

7. Problem of Saint-Venant

Extension, Pure Bending and Torsion

To analysis of the behavior of elastic beams bounded by a cylindrical surface (which is termed lateral surface of the beam) and by a pair of planes normal to the lateral surface.

If one accepts the principle of Saint-Venant and considers a beam whose length is large in comparison with the linear dimensions of its cross section, then the actual distribution of stresses over the ends has no appreciable influence on the character of the solution in portions of the beam sufficiently far removed from the ends.

Thus, our problem can be solved by the principle of superposition of the four elementary problems

1. extension of the cylinder by longitudinal forces applied at the end
2. Bending of the cylinder by couples whose moments are lie in the planes of the bases of the cylinder
3. Torsion of the cylinder by couples whose moments are normal to the bases of the cylinder
4. Flexure of the cylinder by transverse force applied at one end of the cylinder

Our general plan of attack upon the four elementary problems listed above is that of the Saint-Venant semi-inverse method of solution.

Consider a beam of length l bounded by a cylindrical surface

D : domain of the cylinder

B_L : lateral boundary

R : cross section (not necessary simple connected)

C : contour of R

assume no body force $f_i = 0$

boundary condition on B_L : $T_i = 0$

on $x_3 = 0, l$: $\sigma_{3i} = \sigma_{3i}^*$

$T_i = 0$ on B_L means

$$\sigma_{11} \cos(\mathbf{e}_1, \mathbf{n}_i) + \sigma_{12} \cos(\mathbf{e}_2, \mathbf{n}_i) = 0$$

$$\sigma_{21} \cos(\mathbf{e}_1, \mathbf{n}_i) + \sigma_{22} \cos(\mathbf{e}_2, \mathbf{n}_i) = 0$$

$$\sigma_{31} \cos(\mathbf{e}_1, \mathbf{n}_i) + \sigma_{32} \cos(\mathbf{e}_2, \mathbf{n}_i) = 0$$

on end faces

$$\int_R \sigma_{3i} dA = L_i \quad \text{end loading}$$

$$\int_R x_2 \sigma_{33} dA = M_1 \quad \text{bending}$$

$$\int_R x_1 \sigma_{33} dA = -M_2 \quad \text{bending}$$

$$\int_R (x_1 \sigma_{32} - x_2 \sigma_{31}) dA = M_3 \quad \text{torsion}$$

where L_i, M_i are net resultant force and moment due to σ_{3i}^*

four cases are considered

(A) $L_3 = L, L_1 = L_2 = M_i = 0$ pure extension

(B) $M_2 = M, M_1 = M_3 = L_i = 0$ pure bending

(C) $M_3 = M, M_1 = M_2 = L_i = 0$ pure torsion

(D) $L_1 = L, L_2 = L_3 = M_i = 0$ bending due to transverse load

(A) Pure extension (with $f_i = 0$) $L_3 = L$ others = 0

try solution $\sigma_{33} = L / A$

others = 0

clearly satisfy all the boundary conditions and the equation of
equilibrium

by stress-strain relation

$$e_{33} = L / E A \quad e_{11} = e_{22} = -\nu L / E A$$

other $e_{ij} = 0$

by strain-displacement relation

$$u_1 = -\nu L x_1 / E A + C_1$$

$$u_2 = -\nu L x_2 / E A + C_2$$

$$u_3 = L x_3 / E A + C_3$$

C_i are rigid body displacements

all field equations (5.1), (5.2'), (5.3') are satisfied

(A') Beam stretched by its own weight (body force must be considered)

$$f_1 = f_2 = 0 \quad f_3 = -\rho g x_3$$

where ρ is the density of the beam.

the stress acting on each section
of the beam are produced by the
weight of the lower part of the beam,
we shall

suppose that the stresses are distributed uniformly

$$\text{try } \sigma_{33} = \rho g x_3 \quad \text{other } \sigma_{ij} = 0$$

(5.2') is satisfied

$$\text{then} \quad e_{33} = \rho g x_3 / E = u_{3,3} \quad (1)$$

$$e_{11} = e_{22} = -\nu \rho g x_3 / E = u_{1,1} = u_{2,2} \quad (2, 3)$$

$$2 e_{12} = u_{1,2} + u_{2,1} = 0 \quad (4)$$

$$2 e_{13} = u_{1,3} + u_{3,1} = 0 \quad (5)$$

$$2 e_{23} = u_{2,3} + u_{3,2} = 0 \quad (6)$$

$$\text{from (1)} \quad u_3 = \rho g x_3^2 / 2 E + a_3(x_1, x_2) \quad (a)$$

$$\text{from (5)} \quad u_{1,3} = -a_{3,1}$$

$$\text{from (6)} \quad u_{2,3} = -a_{3,2}$$

$$\therefore \quad u_1 = -x_3 a_{3,1} + a_1(x_1, x_2) \quad (b)$$

$$u_2 = -x_3 a_{3,2} + a_2(x_1, x_2) \quad (c)$$

$$\text{from (2)} \quad -x_3 a_{3,11} + a_{1,1} = -\nu \rho g x_3 / E$$

$$\text{then} \quad a_{3,11} = \nu \rho g / E \quad (7)$$

$$a_{1,1} = 0 \quad (8)$$

$$\text{from (3)} \quad a_{3,22} = \nu \rho g / E \quad (9)$$

$$a_{2,2} = 0 \quad (10)$$

$$\text{from (4)} \quad -2 x_3 a_{3,21} + a_{1,2} + a_{1,2} = 0$$

$$\text{then} \quad a_{3,21} = 0 \quad (11)$$

$$a_{1,2} + a_{1,2} = 0 \quad (12)$$

$$(8) \rightarrow a_1 = a_1(x_2)$$

$$(10) \rightarrow a_2 = a_2(x_1)$$

$$(12) \rightarrow a_{1,2}(x_2) + a_{1,2}(x_1) = 0$$

$$\text{then} \quad a_{1,2} = -a_{1,2} = a = \text{constant}$$

$$\therefore \quad a_1 = a x_2 + b$$

$$a_2 = -a x_1 + c$$

$$(7) \text{ and } (9) \quad a_3 = \frac{\nu \rho g}{2E} (x_1^2 + x_2^2) + \beta(x_1, x_2)$$

