

Elliptical Coordinates :

$$z = f(\omega) = c \cosh \omega$$

$$z = x_1 + i x_2 \quad x_1 = c \cosh y_1 \cos y_2$$

$$\omega = y_1 + i y_2 \quad x_2 = c \sinh y_1 \sin y_2$$

along $y_2 = \eta = \text{constant}$ curve

$$dx_1 = \frac{\partial x_1}{\partial y_1} dy_1 \quad dx_2 = \frac{\partial x_2}{\partial y_1} dy_1$$

the arc length is

$$dS_{y_1}^2 = dx_1^2 + dx_2^2 = \left[\left(\frac{\partial x_1}{\partial y_1} \right)^2 + \left(\frac{\partial x_2}{\partial y_1} \right)^2 \right] dy_1^2$$

$$\frac{\partial z}{\partial y_1} = \frac{\partial x_1}{\partial y_1} + i \frac{\partial x_2}{\partial y_1} = \frac{\partial z}{\partial \omega} \frac{\partial \omega}{\partial y_1} = f'(\omega)$$

$$\begin{aligned} \text{let} \quad f'(\omega) &= J e^{ia} \quad J, a \text{ are real} \\ &= J (\cos a + i \sin a) \end{aligned}$$

$$\text{then} \quad \frac{\partial x_1}{\partial y_1} = J \cos a \quad \frac{\partial x_2}{\partial y_1} = J \sin a$$

$$dS_{y_1} = J dy_1$$

$$\text{slope} = \frac{dx_2}{dx_1} = \tan a$$

similarly, along $y_1 = \xi = \text{constant}$ line

$$dS_{y_2} = J d\eta = J dy_2$$

$$\text{slope} = \frac{dx_2}{dx_1} = -\cot a$$

for the elliptic coordinate

$$z = f(\omega) = c \cosh \omega$$

$$\begin{aligned} f'(\omega) &= c \sinh \omega = c \sinh y_1 \cos y_2 + i c \cosh y_1 \sin y_2 \\ &= J e^{ia} = J (\cos a + i \sin a) \end{aligned}$$

then $J \cos a = c \sinh y_1 \cos y_2$

$$J \sin a = c \cosh y_1 \sin y_2$$

and $J^2 = \frac{1}{2} c^2 (\cosh 2y_1 - \cos 2y_2)$

$$\tan a = \coth y_1 \tan y_2$$

the stress components between x_1 - x_2 system and y_1 - y_2 system are

$$\sigma_{y_1} + \sigma_{y_2} = \sigma_{11} + \sigma_{22}$$

$$\begin{aligned} \sigma_{y_2} - \sigma_{y_1} + 2i \tau_{y_1 y_2} &= \\ e^{2ia} (\sigma_{22} - \sigma_{11} + 2i \sigma_{12}) \end{aligned}$$

e^{2ia} can be found by the following procedure

$$f'(\omega) = J e^{ia} \quad \text{and} \quad \overline{f'}(\overline{\omega}) = J e^{-ia}$$

$$\text{then } e^{2ia} = \frac{f'(\omega)}{\overline{f'}(\overline{\omega})} = \frac{\sinh \omega}{\sinh \overline{\omega}}$$

thus the stress components in elliptical coordinate related to the complex functions can be obtained

$$\sigma_{y_1} + \sigma_{y_2} = 4 \operatorname{Re} \psi'(z)$$

$$\sigma_{y_2} - \sigma_{y_1} + 2i \tau_{y_1 y_2} = 2 e^{2ia} [\overline{z} \psi''(z) + \chi''(z)]$$

and the displacement components are

$$u_{y1} + i u_{y2} = e^{-ia} (u_1 + i u_2)$$

for plane stress case

$$2 \mu (u_{y2} - i u_{y1}) = e^{ia} \left[\frac{3 - \nu}{1 + \nu} \bar{\psi}(\bar{z}) - \bar{z} \psi'(z) - \chi'(z) \right]$$

Elliptic Hole in Uniformly Stressed Thin Plate

$$a = c \cosh \xi \quad b = c \sinh \xi$$

if $\xi = 0$, it reduces to a line crack with length $2c$

boundary conditions

$$\sigma_{11} = \sigma_{22} = S \quad \text{at } \infty$$

$$\sigma_{y1} = \tau_{y1y2} = 0 \quad \text{at } y_1 = \xi$$

$$\therefore \quad 2S = \sigma_{11} + \sigma_{22} = 4 \operatorname{Re} [\psi'(z)]$$

$$\text{i.e.} \quad S = 2 \operatorname{Re} [\psi'(z)] \quad \text{at } \infty$$

$$\tau_{y1y2} = 0 \quad \text{at } \infty$$

$$\Rightarrow \quad \bar{z} \psi''(z) + \chi''(z) = 0 \quad \text{at } \infty$$

since the stresses and displacements are, for continuity, to be periodic in y_2 with period 2π , thus ψ and χ will give a stress function with the same periodicity, and such forms are

