefore the systems for which the force $\boldsymbol{R}$ and the couple vector $\boldsymbol{M}_0^R$ are mutually perpendicular

this condition is generally not satisfied except for (1) concurrent forces, (2) coplanar forces, (3) parallel forces

(1) concurrent forces : they can be added directly into their resultant $\boldsymbol{R}$

(2) coplanar forces : all forces lie on the x-y plane, the moment of each force about, $\boldsymbol{M}_0^R$ will be perpendicular to the plane



the force-couple system at O is

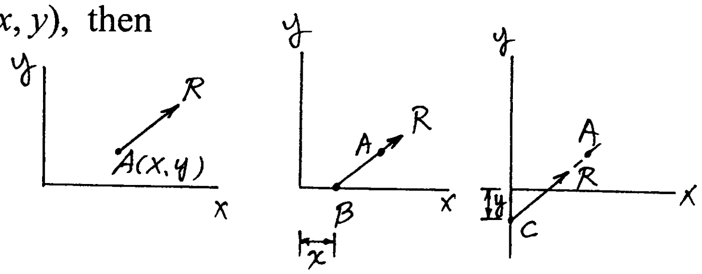
$$R_x = \Sigma F_x, \quad R_y = \Sigma F_y \qquad M_z^R = M_0^R = \Sigma M_0$$

let the position of application is $A(x, y)$, then

$$x R_y - y R_x = M_0^R$$

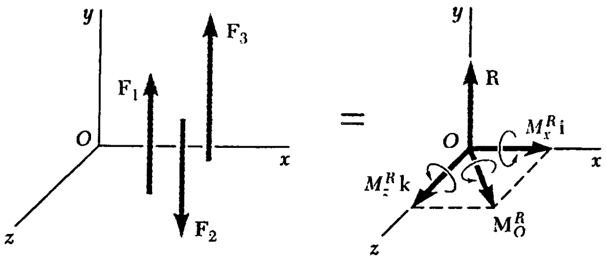
for $y = 0$ $\qquad x = M_0^R / R_y$

for $x = 0$ $\qquad y = M_0^R / R_x$



(3) parallel forces : let all forces parallel to y-axis, of course $\boldsymbol{R}$ should be parallel to y-axis

∵ forces // y-axis, hence the moment about y-axis is zero, only M_x and M_z exist



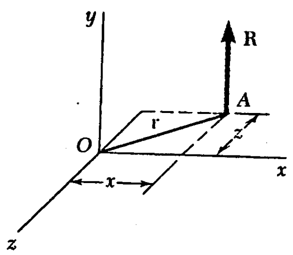
move $\boldsymbol{R}$ to a new point of application $A(x, 0, z)$ to reduced the force-couple system to a single force, then

$$\boldsymbol{r} \times \boldsymbol{R} = \boldsymbol{M}_0^R$$

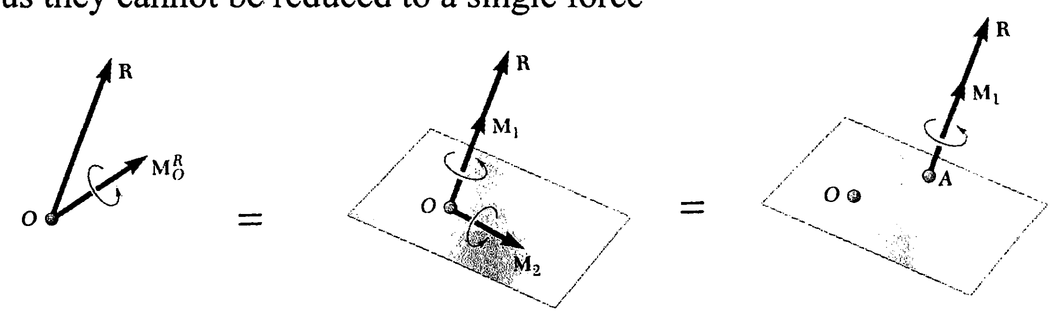
$$(x\,\boldsymbol{i} + z\,\boldsymbol{k}) \times R_y \boldsymbol{j} = M_x^R\,\boldsymbol{i} + M_z^R\,\boldsymbol{k}$$