$$\text{with } \beta_{,11} = \beta_{,22} = 0$$

$$\text{from (11)} \quad \beta_{,21} = 0$$

$$\therefore \quad \beta(x_1, x_2) = a' x_1 + b' x_2 + c'$$

$$\text{i.e.} \quad a_3 = \frac{\nu \rho g}{2E} (x_1^2 + x_2^2) + a' x_1 + b' x_2 + c'$$

from (a) (b) and (c), we can get

$$u_1 = -\frac{\nu \rho g}{E} x_1 x_3 - a' x_3 + a x_2 + b$$

$$u_2 = -\frac{\nu \rho g}{E} x_2 x_3 - b' x_3 - a x_1 + c$$

$$u_3 = \frac{\rho g}{2E} (x_3^2 + \nu x_1^2 + \nu x_2^2) + a' x_1 + b' x_2 + c'$$

the linear part represents rigid body motion

if we set $u_i = 0$ at $(0, 0, l)$

and prevent the possibility of rotations at $(0, 0, l)$

$$\text{i.e.} \quad \omega_i = \varepsilon_{ijk} u_{k,j} = 0$$

$$u_{2,1} - u_{1,2} = 0$$

$$u_{3,1} - u_{1,3} = 0$$

$$u_{2,3} - u_{3,2} = 0$$

these six conditions enable us to eliminate six constants

$$\text{i.e.} \quad a = b = c = a' = b' = c' = 0$$

we have

$$u_1 = -\frac{\nu \rho g}{E} x_1 x_3$$

$$u_2 = -\frac{\nu \rho g}{E} x_2 x_3$$

$$u_3 = \frac{\rho g}{2 E} (x_3^2 + \nu x_1^2 + \nu x_2^2 - l^2)$$

on x_3 axis

$$u_3 = \frac{\rho g}{2 E} (x_3^2 - l^2)$$

at $x_3 = 0$

$$u_3 = -\rho g l^2 / 2 E = -W l / 2 E A$$

same as in strength of materials

(B) Pure Bending $M_2 = M$ others = 0

let x_1, x_2 be principal axes of inertia

try $\sigma_{33} = -M x_1 / I$ others = 0

where $I = \int_{\mathcal{R}} x_1^2 dA$ is the moment of area of the section

the stress components satisfy the equilibrium equation and boundary conditions, then the strain components are

$$e_{33} = -M x_1 / E I \quad e_{11} = e_{22} = \nu M x_1 / E I$$

$$e_{12} = e_{23} = e_{13} = 0$$

$$\therefore u_{1,1} = \nu M x_1 / E I \quad (1)$$

$$u_{2,2} = \nu M x_1 / E I \quad (2)$$

$$u_{3,3} = -M x_1 / E I \quad (3)$$

$$u_{1,2} + u_{2,1} = 0 \quad (4)$$

$$u_{1,3} + u_{3,1} = 0 \quad (5)$$

$$u_{2,3} + u_{3,2} = 0 \quad (6)$$

$$\text{from (3)} \quad u_3 = -M x_1 x_3 / E I + u_3^0(x_1, x_2)$$

$$(5) (6) \rightarrow u_{1,3} = M x_3 / E I - u_{3,1}^0 \quad (7)$$

$$u_{2,3} = -u_{3,2}^0 \quad (8)$$

$$(7) (8) \rightarrow u_1 = M x_3^2 / 2 E I - u_{3,1}^0 x_3 + u_1^0(x_1, x_2)$$

$$u_2 = -u_{3,2}^0 x_3 + u_2^0(x_1, x_2)$$

$$(1) \rightarrow u_{1,1} = -u_{3,11}^0 x_3 + u_{1,1}^0 = \nu M x_1 / E I$$

$$(2) \rightarrow u_{2,2} = -u_{3,22}^0 x_3 + u_{2,2}^0 = \nu M x_1 / E I$$

these two equations are held for all x_3 , then

$$u_{3,11}^0 = u_{3,22}^0 = 0 \quad (a)$$

$$u_{1,1}^0 = \nu M x_1 / E I$$

$$u_{2,2}^0 = \nu M x_1 / E I$$

$$\therefore u_1^0 = \nu M x_1^2 / 2 E I + f_1(x_2)$$

$$u_2^0 = \nu M x_1 x_2 / E I + f_2(x_1)$$

$$\text{then } u_1 = M x_3^2 / 2 E I - u_{3,1}^0 x_3 + \nu M x_1^2 / 2 E I + f_1(x_2)$$

$$u_2 = -u_{3,2}^0 x_3 + \nu M x_1 x_2 / EI + f_2(x_1)$$

from (4)

$$-2u_{3,12}^0 x_3 + f_{1,2} + \nu M x_2 / EI + f_{2,1} = 0$$

it is true for all x_3 , then

$$u_{3,12}^0 = 0 \quad (b)$$

$$\nu M x_2 / EI + f_{1,2} + f_{2,1} = 0$$

$$\text{let } f_{2,1} = -a \quad \text{then} \quad \nu M x_2 / EI + f_{1,2} = a$$