$$\sinh n \omega = \sinh n y_1 \cos n y_2 + i \cosh n y_1 \sin n y_2$$

$$\cosh n \omega = \cosh n y_1 \cos n y_2 + i \sinh n y_1 \sin n y_2$$

where n is integer

also the function $\chi(z) = B c^2 \omega$ is suitable to the problem

$$\therefore \chi'(z) \text{ and } \chi''(z) \Rightarrow \sinh \omega \quad \text{and} \quad \cosh \omega$$

$$\text{since } \chi(z) = B c^2 \omega$$

$$\begin{aligned} \chi'(z) &= \frac{\partial \chi}{\partial \omega} \frac{\partial \omega}{\partial z} & [z = c \cosh \omega] \\ &= B c^2 \frac{1}{c \sinh \omega} & \partial z / \partial \omega = c \sinh \omega \end{aligned}$$

$\therefore$ stress only in terms of χ'' , and displacement only in terms of χ' , so they have the same value for $y_1 = 0$ and 2π

$$\text{let } \psi(z) = A c \sinh \omega$$

$$\chi(z) = B c^2 \omega \quad A, B \text{ are constants}$$

$$\text{then } \psi'(z) = A c \cosh \omega \frac{\partial \omega}{\partial z} = A \coth \omega$$

$$\text{at } y_1 = \infty \quad \coth \omega \rightarrow 1$$

$$\therefore S = 2A \quad \text{or} \quad A = \frac{1}{2} S$$

$$\text{and } \psi''(z) = -\frac{A}{c} \frac{1}{\sinh^3 \omega}$$

$$\frac{\partial}{\partial z} \psi''(z) = -A \frac{\cosh \omega}{\sinh^3 \omega}$$

$$\therefore \quad \chi(z) = B c^2 \omega$$

$$\text{then} \quad \chi'(z) = \frac{B c}{\sinh \omega} \quad \chi''(z) = -B \frac{\cosh \omega}{\sinh^3 \omega}$$

$$\text{at } \infty \quad \bar{z} \psi''(z) \quad \text{and} \quad \chi''(z) \Rightarrow 0$$

$$\therefore \quad \tau_{y_1 y_2} = 0 \quad \text{at } \infty \quad \text{is satisfied}$$

$$\therefore \quad \sigma_{y_1} + \sigma_{y_2} = 2 [\psi'(z) + \bar{\psi}'(\bar{z})]$$

$$\sigma_{y_2} - \sigma_{y_1} + 2i \tau_{y_1 y_2} = 2 e^{2i\alpha} [\bar{z} \psi''(z) + \chi''(z)]$$

$$\therefore \quad \sigma_{y_1} - i \tau_{y_1 y_2} = \psi'(z) + \bar{\psi}'(\bar{z}) - e^{2i\alpha} [\bar{z} \psi''(z) + \chi''(z)]$$

$$= A \left(\frac{\cosh \omega}{\sinh \omega} + \frac{\cosh \bar{\omega}}{\sinh \bar{\omega}} \right) + \frac{\sinh \omega}{\sinh \bar{\omega}} \left(A \frac{\cosh \bar{\omega}}{\sinh^3 \omega} + B \frac{\cosh \omega}{\sinh^3 \omega} \right)$$

at the boundary of the elliptic hole, $y_1 = \xi$

$$\text{then} \quad \omega + \bar{\omega} = 2\xi \quad \text{or} \quad \bar{\omega} = 2\xi - \omega$$

$$\sigma_{y_1} - i \tau_{y_1 y_2} = \frac{1}{\sinh^2 \omega \sinh \bar{\omega}} (A \cosh 2\xi + B) \cosh \omega$$

$$\text{at } y_1 = \xi, \quad \sigma_{y_1} = \tau_{y_1 y_2} = 0,$$

$$\therefore \quad B = -A \cosh 2\xi = -\frac{1}{2} S \cosh 2\xi$$

thus we have

$$\psi(z) = \frac{1}{2} S c \sinh \omega$$

$$\chi(z) = -\frac{1}{2} S c^2 \cosh 2\xi \omega$$

the normal stresses are

$$\begin{aligned}\sigma_{y1} + \sigma_{y2} &= 4 \operatorname{Re} \psi'(z) \\ &= 2 S \operatorname{Re} (\coth \omega) \\ &= \frac{2 S \sinh 2y_1}{\cosh 2y_1 - \cos 2y_2}\end{aligned}$$