$$-z R_y\,\boldsymbol{i} + x R_y\,\boldsymbol{k} = M_x^R\,\boldsymbol{i} + M_z^R\,\boldsymbol{k}$$

then $\qquad x = M_z^R / R_y \qquad z = -M_x^R / R_y$



in general case, the resultant $\boldsymbol{R}$ and the couple are not perpendicular, thus they cannot be reduced to a single force



resolving $\boldsymbol{M}_0^R$ into components $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, where $\boldsymbol{M}_1$ along the direction of $\boldsymbol{R}$ and $\boldsymbol{M}_2$ is perpendicular to $\boldsymbol{R}$

move $\boldsymbol{R}$ to a new point of application A, such that the original system of force thus reduces to $\boldsymbol{R}$ and the couple $\boldsymbol{M}_1$, where $\boldsymbol{R}$ and $\boldsymbol{M}_1$ has the same direction, i.e. the couple acting in the plane perpendicular to $\boldsymbol{R}$

this combination of force-couple is called wrench, the line of action of $\boldsymbol{R}$ is known as the axis of the wrench

the magnitude of $\boldsymbol{M}_1$ is the projection of $\boldsymbol{M}_0^{\mathrm{R}}$ on the line of action of $\boldsymbol{R}$, such that

$$M_1 = \boldsymbol{R} \cdot \boldsymbol{M}_0^{\mathrm{R}} / |\boldsymbol{R}|$$

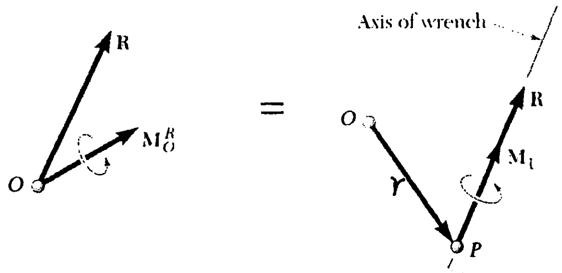
and the ratio $\dfrac{M_1}{R} = \dfrac{\boldsymbol{R} \cdot \boldsymbol{M}_0^{\mathrm{R}}}{R^2} = p$ is called pitch of wrench

i.e.

$$\boldsymbol{M}_1 = p\,\boldsymbol{R}$$

$$\boldsymbol{M}_1 + \boldsymbol{r} \times \boldsymbol{R} = \boldsymbol{M}_0^{\mathrm{R}}$$

$$p\,\boldsymbol{R} + \boldsymbol{r} \times \boldsymbol{R} = \boldsymbol{M}_0^{\mathrm{R}}$$



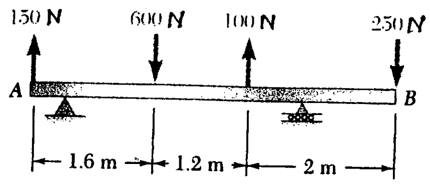
if $\Sigma \boldsymbol{F} = \boldsymbol{R} = 0$, $\Sigma \boldsymbol{M}_0^{\mathrm{R}} = 0$ (particular case), the rigid body is said to be equilibrium

Sample Problem 3.8

(a) equivalent force-couple at A

(b) equivalent force-couple at B

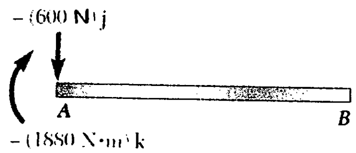
(c) a single force or resultant



(a) $\boldsymbol{R} = \Sigma \boldsymbol{F}$

$= (150 - 600 + 100 - 250)\boldsymbol{j}$

$= -600\boldsymbol{j}$ (N)



$\boldsymbol{M}_A^{\mathrm{R}} = \Sigma(\boldsymbol{r} \times \boldsymbol{F})$

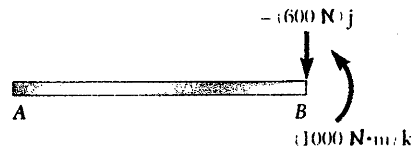
$= 1.6\,\boldsymbol{i} \times (-600\boldsymbol{j}) + 2.8\,\boldsymbol{i} \times 100\boldsymbol{j} + 4.8\,\boldsymbol{i} \times (-250\boldsymbol{j})$

$= -1880\,\boldsymbol{k}$ (N-m)