$$\therefore f_2 = -a x_1 + b$$

$$f_1 = -\nu M x_2^2 / 2EI + a x_2 + a$$

from (a) and (b), u_3^0 is linear function of x_1 and x_2

$$\text{then} \quad u_3^0 = \beta x_1 + \gamma x_2 + c$$

$$\text{thus} \quad u_1 = M(x_3^2 + \nu x_1^2 - \nu x_2^2) / 2EI + a x_2 - \beta x_3 + a$$

$$u_2 = \nu M x_1 x_2 / EI - a x_1 - \gamma x_3 + b$$

$$u_3 = -M x_1 x_3 / EI + \beta x_1 + \gamma x_2 + c$$

in order to uniquely determine u_i , assume

$$u_i = 0 \quad u_{1,3} = u_{2,3} = u_{2,1} = 0 \quad \text{at } (0, 0, 0)$$

$$\text{then} \quad a = b = c = a = \beta = \gamma = 0$$

the solution can be written in the form

$$u_1 = M(x_3^2 + \nu x_1^2 - \nu x_2^2) / 2EI$$

$$u_2 = \nu M x_1 x_2 / EI$$

$$u_3 = -M x_1 x_3 / EI$$

along the centroid axis ($x_1 = x_2 = 0$)

$$u_1 = M x_3^2 / 2 E I \quad u_2 = u_3 = 0$$

solution from strength of materials

$$\Delta = M x_3^2 / 2 E I = u_1(0, 0, x_3)$$

$$\text{slope} = \theta = u_{1,3} = M x_3 / E I$$

(C) Torsion

Consider a circular cylinder, of length l , with one of its bases fixed in the x_1 - x_2 plane, while the other base ($x_3 = l$) is acted upon by a couple whose moment lies along the x_3 axis

$$M_1 = M_2 = 0 \quad M_3 = M$$

$$\text{on } B_L, T_i = 0 \quad \sigma_{31} n_1 + \sigma_{32} n_2 = 0$$

$$\sigma_{21} n_1 + \sigma_{22} n_2 = 0$$

$$\sigma_{11} n_1 + \sigma_{12} n_2 = 0$$

$$\text{on } x_3 = l \quad \int_R \sigma_{31} dA = 0 \quad \int_R \sigma_{32} dA = 0 \quad \int_R \sigma_{33} dA = 0$$

$$\int_R x_2 \sigma_{32} dA = M_1 = 0$$

$$\int_R x_1 \sigma_{33} dA = M_2 = 0$$

$$\int_R (x_1 \sigma_{32} - x_2 \sigma_{31}) dA = M_3 = M$$

first assume R is circular : suppose that circular cross sections remain plane and circular after twist (will check later), and each section rotates rigidly relative to $x_3 = 0$, through an angle proportional to its distance from the end section

after twist

$$x_1' = x_1 + u_1$$

$$x_2' = x_2 + u_2$$

$$x_3' = x_3 \rightarrow u_3 = 0$$

β : angle of twist

let a be the constant of proportionality between β and x_3 (angle of twist per unit length)

then $\beta = a x_3$

thus

$$\begin{aligned} u_1 &= x_1' - x_1 = \rho \cos(\theta + \beta) - \rho \cos \theta \\ &= x_1 (\cos \beta - 1) - x_2 \sin \beta \\ u_2 &= x_2' - x_2 = \rho \sin(\theta + \beta) - \rho \sin \theta \\ &= x_2 (\cos \beta - 1) - x_1 \sin \beta \\ u_3 &= 0 \end{aligned}$$

for u_i small $\rightarrow \beta$ small, $\cos \beta \simeq 1$ $\sin \beta \simeq \beta$

then

$$\begin{aligned} u_1 &= -x_2 \beta = -a x_2 x_3 \\ u_2 &= x_1 \beta = a x_1 x_3 \\ u_3 &= 0 \end{aligned}$$

$$\begin{aligned} \therefore e_{11} &= e_{22} = e_{33} = 0 & e_{kk} &= 0 \\ e_{12} &= \frac{1}{2} (u_{1,2} + u_{2,1}) = \frac{1}{2} (-a x_3 + a x_3) = 0 \\ e_{13} &= -\frac{1}{2} a x_2 & e_{23} &= \frac{1}{2} a x_1 \end{aligned}$$

$$\sigma_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk}$$

$$\begin{aligned} \therefore \quad \sigma_{11} &= \sigma_{22} = \sigma_{33} = 0 & \sigma_{12} &= 0 \\ \sigma_{13} &= 2\mu e_{13} = -a\mu x_2 \\ \sigma_{23} &= 2\mu e_{23} = a\mu x_1 \end{aligned}$$

equation of equilibrium is identically satisfied

boundary conditions

$$\text{on } B_L \quad \therefore \quad \sigma_{11} = \sigma_{22} = \sigma_{12} = 0$$

the last two equations are identically satisfied

the first one is

$$\sigma_{31} n_1 + \sigma_{32} n_2 = 0$$

$$\text{i.e.} \quad -x_2 n_1 + x_1 n_2 = 0 \quad \text{on } C$$

$$\therefore \quad n_1 = \cos \theta$$

$$n_2 = \sin \theta$$

$$x_2 = a \sin \theta$$

$$x_1 = a \cos \theta$$

$$\therefore \quad -x_2 n_1 + x_1 n_2 = 0 \quad \text{is satisfied}$$

$$\text{on } R \quad \int_R \sigma_{31} dA = -a\mu \int_R x_2 dA = 0 \quad [x_2 \text{ is a centroidal axis}]$$

$$\text{similarly for} \quad \int_R \sigma_{32} dA = 0$$

$$\text{and} \quad \int_R \sigma_{33} dA = \int_R \sigma_{33} x_1 dA = \int_R \sigma_{33} x_2 dA = 0$$

$$\begin{aligned} M &= \int_R (x_1 \sigma_{32} - x_2 \sigma_{31}) dA \\ &= a\mu \int_R (x_1^2 + x_2^2) dA = a\mu I_0 \end{aligned}$$

where I_0 is the polar moment of inertia

so we have $\alpha = M / \mu I_0$

on $x_3 = \text{constant}$ (plane) $T_i = \sigma_{ji} n_j$ $n_j = (0, 0, 1)$

then $N = T_3 = \sigma_{j3} n_j = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 = 0$