at $y_1 = \xi \quad \sigma_{y1} = 0$

$$\sigma_{y2} = \frac{2 S \sinh 2\xi}{\cosh 2\xi - \cos 2y_2}$$

maximum σ_{y2} occurs at $\cos 2y_2 = 1 \quad [y_2 = 0^\circ]$

minimum σ_{y2} occurs at $\cos 2y_2 = -1 \quad [y_2 = 90^\circ]$

$$(\sigma_{y2})_{\max} = \frac{2 S \sinh 2\xi}{\cosh 2\xi - 1} = 2 S \frac{a}{b}$$

$$(\sigma_{y2})_{\min} = \frac{2 S \sinh 2\xi}{\cosh 2\xi + 1} = 2 S \frac{b}{a}$$

for $a = b$ [circular hole], $\sigma_{y2} = 2 S = \text{constant along the hole boundary}$

Elliptic Hole in a plate under simple tension

stress transformation

$$\begin{aligned}\sigma_{11}' + \sigma_{22}' &= \sigma_{11} + \sigma_{22} \\ \sigma_{22}' - \sigma_{11}' + 2i\sigma_{12}' &= e^{2i\beta} (\sigma_{22} - \sigma_{11} + 2i\sigma_{12})\end{aligned}$$

at $\infty \quad \sigma_{11}' = S \quad \sigma_{22}' = \sigma_{12}' = 0$

thus we have

$$\sigma_{11} + \sigma_{22} = S$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = -S e^{-2i\beta}$$

i.e. $4 \operatorname{Re} [\psi'(z)] = S$

$$2 [\bar{z} \psi''(z) + \chi''(z)] = -S e^{-2i\beta}$$

at boundary of the hole

$$y_1 = \xi \quad \sigma_{y1} = \tau_{y1y2} = 0$$

Stevenson proposed a pair of complex functions in 1945

$$4 \psi(z) = A c \cosh \omega + B c \sinh \omega$$

$$4 \chi(z) = C c^2 \omega + D c^2 \cosh 2\omega + E c^2 \sinh 2\omega$$

the five constants A, B, C, D and E are solved by the five boundary conditions

$$\sigma_{11}' = S \quad \sigma_{22}' = \sigma_{12}' = 0 \quad \text{at } \infty$$

$$\sigma_{y1} = \tau_{y1y2} = 0 \quad \text{at } y_1 = \xi$$

then we have

$$A = S e^{2\xi} \cos 2\beta$$

$$B = S (1 - e^{2\xi - 2i\beta})$$

$$C = -S (\cosh 2\xi - \cos 2\beta)$$

$$D = -\frac{1}{2} S e^{2\xi} \cosh 2(\xi + i\beta)$$

$$E = \frac{1}{2} S e^{2\xi} \sinh 2(\xi + i\beta)$$

at elliptic hole boundary $y_1 = \xi$

$$\sigma_{y1} = 0$$

$$\sigma_{y2} = 4 \operatorname{Re} \psi'(z) = S \frac{\sinh 2\xi + \cos 2\beta - e^{2\xi} \cos 2(\beta - y_2)}{\cosh 2\xi - \cos 2y_2}$$

$$\text{if } \beta = \pi/2$$

$$\sigma_{y2} = S e^{2\xi} \left[\frac{\sinh 2\xi (1 + e^{-2\xi})}{\cosh 2\xi - \cos 2y_2} - 1 \right]$$

maximum values occur at $\cos 2y_2 = 1$, i.e. $y_2 = 0, \pi$

thus the maximum normal stress is

$$(\sigma_{y2})_{\max} = S \left(1 + 2 \frac{a}{b} \right)$$

$$\text{if } b \rightarrow 0, \text{ elliptic hole} \rightarrow \text{crack} \quad \Rightarrow \quad \text{Williams Solution}$$

similarly, for $\beta = 0^\circ$

$$(\sigma_{y2})_{\max} = S \left(1 + 2 \frac{b}{a} \right)$$

$$\text{if } b \rightarrow 0, \quad (\sigma_{y2})_{y1=\xi} = S$$

for circular hole $a = b$

$$\sigma_{\max} = 3S$$

Williams Solution (series expansion)

expand $\psi(z)$ and $\chi(z)$ as Laurent series

$$\psi(z) = \sum_{n=0}^{\infty} A_n z^{\lambda_n}$$

$$\chi(z) = \sum A_n z^{\lambda_n+1}$$

$$n=0$$

$$\text{where } z = x_1 + i x_2 = r e^{i\theta}$$

$$\begin{aligned} \therefore \Phi &= \operatorname{Re} [\bar{z} \psi(z) + \chi(z)] \\ &= \operatorname{Re} \left[r e^{-i\theta} \sum_{n=0}^{\infty} A_n r^{\lambda_n} e^{i\lambda_n \theta} + \sum_{n=0}^{\infty} B_n r^{\lambda_n+1} e^{i(\lambda_n+1)\theta} \right] \\ &= \operatorname{Re} \left\{ \sum_{n=0}^{\infty} r^{\lambda_n+1} [A_n \cos(\lambda_n-1)\theta + i A_n \sin(\lambda_n-1)\theta + \right. \\ &\quad \left. B_n \cos(\lambda_n+1)\theta + i B_n \sin(\lambda_n+1)\theta] \right\} \end{aligned}$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 2 [\psi'(z) + \bar{\psi}'(\bar{z})] \quad (1)$$