(b) $\boldsymbol{M}_B^R = \boldsymbol{M}_A^R + \boldsymbol{BA} \times \boldsymbol{R}$

$= -1880\,\boldsymbol{k} + (-4.8\,\boldsymbol{i}) \times (-600\,\boldsymbol{j})$

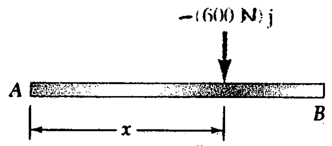
$= 1000\,\boldsymbol{k}$ (N-m)



(c) $\boldsymbol{r} \times \boldsymbol{R} = \boldsymbol{M}_A^R$

$x\,\boldsymbol{i} \times (-600\,\boldsymbol{j}) = -1880\,\boldsymbol{k}$

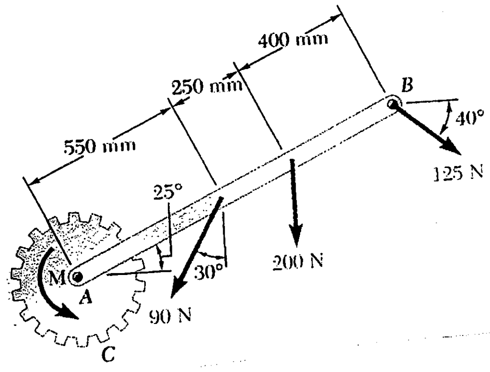
$x = 3.13$ m



Sample Problem 3.9

determine the equivalent force and couple

resolve 125 N and 90 N forces in x and y directions



125 N $\quad$ x-dir. $= 125 \cos 40^{o} = 95.756$ N

y-dir. $= -125 \sin 40^{o} = 80.348$ N

90 N $\quad$ x-dir. $= -45$ N

y-dir. $= -77.942$ N

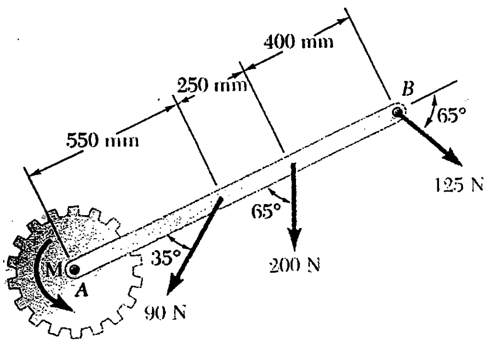
$R_x = \Sigma F_x = 95.756 - 45 = 50.756$ N

$R_y = \Sigma F_y = -80.348 - 200 - 77.942 = -358.29$ N

$R = (50.756^2 + 358.29^2)^{½} = 361.87$

$\tan\theta = 358.29/50.756 \qquad \theta = 81.9^{o}$

$R = 362$ N $\qquad 81.9^{o}$



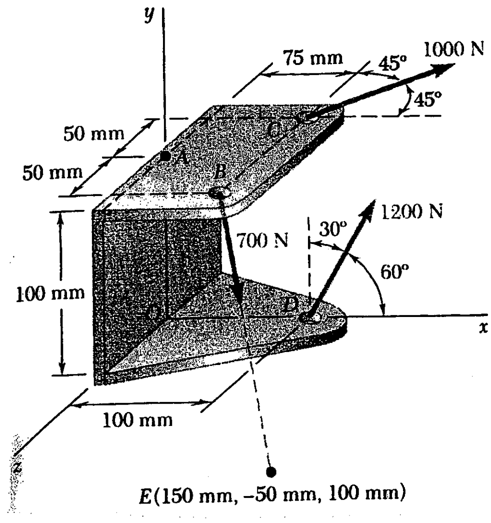
$M_{eq} = -550 \times 90 \sin 35^{o} - 800 \times 200 \sin 65^{o}$

$- 1200 \times 125 \sin 65^{o}$

$= -309$ N-m

Sample Problem 3.10

replace the forces by a force-couple system at A



$B(75, 100, 50) \qquad E(150, -50, 100)$

$BE = 75\,\boldsymbol{i} - 150\,\boldsymbol{j} + 50\,\boldsymbol{k}$ $BE = 175$

$$\lambda_{BE} = \frac{3}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} + \frac{2}{7}\boldsymbol{k}$$