$\therefore T_i$ is a stress vector lie on the plane

i.e. $\mathbf{S} = \mathbf{T} - N = \mathbf{T}$

$$\begin{aligned} S &= |\mathbf{S}| = (\sigma_{13}^2 + \sigma_{23}^2)^{1/2} \\ &= \mu |\alpha| (x_1^2 + x_2^2)^{1/2} = \mu |\alpha| \rho \\ &= \mu [|M| / \mu I_0] \rho \\ &= |M| \rho / I_0 \end{aligned}$$

then $S_{\max}$ occurs at $\rho = a$

if we try the same solution for noncircular cross section, we find that

$$\sigma_{31} n_1 + \sigma_{32} n_2 \neq 0$$

this equation is not satisfied, so the solution is not correct.

For noncircular cross section

$$\begin{aligned} \text{assume } u_1 &= -\alpha x_2 x_3 \\ u_2 &= \alpha x_1 x_3 \\ u_3 &= \alpha \phi(x_1, x_2) \end{aligned}$$

$\phi(x_1, x_2) \neq 0$ means plane is not remain plane

$\phi(x_1, x_2)$ is called the warping function

by strain-displacement and constitutive relations

$$\begin{aligned} e_{11} &= e_{22} = e_{33} = e_{12} = 0 \\ \sigma_{11} &= \sigma_{22} = \sigma_{33} = \sigma_{12} = 0 \end{aligned}$$

$$\text{and} \quad \begin{aligned} 2 \mu e_{13} &= \sigma_{13} = a \mu (\phi_{,1} - x_2) \\ 2 \mu e_{23} &= \sigma_{23} = a \mu (\phi_{,2} + x_1) \end{aligned}$$

equation of equilibrium

$$\begin{aligned} \sigma_{31,1} + \sigma_{32,2} &= 0 \\ a \mu (\phi_{,11} + \phi_{,22}) &= 0 \quad \text{in } R \end{aligned}$$

i.e. ϕ must be a harmonic function in R

on B_L the last two equations are identically satisfied
the first one becomes

$$\begin{aligned} \sigma_{31} n_1 + \sigma_{32} n_2 &= 0 && \text{on } C \\ (\phi_{,1} - x_2) n_1 + (\phi_{,2} + x_1) n_2 &= 0 && \text{on } C \\ \text{i.e.} \quad d\phi / dn &= \phi_{,1} n_1 + \phi_{,2} n_2 = x_2 n_1 - x_1 n_2 && \text{on } C \\ \therefore \quad \mathbf{n} &= n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 \end{aligned}$$

$$\begin{aligned} n_1 &= dx_2 / dS & n_2 &= -dx_1 / dS \\ \mathbf{n} &= (dx_2 / dS) \mathbf{e}_1 - (dx_1 / dS) \mathbf{e}_2 \end{aligned}$$

$\mathbf{t}$ has the same direction as dS

$$\begin{aligned} \mathbf{t} &= (dx_1 / dS) \mathbf{e}_1 + (dx_2 / dS) \mathbf{e}_2 \\ \text{note that} \quad \mathbf{dS} \cdot \mathbf{n} &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \quad \phi_{,n} &= \phi_{,1} n_1 + \phi_{,2} n_2 = x_2 n_1 - x_1 n_2 \quad \text{on } C \\ &= x_2 (dx_2 / dS) + x_1 (dx_1 / dS) \\ &= \frac{1}{2} \frac{d}{dS} (x_1^2 + x_2^2) = \frac{1}{2} \frac{d(\rho^2)}{dS} \quad \text{on } C \end{aligned}$$

thus, the problem of determining the warping function $\phi(x_1, x_2)$ is a special case of the second boundary value problem of Potential Theory, it is called Neumann's problem : given a function ϕ that is harmonic in a given

region and whose normal derivative is prescribed on the boundary of the region. i.e.

$$\begin{aligned}\phi_{,ii} &= 0 && \text{in } R && [i = 1, 2] \\ \phi_{,n} &= \frac{1}{2} d(\rho^2) / dS && \text{on } C\end{aligned}$$

the integral of the normal derivative over the entire boundary is

$$\begin{aligned}\oint_C \phi_{,n} dS &= \oint_C \phi_{,i} n_i dS = \oint_R \phi_{,ii} dA \\ &= \frac{1}{2} \oint_C [d(\rho^2) / dS] dS = \frac{1}{2} \oint_C d(\rho^2) \\ &= 0 && [\text{exact differential}]\end{aligned}$$