$$\sigma_{\theta\theta} - \sigma_{rr} + 2i \tau_{r\theta} = 2 e^{2i\theta} [\bar{z} \psi''(z) + \chi''(z)] \quad (2)$$

$$2\mu (u_r + i u_\theta) = e^{-i\theta} [\kappa \psi(z) - \bar{z} \bar{\psi}'(\bar{z}) - \bar{\chi}'(\bar{z})] \quad (3)$$

$$\kappa = 3 - 4\nu \quad \text{for plane strain}$$

$$\kappa = \frac{3 - \nu}{1 + \nu} \quad \text{for plane stress}$$

$$(1) + (2)$$

$$\begin{aligned} \sigma_{\theta\theta} + i \tau_{r\theta} &= \sum_{n=0}^{\infty} \lambda_n r^{\lambda_n-1} \{ e^{i\lambda_n \theta} [e^{i\theta} + (\lambda_n - 1) e^{-i\theta}] A_n \\ &\quad + e^{i\theta} [e^{-i\lambda_n \theta} + (\lambda_n + 1) e^{i\lambda_n \theta}] B_n \} = 0 \end{aligned}$$

$$\text{on boundary } \theta = \pm \pi$$

this is an eigenvalue problem, the characteristic equation is

$$\sin 2 \pi \lambda_n = 0$$

i.e. $\lambda_n = \frac{1}{2} n \quad n = 0, 1, 2, \dots$

but $n = 0 \Rightarrow \text{stresses} = 0 \quad [\text{trivial solution}]$

$\therefore n = 1, 2, \dots$

$$\lambda_n A_n + (-1)^n \bar{A}_n + (\lambda_n + 1) B_n = 0$$

for $n = 1$

$$3 B_1 = 2 A_1 - \bar{A}_1 \quad A_1 = a_1 + i a_2$$

$$\psi(z) = A_1 z^{1/2}$$

$$\chi(z) = B_1 z^{3/2}$$

for $n = 1, 2, \dots$

$$\sigma_{rr} = \frac{1}{2 r^{1/2}} [a_1 (3 - \cos \theta) \cos \frac{\theta}{2} - a_2 (3 \cos \theta - 1) \sin \frac{\theta}{2}] + \dots$$

$$\sigma_{\theta\theta} = \frac{1}{2 r^{1/2}} [a_1 (1 + \cos \theta) \cos \frac{\theta}{2} - a_2 (3 \sin \theta) \sin \frac{\theta}{2}] + \dots$$

$$\sigma_{r\theta} = \frac{1}{2 r^{1/2}} [a_1 \sin \theta \cos \frac{\theta}{2} - a_2 (3 \cos \theta - 1) \cos \frac{\theta}{2}] + \dots$$

and

$$u_r = \frac{r^{1/2}}{4 \mu} \{ a_1 [(2 \kappa - 1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2}] + a_2 [(2 \kappa - 1) \sin \frac{\theta}{2} - 3 \sin \frac{3\theta}{2}] \} + \dots$$

$$u_{\theta} = \frac{r^{1/2}}{4\mu} \left\{ a_1 \left[-(2\kappa + 1) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right] + a_2 \left[(2\kappa + 1) \cos \frac{\theta}{2} - 3 \cos \frac{3\theta}{2} \right] \right\} + \dots$$

for symmetric problem

$$\begin{aligned} \sigma_{11} &= \frac{a_1}{r^{1/2}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] + \dots \\ \sigma_{22} &= \frac{a_1}{r^{1/2}} \cos \frac{\theta}{2} \left[1 + \cos \frac{\theta}{2} \sin \frac{3\theta}{2} \right] + \dots \\ \sigma_{12} &= \frac{a_1}{r^{1/2}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \sin \frac{3\theta}{2} + \dots \end{aligned}$$

for skew-symmetric problem

$$\begin{aligned} \sigma_{11} &= \frac{a_1}{r^{1/2}} \sin \frac{\theta}{2} \left[1 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right] + \dots \\ \sigma_{22} &= -\frac{a_1}{r^{1/2}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + \dots \\ \sigma_{12} &= -\frac{a_1}{r^{1/2}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] + \dots \end{aligned}$$

and the displacements for symmetric and skew-symmetric problems can also be obtained.