

4. Constitutive Equation (Infinitesimal)

Definition : Regular Stresses and Displacements

$\sigma_{ij}(x_i, t)$, $u_i(x_i, t)$ are regular in D (region) and B (boundary) for $0 \leq t \leq \infty$ if

$$\sigma_{ij} \in C_1 \quad \text{w. r. t. } x_i$$

$$u_i \in C_1 \quad \text{w. r. t. } x_i \quad \text{and} \quad C_2 \quad \text{w. r. t. } t$$

Definition : Power P (the rate of work done)

$$P =$$

$$\text{where } \dot{u}_i =$$

Definition : Kinetic Energy K

$$K =$$

Postulate : There exists an internal energy function ε and an internal energy density e related by

$$\varepsilon = \int_V e \, dV \quad e : \text{energy density per unit volume}$$

Postulate : Energy Balance

$$\dot{K} + \dot{\varepsilon} =$$

where L is the rate of loss of energy in a non mechanical manner, such as heat transfer, chemical reaction, etc.

Definition : Elastic Material

an elastic material is one of which $L = 0$ (no energy loss)

Theorem : Given an continuous elastic medium in $D + B$, with density $\rho(x_i)$, having body force $f_i(x_i)$ and surface traction $T_i(x_i)$ applied, and undergoing a motion $u_i(x_i, t)$, with associated stress $\sigma_{ij}(x_i, t)$. Let σ_{ij} and u_i be regular, then

$$\dot{\mathcal{E}} =$$

or $P =$

proof $\int_B T_i \dot{u}_i dA =$

$$=$$

$$=$$

$$\therefore \dot{u}_{i,j} = \dot{e}_{ij} + \dot{\omega}_{ij} \quad \text{and} \quad \sigma_{ij} \dot{\omega}_{ij} = 0$$

$$\sigma_{ij,j} = \rho \ddot{u}_i - f_i \quad (\text{eq. of motion, } \rho \ddot{u}_i \text{ is inertia term})$$

$$\therefore \int_B T_i \dot{u}_i dA =$$

$$\therefore P =$$

$$=$$

$$=$$

$$=$$

$$\therefore \dot{\varepsilon} =$$

Theorem : Given the condition same as the above theorem, then

$$e = e(e_{ij}) \quad \text{and} \quad \sigma_{ij} = e / e_{ij}$$

proof $\therefore \mathcal{E}(D^*) =$

where D^* is any subset of the region D

since D^* is arbitrary, thus

$$\sigma_{ij} \dot{e}_{ij} =$$

or $\sigma_{ij} d e_{ij} =$ [exact differential]

$$\therefore e =$$

$$d e =$$

Note : this relation is usually used to define an elastic material,
since it is independent of regularity of D

Generalized Hooke's Law

when an elastic medium is maintained at a fixed temperature, there is a one-to-one analytic relation between stress tensor σ_{ij} and strain tensor e_{ij}

i.e. $\sigma_{ij} = f_{ij}(e_{ij}, x_i)$

and that σ_{ij} vanish when all strain e_{ij} are zero, i.e. initial unstrained state is unstressed

Definition : Homogeneous

A medium is said to be homogeneous if f_{ij} does not depend explicitly on location x_i

$$\text{i.e.} \quad \sigma_{ij} = f_{ij}(e_{ij})$$

Definition : Isotropic Medium

A material is isotropic if f_{ij} are invariant under rotation at each location x_i .

$$\text{i.e.} \quad \sigma_{ij} =$$

$$\sigma'_{ij} = \quad \text{[rotation]}$$

$$\text{or} \quad f_{ij} = f'_{ij} \quad \text{but} \quad \sigma_{ij} \neq \sigma'_{ij}$$

$$\text{e.g.} \quad \sigma_{11} = E e_{11} \quad \text{and} \quad \sigma'_{11} = E e'_{11}$$

$$\text{but} \quad \sigma_{11} \neq \sigma'_{11} \quad \text{or} \quad e_{11} \neq e'_{11}$$

isotropic means same relation between σ_{ij} and e_{ij}

If the function f_{ij} are expanded in the series in e_{ij} and only the linear terms retained in the expansions

$$\sigma_{ij} =$$

the coefficients C_{ijkl} may be vary from point to point

for homogeneous medium, C_{ijkl} are independent of the position of the point

for isotropic medium, C_{ijkl} are the components of an isotropic tensor

Theorem : For a linear medium

$$C_{ijkl} = C_{jikl} = C_{ijlk}$$

$$\text{proof (a)} \quad \sigma_{ij} = \quad \text{and} \quad \sigma_{ji} =$$

$$(\sigma_{ij} - \sigma_{ji}) =$$

for arbitrary e_{kl}

$$\therefore C_{ijkl} = C_{jikl}$$

$$(b) \quad C_{ijkl} =$$

sym. w.r.t k and l skew-sym.