so we can expect that ϕ exists

on $x_3 = l$, we have

$$\begin{aligned}\int_R \sigma_{31} dA &= a\mu \int_R (\phi_{,1} - x_2) dA \\ &= a\mu \int_R (\phi_{,1} - x_2 + x_1 \phi_{,ii}) dA \\ &= a\mu \int_R \{[x_1 (\phi_{,1} - x_2)],_1 + [x_1 (\phi_{,2} + x_1)],_2\} dA \\ &= a\mu \int_R v_{i,i} dA \\ &= a\mu \oint_C v_i n_i dS \\ &= a\mu \oint_C [x_1 (\phi_{,1} - x_2) n_1 + x_1 (\phi_{,2} + x_1) n_2] dS \\ &= a\mu \oint_C x_1 (\phi_{,n} - x_2 n_1 + x_1 n_2) dS = 0 && [\text{O. K.}]\end{aligned}$$

$$\text{similarly} \quad \int_R \sigma_{32} dA = 0 \quad [\text{O. K.}]$$

$$\begin{aligned}\text{and} \quad M &= \int_R (x_1 \sigma_{32} - x_2 \sigma_{31}) dA \\ &= a\mu \int_R [x_1 (\phi_{,2} + x_1) - x_2 (\phi_{,1} - x_2)] dA\end{aligned}$$

define K , the torsional stiffness by

$$K = \mu \int_R [x_1 (\phi_{,2} + x_1) - x_2 (\phi_{,1} - x_2)] dA$$

$$\text{then} \quad a = M / K$$

Theorem : The solution of the torsional problem subjected to the boundary conditions is

$$u_1 = -\alpha x_2 x_3$$

$$u_2 = \alpha x_1 x_3$$

$$u_3 = \alpha \phi(x_1, x_2)$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0$$

$$\sigma_{31} = \alpha \mu (\phi_{,1} - x_2)$$

$$\sigma_{32} = \alpha \mu (\phi_{,2} + x_1)$$

$$\text{where } \phi_{,ii} = 0 \quad \text{in } R \quad [i = 1, 2]$$

$$\phi_{,n} = \frac{1}{2} d(\rho^2) / dS \quad \text{on } C$$

[Neumann Problem]

$$\text{and } \alpha = M / K \quad \text{angle of twist per unit length}$$

$$\text{where } K = \mu \int_R [x_1 (\phi_{,2} + x_1) - x_2 (\phi_{,1} - x_2)] dA$$

Note : the torsion problem has been reduced to solving the Neumann Problem

Conjugate Warping function $\psi(x_1, x_2)$

if a harmonic function $f(z)$ is analytic in R , and $f = \phi + i \psi$, where ϕ and ψ are real harmonics and obey Cauchy-Riemann condition

$$\phi_{,1} = \psi_{,2} \quad \text{and} \quad \phi_{,2} = -\psi_{,1}$$

then ψ is called harmonic conjugate to ϕ

$$\begin{aligned} \text{on } C : \quad \phi_{,n} &= \phi_{,1} n_1 + \phi_{,2} n_2 \\ &= \psi_{,2} (dx_2 / dS) + (-\psi_{,1}) (-dx_1 / dS) \\ &= \psi_{,2} t_2 + \psi_{,1} t_1 \end{aligned}$$

$$= d\psi / dS = \frac{1}{2} d(x_1^2 + x_2^2) / dS$$

thus we have $\psi_{,ii} = 0$ in R

$$\psi = \frac{1}{2} (x_1^2 + x_2^2) + \text{constant} \quad \text{on } C$$

[note that the constant may set to zero, because it does not affect σ_{ij}]

this is the problem of Dirichlet, the solution of this problem is unique

and we have

$$\sigma_{31} = a\mu (\psi_{,2} - x_2)$$

$$\sigma_{32} = a\mu (-\psi_{,1} + x_1)$$

Prandtl introduced a function $\Psi(x_1, x_2)$ such that

$$\Psi(x_1, x_2) = \psi(x_1, x_2) - \frac{1}{2} (x_1^2 + x_2^2)$$

it is called Prandtl stress function

then we have $\Psi_{,ii} = -2$ in R

$$\Psi = 0 \quad \text{on } C$$

solve for $\Psi(x_1, x_2)$ and obtain

$$\sigma_{31} = a\mu \Psi_{,2}$$

$$\sigma_{32} = -a\mu \Psi_{,1}$$

also the stress vector $\tau = \sigma_{31} \mathbf{e}_1 + \sigma_{32} \mathbf{e}_2$

$$\begin{aligned} \text{and} \quad |\tau| &= (\sigma_{31}^2 + \sigma_{32}^2)^{1/2} \\ &= a\mu (\Psi_{,1}^2 + \Psi_{,2}^2)^{1/2} \end{aligned}$$

Theorem : $|\tau|$ attain its maximum on C

proof see pp. 117-118 on Sokolnikoff which shows

$|\nabla \Psi|$ attains its maximum on C by the property of subharmonic functions

$$\text{and } |\tau| = \tau = (\sigma_{31}^2 + \sigma_{32}^2)^{1/2} = |\alpha| \mu |\nabla \Psi|$$

also attains its maximum on C

Theorem : $|\tau| = \sigma_{31} \mathbf{e}_1 + \sigma_{32} \mathbf{e}_2$ is directed along the line $\Psi = \text{constant}$

let $\Psi = \text{constant}$

$$d\Psi = \Psi_{,1} dx_1 + \Psi_{,2} dx_2 = 0$$

$$\text{or } dx_2 / dx_1 = -\Psi_{,1} / \Psi_{,2} = \sigma_{32} / \sigma_{31}$$

$$\therefore \mathbf{t} = dx_1 \mathbf{e}_1 + dx_2 \mathbf{e}_2$$

$$\text{and } \mathbf{T} = \sigma_{31} \mathbf{e}_1 + \sigma_{32} \mathbf{e}_2$$

$$\therefore \mathbf{T} // \mathbf{t} \quad \text{i.e. } \mathbf{T} \text{ directed along the line } \Psi = \text{constant}$$