$\boldsymbol{F}_B = 700\,\lambda_{BE} = 300\,\boldsymbol{i} - 600\,\boldsymbol{j} + 200\,\boldsymbol{k}$

$\boldsymbol{F}_C = 707\,\boldsymbol{i} - 707\,\boldsymbol{k}$

$\boldsymbol{F}_D = 600\,\boldsymbol{i} + 1039\,\boldsymbol{j}$

$\boldsymbol{R} = \Sigma\,\boldsymbol{F} = 1067\,\boldsymbol{i} + 439\,\boldsymbol{j} - 507\,\boldsymbol{k}$

$\boldsymbol{AB} = 0.075\,\boldsymbol{i} + 0.05\,\boldsymbol{k} \qquad \boldsymbol{AC} = 0.075\,\boldsymbol{i} - 0.05\,\boldsymbol{k}$

$\boldsymbol{AD} = 0.1\,\boldsymbol{i} - 0.1\,\boldsymbol{j}$

$\boldsymbol{M}_A^R = \boldsymbol{AB} \times \boldsymbol{F}_B + \boldsymbol{AC} \times \boldsymbol{F}_C + \boldsymbol{AD} \times \boldsymbol{F}_D$

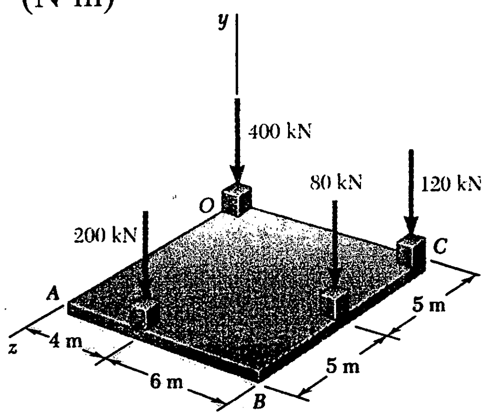
$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0.075 & 0 & 0.05 \\ 300 & -600 & 200 \end{vmatrix} + \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0.075 & 0 & -0.05 \\ 707 & 0 & -707 \end{vmatrix} + \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0.1 & -0.1 & 0 \\ 600 & 1039 & 0 \end{vmatrix}$$

$$= (30\,\boldsymbol{i} - 45\,\boldsymbol{k}) + (17.68\,\boldsymbol{j}) + (163.9\,\boldsymbol{k})$$

$$= 30\,\boldsymbol{i} + 17.68\,\boldsymbol{j} + 118.9\,\boldsymbol{k} \qquad \text{(N-m)}$$

Sample Problem 3.11

determine the magnitude and point of application of the resultant $\boldsymbol{R}$



$\boldsymbol{R} = \Sigma\,\boldsymbol{F}$

$\boldsymbol{M}_0^R = \Sigma\,(\boldsymbol{r} \times \boldsymbol{F})$

$\boldsymbol{r}$ (m)	$\boldsymbol{F}$ (kN)	$\boldsymbol{r} \times \boldsymbol{F}$ (kN-m)
0	$-180\,\boldsymbol{j}$	0
3 i	$-54\,\boldsymbol{j}$	$-162\,\boldsymbol{k}$
$3\,\boldsymbol{i} + 1.5\,\boldsymbol{k}$	$-36\,\boldsymbol{j}$	$54\,\boldsymbol{i} - 108\,\boldsymbol{k}$

$1.2\,\boldsymbol{i} + 3\,\boldsymbol{k} \qquad -90\,\boldsymbol{j} \qquad 270\,\boldsymbol{i} - 108\,\boldsymbol{k}$

$\boldsymbol{R} = -360\,\boldsymbol{j} \qquad \boldsymbol{M}_0^{\mathrm{R}} = 324\,\boldsymbol{i} - 378\,\boldsymbol{k}$

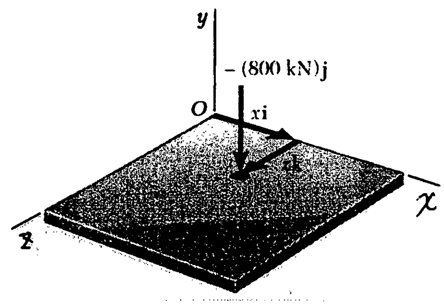
$\boldsymbol{r} = x\,\boldsymbol{i} + z\,\boldsymbol{k}$