$$C_{ijkl} e_{kl} =$$

$$=$$

$$\text{thus} \quad C_{ijkl} = C_{ijlk}$$

for an arbitrary strain tensor e_{kl}

$$\sigma_{11} =$$

similarly for others

there are six stress components in terms of six strain components

so we see that the 81 C_{ijkl} reduce to at most 36 independent C_{ijkl}

$$\text{if } \det | C_{ijkl} | \neq 0$$

then we can solve for e_{ij} , i.e.

$$e_{ij} =$$

$$\text{with} \quad K_{ijkl} = K_{jikl} = K_{ijlk}$$

Theorem : C_{ijkl} is a component of 4th order tensor

proof $\sigma'_{\alpha\beta} =$

$$=$$

$$=$$

$$=$$

$$\therefore C_{\alpha\beta\gamma\delta} =$$

i.e. C_{ijkl} is a 4th order tensor

Theorem : A necessary and sufficient condition that a linear medium be elastic is that

$$C_{ijkl} = C_{klij}$$

in this case, $e(e_{ij}) =$

proof : necessary : linear + elastic $\rightarrow C_{ijkl} = C_{klij}$

$\therefore$ medium is elastic $\sigma_{ij} = e / e_{ij}$

$$\frac{\sigma_{ij}}{e_{kl}} = \text{_____}$$

$$C_{ijkl} = \text{_____}$$

also $C_{klij} = \text{_____} =$

sufficiency : linear + $(C_{ijkl} = C_{klij}) \rightarrow$ elastic

assume $C_{ijkl} = C_{klij}$

linear medium $\sigma_{ij} =$

consider $e =$

$$\begin{aligned}
 \frac{e}{e_{ij}} &= \\
 &= \\
 &= \\
 \therefore \frac{e}{e_{ij}} &=
 \end{aligned}$$

$\therefore$ the medium is elastic

For general case of a linear elastic medium

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$$

i.e. 21 independent constants (functions) for C_{ijkl} for anisotropic linear elastic medium

Theorem : For a linear elastic medium

$$e(\sigma_{ij}) = \frac{1}{2} K_{ijkl} \sigma_{ij} \sigma_{kl} \quad \text{and} \quad e_{ij} = e / \sigma_{ij}$$

proof linear $e_{ij} =$

$$e =$$

also

$$\begin{aligned}
 \frac{e}{\sigma_{\alpha\beta}} &= \\
 &= \\
 &=
 \end{aligned}$$

Definition : A function $F(x_i)$ is said to be positive definite if $F > 0$ for all $x_i \neq 0$ and $F = 0$ for all $x_i = 0$

Theorem : The strain energy is positive definite

proof : the external forces and body forces acting on any subregion D^* of D must do positive work

$$\geq 0$$

$$\geq 0$$

$$\geq 0$$

$$\geq 0$$

$$\geq 0 \quad [\sigma_{ij} \omega_{ij} = 0]$$

for arbitrary D^* $\sigma_{ij} e_{ij} \geq 0$

so that $e \geq 0$ places still restrictions of C_{ijkl} as follows

$$\begin{aligned} e(e_{ij}) &= & \text{for } e_{ij} \neq 0 \\ &= & \text{for } e_{ij} = 0 \end{aligned}$$

$\therefore$ the necessary and sufficient condition that the quadratic form $e(e_{ij})$ be positive definite is that each minor det. of C_{ijkl} is positive

Theorem : For a linear isotropic medium, the constitutive equation can be written as

$$\sigma_{ij} =$$

where μ and λ are called Lamé' constants

proof : for isotropic medium, C_{ijkl} is a 4th order isotropic tensor

$$C_{ijkl} =$$

$$\sigma_{ij} =$$

$$=$$

if x_i are direction along the principle strain axes, then $e_{12} = e_{23} = e_{13} = 0$,
then for $i \neq j$

$$\sigma_{ij} =$$

i.e. the principal axes of stress are coincident with the principal axes of strain, if the medium is isotropic