Theorem : $K = 2\mu \int_R \Psi dA$

$$\begin{aligned} \text{proof } \therefore K &= \mu \int_R (x_1^2 + x_2^2 + x_1 \phi_{,2} - x_2 \phi_{,1}) dA \\ &= \mu \int_R (x_1^2 + x_2^2 - x_1 \psi_{,1} - x_2 \psi_{,2}) dA \\ &= \mu \int_R [(x_1 (x_1 - \psi_{,1}) + x_2 (x_2 - \psi_{,2}))] dA \end{aligned}$$

$$\therefore \Psi = \psi - \frac{1}{2} (x_1^2 + x_2^2)$$

$$\Psi_{,1} = \psi_{,1} - x_1$$

$$\Psi_{,2} = \psi_{,2} - x_2$$

$$\begin{aligned} \therefore K &= -\mu \int_R (x_1 \Psi_{,1} + x_2 \Psi_{,2}) dA \\ &= -\mu \int_R (x_i \Psi)_{,i} dA + \mu \int_R 2\Psi dA \\ &= -\mu \oint_C x_i \Psi n_i dS + 2\mu \int_R \Psi dA \end{aligned}$$

$$= 2\mu \int_R \Psi dA$$

Methods of Solutions

1. Numerical method -- optimization
2. Inverse method
3. Direct method
4. Approximate method -- membrane analogy

1. Optimization Approach

Consider the functional

$$J(\phi) = \int_R \phi_{,i} \phi_{,i} dA - 2 \oint_C g(s) \phi(s) dS$$

where $g(s)$ is known

is there a function $\phi \in C_2$ that minimizes $J(\phi)$

take a variation of $J(\phi)$

$$\delta J = \int_R 2 (\delta \phi)_{,i} \phi_{,i} dA - 2 \oint_C g(s) \delta \phi(s) dS$$

for a minimum $J(\phi)$, $\delta J = 0$ for all $\delta \phi$

$$\therefore 2 \int_R (\phi_{,i} \delta \phi)_{,i} dA - 2 \int_R \phi_{,ii} \delta \phi dA - 2 \oint_C g(s) \delta \phi(s) dS = 0$$

$$\int_R (\phi_{,i} \delta \phi)_{,i} dA = \oint_C \phi_{,i} \delta \phi n_i dS = \oint_C \phi_{,n} \delta \phi dS$$

then, we have

$$2 \oint_C (\phi_{,n} - g) \delta \phi dS - 2 \int_R \phi_{,ii} \delta \phi dA = 0$$

for all $\delta \phi$

$$\begin{aligned} \text{i.e.} \quad \phi_{,ii} &= 0 && \text{in } R \\ \phi_{,n} &= g && \text{on } C \end{aligned} \quad (a)$$

this is called external condition, i.e. necessary condition for minimum $J(\phi)$

we now show that if ϕ satisfies (a), then $J(\phi)$ is an absolute minimum

let ϕ_1 satisfies (a)

ϕ_2 be any other function $\in C_2$

$$\begin{aligned}\Delta J &= J(\phi_2) - J(\phi_1) \\ &= \int_R (\phi_{2,i} \phi_{2,i} - \phi_{1,i} \phi_{1,i}) dA - 2 \oint_C g (\phi_2 - \phi_1) dS\end{aligned}$$

$$\text{let } \Delta \phi = \phi_2 - \phi_1$$

$$\begin{aligned}\text{then } \Delta J &= \int_R [(\phi_{1,i} + \Delta \phi_{,i})(\phi_{1,i} + \Delta \phi_{,i}) - \phi_{1,i} \phi_{1,i}] dA - 2 \oint_C g \Delta \phi dS \\ &= 2 \int_R \phi_{1,i} \Delta \phi_{,i} dA + \int_R (\Delta \phi)_{,i} (\Delta \phi)_{,i} dA - 2 \oint_C g \Delta \phi dS \\ &= 2 \int_R (\phi_{1,i} \Delta \phi)_{,i} dA - 2 \int_R \phi_{1,ii} \Delta \phi dA \\ &\quad + \int_R (\Delta \phi)_{,i} (\Delta \phi)_{,i} dA - 2 \oint_C g \Delta \phi dS \\ &= 2 \oint_C \phi_{1,n} \Delta \phi dS - 2 \oint_C g \Delta \phi dS + \int_R (\Delta \phi)_{,i} (\Delta \phi)_{,i} dA \\ &= \int_R (\Delta \phi)_{,i} (\Delta \phi)_{,i} dA \geq 0\end{aligned}$$

so for the torsion problem, the warping function $\phi(x_1, x_2)$ is the one that minimizes the function

$$\int_R \phi_{,i} \phi_{,i} dA - 2 \oint_C \phi (x_2 n_1 - x_1 n_2) dS$$

2. Inverse Method

Consider a complex function

$$f(z) = i(c^2 z^2 + k^2)$$

where $z = x_1 + i x_2$ c, k are constants

$$\text{then } f(z) = i[c^2(x_1^2 - x_2^2) + 2 i c x_1 x_2 + k^2]$$

$$\text{and } \operatorname{Re}\{f\} = \phi = -2 c x_1 x_2 \quad \text{harmonic}$$

$$\operatorname{Im}\{f\} = \psi = c^2 (x_1^2 - x_2^2) + k^2 \quad \text{harmonic conjugate}$$

does this exist a closed contour C so that

$$\psi = \frac{1}{2} (x_1^2 - x_2^2) \quad \text{on } C$$

$$\text{let } \psi = c^2 (x_1^2 - x_2^2) + k^2 = \frac{1}{2} (x_1^2 - x_2^2) \quad \text{on } C$$