$\boldsymbol{r} \times \boldsymbol{R} = \boldsymbol{M}_0^{\mathrm{R}}$

$(x\,\boldsymbol{i} + z\,\boldsymbol{k}) \times (-360\,\boldsymbol{j}) = 324\,\boldsymbol{i} - 378\,\boldsymbol{k}$

$360\,z\,\boldsymbol{i} - 360\,x\,\boldsymbol{k} = 324\,\boldsymbol{i} - 378\,\boldsymbol{k}$

$x = 1.05\ \mathrm{m} \qquad z = 0.9\ \mathrm{m}$

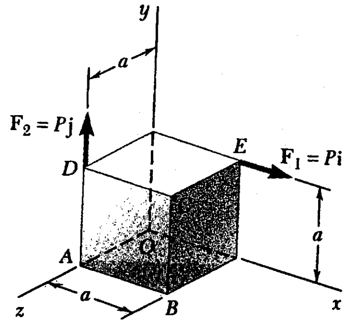


Sample Problem 3.12

replace the two forces into a wrench

determine $\boldsymbol{R}$, p and the intersection

of the wrench axis on y-z plane



$\boldsymbol{r}_{\mathrm{D}} = a\boldsymbol{j} + a\,\boldsymbol{k} \qquad \boldsymbol{r}_{\mathrm{E}} = a\,\boldsymbol{i} + a\boldsymbol{j}$

force-couple at O

$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 = P\,(\boldsymbol{i} + \boldsymbol{j})$

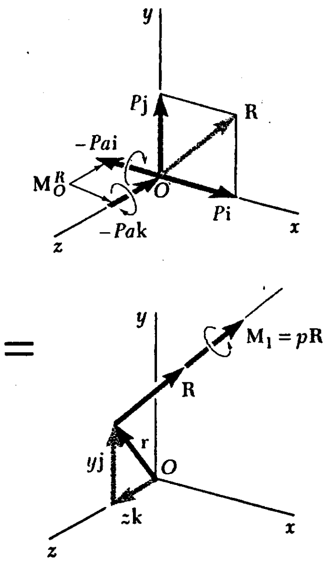
$\boldsymbol{M}_0^{\mathrm{R}} = \boldsymbol{r}_{\mathrm{E}} \times \boldsymbol{F}_1 + \boldsymbol{r}_{\mathrm{D}} \times \boldsymbol{F}_2$

$= a\,P\,(\boldsymbol{i} + \boldsymbol{j}) \times \boldsymbol{i} + a\,P\,(\boldsymbol{j} + \boldsymbol{k}) \times \boldsymbol{j}$

$= -\,a\,P\,(\boldsymbol{i} + \boldsymbol{k})$

$$\text{pitch } p = \frac{\mathrm{R} \cdot \boldsymbol{M}_0^{\mathrm{R}}}{R^2} = \frac{P\,(\boldsymbol{i} + \boldsymbol{j}) \cdot [-\,a\,P\,(\boldsymbol{i} + \boldsymbol{k})]}{(\sqrt{2}\,P)^2}$$

$$= \frac{-\,a\,P^2}{2\,P^2} = -\frac{a}{2}$$



$\boldsymbol{M}_1 = p\,\boldsymbol{R} = -\,a\,P\,(\boldsymbol{i} + \boldsymbol{k})/2$

$\boldsymbol{M}_1 + \boldsymbol{r} \times \boldsymbol{R} = \boldsymbol{M}_0^{\mathrm{R}}$

$\boldsymbol{r} = y\boldsymbol{j} + z\,\boldsymbol{k}$ (on y-z plane)

$-\,a\,P\,(\boldsymbol{i} + \boldsymbol{k})/2 + (y\boldsymbol{j} + z\,\boldsymbol{k}) \times P\,(\boldsymbol{i} + \boldsymbol{j}) = -\,a\,P\,(\boldsymbol{i} + \boldsymbol{k})$

$(-\,a\,P/2 - P\,z)\,\boldsymbol{i} - (a\,P/2 - P\,z)\,\boldsymbol{j} - P\,y\,\boldsymbol{k} = -\,a\,P\,(\boldsymbol{i} + \boldsymbol{k})$

$\Rightarrow \quad y = a \qquad z = a/2$ the position of G is $G(0, a, a/2)$