Theorem : A linear isotropic medium is elastic

$$\text{proof : } \quad C_{ijkl} =$$

$$C_{klij} =$$

$$\therefore C_{ijkl} =$$

Theorem : For an linear elastic isotropic medium, the strain tensor can be expressed in stress tensor as

$$e_{ij} =$$

where E is modulus of elasticity (Young' s modulus)

ν is called the Poisson' s ratio

$$\text{proof : } \quad \therefore \sigma_{ij} =$$

let $i = j$ $\sigma_{ii} =$ _____

$\therefore$ $e_{ii} =$ _____

also $e_{ij} =$ _____

if $\sigma_{11} = \sigma$, others = 0, then $\sigma_{kk} = \sigma$

$\therefore$ $e_{11} =$ _____ - _____ - _____

$=$ _____

or $\sigma_{11} =$ _____

$\therefore$ $E =$ _____ Young's modulus

and $e_{22} =$ - _____ - _____

- _____ = $\frac{\text{_____} \lambda}{\text{_____}}$ = _____ =

where ν is the Poisson's ratio

for $\sigma_{12} \neq 0$, others = 0

$e_{12} =$ _____ or $\sigma_{12} =$ _____

where μ is shear modulus

also $\sigma_{ii} =$

$\therefore K =$ is the bulk modulus

$$\frac{E}{2(1+\nu)} = \underline{\hspace{2cm}}$$

$$\frac{E}{3(1-2\nu)} = \underline{\hspace{2cm}}$$

$\therefore \mu =$

$$K =$$

if the Lamé' constants λ and μ as independent constants

then $E =$

$$\nu =$$

$$G =$$

$$K =$$

if E and ν are chosen as independent constants

then $G =$

$$K =$$

only two independent constants may be used, we can use “ λ and μ ” or “ E and ν ”

$$\begin{aligned} \therefore e_{ij} &= \\ &= \\ &= \\ &= \end{aligned}$$

$$\begin{aligned} \text{let } \sigma_{ij} &= \\ \text{and } \sigma_{ii} &= \\ &= -3p \end{aligned}$$

$$\frac{\Delta V}{V} = e_{ii} = - \frac{p}{K} = - \frac{1}{3} \frac{p}{K} \implies K > 0$$

if $e_{ii} = 0$ for p fixed, $\rightarrow K = \infty$ incompressibility
but it is impossible

$$\begin{aligned} \therefore K &= \\ K = \infty &\rightarrow \nu = 1/2 \\ \therefore \text{ we expect } \nu &< 1/2 \end{aligned}$$

Theorem : A necessary and sufficient condition for $e(e_{ij})$ be positive definite is that

$$E > 0 \quad -1 < \nu < 1/2$$

for linear isotropic materials

proof : necessary $e(e_{ij}) > 0$ unless all $\sigma_{ij} = 0$

$$e(e_{ij}) =$$

$$=$$

$$=$$

$$=$$

$$\text{case i} \quad \sigma_I \neq 0 \quad \sigma_{II} = \sigma_{III} = 0$$

$$e =$$

$$\text{case ii} \quad \sigma_I = -(\sigma_{II} + \sigma_{III}) \neq 0$$

$$e =$$

$$\text{case iii} \quad \sigma_I = \sigma_{II} = \sigma_{III} = \sigma$$

$$e =$$

sufficiency: assume $E > 0$ and $-1 < \nu < 1/2$

$$\therefore \mu =$$

we write e in terms of invariant I_1

$$e =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$> 0$$

unless $I_1 = 0$ and $\sigma_I = \sigma_{II} = \sigma_{III}$

i.e. all $\sigma_{ij} = 0$, $e = 0$ for all e-value vanish

Theorem : Let (σ_{ij}', e_{ij}') and $(\sigma_{ij}'', e_{ij}'')$ both satisfy $\sigma_{ij} = C_{ijkl} e_{kl}$ for same C_{ijkl} , and the medium be elastic, then

$$\sigma_{ij}'' e_{ij}' = \sigma_{ij}' e_{ij}''$$

proof $\sigma_{ij}'' e_{ij}' =$

$$=$$

$$=$$

$$=$$

Theorem : Let (σ_{ij}', e_{ij}') and $(\sigma_{ij}'', e_{ij}'')$ meet hypothesis of above theorem, and

let $\sigma_{ij} = \sigma_{ij}' + \sigma_{ij}''$

$$e_{ij} = e_{ij}' + e_{ij}''$$

then $e(e_{ij}) = e(e_{ij}') + e(e_{ij}'') + 2 e(e_{ij}', e_{ij}'')$

proof $e =$

$$=$$

$$=$$