$$\text{i.e. } (\frac{1}{2} - c^2) x_1^2 + (\frac{1}{2} + c^2) x_2^2 = k^2$$

$$c^2 < \frac{1}{2} \quad \text{ellipse, closed curve}$$

$$c^2 > \frac{1}{2} \quad \text{hyperbolic, not a closed curve}$$

$$c^2 = 0 \quad \text{circular, closed curve, special case}$$

$$\text{take } a = \frac{k}{(2 - c^2)^{1/2}} \quad b = \frac{k}{(2 - c^2)^{1/2}}$$

$$\text{or } c^2 = \frac{a^2 - b^2}{2(a^2 + b^2)} \quad k^2 = \frac{a^2 b^2}{a^2 + b^2}$$

then the contour C is

$$C: \quad \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

$$\phi = \frac{a^2 - b^2}{a^2 + b^2} x_1 x_2$$

$$\psi = \frac{1}{a^2 + b^2} [\frac{1}{2} (a^2 - b^2) (x_1^2 - x_2^2) + a^2 b^2]$$

$$\begin{aligned} \text{and } \Psi &= \psi - \frac{1}{2} (x_1^2 + x_2^2) \\ &= - \frac{a^2 - b^2}{a^2 + b^2} \left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1 \right) \end{aligned}$$

then the stresses are

$$\sigma_{31} = \mu a \Psi_{,2} = -2 \mu a \frac{a^2 x_2}{a^2 + b^2}$$

$$\sigma_{32} = -\mu a \Psi_{,1} = 2 \mu a \frac{b^2 x_1}{a^2 + b^2}$$

$$a^2 + b^2$$

and the torsional stiffness is

$$\begin{aligned} K &= 2\mu \int \Psi dA \\ &= -\frac{2\mu}{a^2 + b^2} (b^2 \int_{\mathcal{R}} x_1^2 dA + a^2 \int_{\mathcal{R}} x_2^2 dA - a^2 b^2 \int_{\mathcal{R}} dA) \\ &= \frac{2\mu}{a^2 + b^2} (\pi a^3 b^3 - \pi a^3 b^3 / 4 - \pi a^3 b^3 / 4) \\ &= \frac{2\mu a^3 b^3}{a^2 + b^2} \end{aligned}$$

$$M = K a$$

$$\therefore a = \frac{M(a^2 + b^2)}{\pi \mu a^3 b^3}$$

$$\sigma_{31} = -\frac{2 M x_2}{\pi a b^3}$$

$$\sigma_{31} = \frac{2 M x_1}{\pi a^3 b}$$

$$S = (\sigma_{31}^2 + \sigma_{32}^2)^{1/2} = \frac{2 M}{\pi a^3 b^3} (a^4 x_2^2 + b^4 x_1^2)^{1/2}$$

on C $b^2 x_1^2 + a^2 x_2^2 = a^2 b^2$

$$\begin{aligned} \therefore (a^4 x_2^2 + b^4 x_1^2)^{1/2} &= [a^2 (a^2 b^2 - b^2 x_1^2) + b^4 x_1^2]^{1/2} \\ &= [a^4 b^2 - (a^2 b^2 - b^4) x_1^2]^{1/2} \\ &= a b [a^2 - (1 - b^2 / a^2) x_1^2]^{1/2} \end{aligned}$$

let $e^2 = 1 - b^2 / a^2 > 0$ for $b < a$

$$\therefore S = \frac{2 M}{\pi a^3 b^3} (a^2 - e^2 x_1^2)^{1/2}$$

$$\pi a^2 b^2$$

then $S_{\max}$ occurs at $x_1 = 0$

$$\text{i.e. } S_{\max} = 2 M / \pi a b^2$$

for circular cross-section $a = b$

$$S_{\max} = \frac{M}{\pi a^3/2} = \frac{M a}{\pi a^4/2} = \frac{M a}{I_0}$$

$$\therefore \phi = \frac{a^2 - b^2}{a^2 + b^2} x_1 x_2$$

$$\therefore u_3 = a \phi(x_1, x_2) = \frac{M(a^2 - b^2)}{\pi \mu a^3 b^3} x_1 x_2$$

if $a > b$ for $x_1 x_2 > 0$ $u_3 > 0$
 $x_1 x_2 < 0$ $u_3 < 0$
 and $u_3 = 0$ on x_1 and x_2 axes

Consider a Complex Function

$$\begin{aligned} f(z) &= i(c z^3 + k) \\ &= c(x_2^3 - 3x_1^2 x_2) + i[c(x_1^3 - 3x_1 x_2^2) + k] \end{aligned}$$

$$\begin{aligned} \text{then } \phi &= c(x_2^3 - 3x_1^2 x_2) \\ \psi &= c(x_1^3 - 3x_1 x_2^2) + k \end{aligned}$$

$$\text{on boundary } \psi = \frac{1}{2}(x_1^2 + x_2^2)$$

$$\therefore c(x_1^3 - 3x_1 x_2^2) + k = \frac{1}{2}(x_1^2 + x_2^2)$$

set $c = -a/6$ and $k = \frac{2}{3}a^2$

then the above equation becomes

$$(x_1 - a)(x_1 - \sqrt{3}x_2 + 2a)(x_1 + \sqrt{3}x_2 + 2a) = 0$$

the boundary of the region

is an equilateral triangle

$$\therefore \phi = (x_2^3 - 3x_1^2x_2)/6a$$

$$\psi = -(x_1^3 - 3x_1x_2^2)/6a + \frac{2}{3}a^2$$

$$\text{and } \Psi = \psi - \frac{1}{2}(x_1^2 + x_2^2)$$

$$= -(x_1^3 - 3x_1x_2^2)/6a + \frac{2}{3}a^2 - \frac{1}{2}(x_1^2 + x_2^2)$$

then the stresses are

$$\begin{aligned}\sigma_{31} &= \mu a \Psi_{,2} = \mu a (x_1x_2/a - x_2) \\ &= \frac{\mu a}{a} (x_1 - a)x_2\end{aligned}$$

$$\sigma_{32} = -\mu a \Psi_{,1} = \frac{\mu a}{2a} (x_1^2 + 2ax_1 - x_2^2)$$

at $x_1 = a$

$$\sigma_{31} = 0$$

$$\sigma_{32} = \frac{\mu a}{2a} (3a^2 - x_2^2)$$

$$\text{and } \max \sigma_{32} = 3a\mu a/2 \quad \text{at } x_2 = 0$$

along $x_2 = 0$

$$\sigma_{32} = -\frac{\mu a}{2a} (x_1^2 + 2ax_1)$$

$$K = 2\mu \int_R \Psi dA$$

$$\begin{aligned}
&= \mu \int_R (x_1^2 + x_2^2 + x_1 \phi_{,2} - x_2 \phi_{,1}) dA \\
&= \mu (3a)^4 / 15 \sqrt{3} = 3\mu I_0 / 5
\end{aligned}$$

where $I_0 = 3 \sqrt{3} a^4$ is the polar moment of inertia of the triangle

$$\begin{aligned}
\therefore M &= K a \\
\therefore a &= \frac{M}{K} = \frac{15 \sqrt{3} M}{\mu (3a)^4}
\end{aligned}$$

$$\text{and } (\sigma_{32})_{\max} = \tau_{\max} = 3 a \mu a / 2 = \frac{15 \sqrt{3} M}{2 (3a)^3}$$

$$\begin{aligned}
u_3 &= a \phi(x_1, x_2) = -a a (x_2^3 - 3 x_1^2 x_2) / 6 \\
&= -a a x_2 (x_2^2 - 3 x_1^2) / 6
\end{aligned}$$

Consider a pair of harmonic functions

$$x_1 \quad \frac{x_1}{x_1^2 + x_2^2}$$

in polar coordinate $x_1 = r \cos \theta$ $x_2 = r \sin \theta$

construct a harmonic function ψ , such that

$$\begin{aligned}
\psi &= a \left(x_1 - b^2 \frac{x_1}{x_1^2 + x_2^2} \right) + \frac{1}{2} b^2 \\
&= a \left(r \cos \theta - \frac{b^2 \cos \theta}{r} \right) + \frac{1}{2} b^2
\end{aligned}$$

$$\therefore \psi = \frac{1}{2} (x_1^2 + x_2^2) = \frac{1}{2} r^2 \quad \text{on } C$$

∴ the boundary of the section is

$$a (r \cos \theta - b^2 \cos \theta / r) + \frac{1}{2} b^2 = \frac{1}{2} r^2$$

or
$$r^2 - b^2 - 2 a (r^2 - b^2) \cos \theta / r = 0$$

$$(r^2 - b^2) (1 - 2 a \cos \theta / r) = 0$$

the boundary is made up of two circles

$$r = b \quad \text{center at } O$$

$$r = a \quad \text{center at } (a, 0)$$

$$\Psi = \psi - \frac{1}{2} (x_1^2 + x_2^2)$$

$$= a (x_1 - b^2 \frac{x_1}{x_1^2 + x_2^2}) + \frac{1}{2} b^2 - \frac{1}{2} (x_1^2 + x_2^2)$$

$$\sigma_{31} = \mu a \Psi_{,2} = \mu a [a b^2 \frac{x_1 \cdot 2 x_2}{(x_1^2 + x_2^2)^2} - x_2]$$

$$= \mu a [a b^2 (2 \sin \theta \cos \theta) / r^2 - r \sin \theta]$$

$$\sigma_{32} = -\mu a \Psi_{,1} = -\mu a [a + a b^2 \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2} - x_1]$$

$$= -\mu a [a + a b^2 (\cos^2 \theta - \sin^2 \theta) / r^2 - r \cos \theta]$$

on boundary $r = 2 a \cos \theta$

$$\sigma_{31} = \mu a [a b^2 (2 \sin \theta \cos \theta) / 4 a^2 \cos^2 \theta - 2 a \cos \theta \sin \theta]$$

$$= \mu a (b^2 - 4 a^2 \cos^2 \theta) \sin \theta / 2 a \cos \theta$$

$$\sigma_{32} = -\mu a [a + a b^2 (\cos^2 \theta - \sin^2 \theta) / 4 a^2 \cos^2 \theta - 2 a \cos^2 \theta]$$

$$= \mu a [8 a^2 \cos^4 \theta - 4 a^2 \cos^2 \theta - b^2 (\cos 2\theta - 1)] / 4 a \cos^2 \theta$$

$$S = (\sigma_{31}^2 + \sigma_{32}^2)^{1/2}$$

at $\theta = 0 \quad \sigma_{31} = 0$

$$\begin{aligned}\sigma_{32} &= \mu a (8 a^2 - 4 a^2 - b^2) / 4 a \\ &= \mu a (4 a^2 - b^2) / 4 a\end{aligned}$$

$$\text{for } a \gg b \quad \sigma_{32} = \mu a a$$

same as circular cross section

on boundary $r = b$

$$\sigma_{31} = \mu a (2 a \cos \theta - b) \sin \theta$$

$$\sigma_{32} = -\mu a (2 a \cos \theta - b) \cos \theta$$

$$S = (\sigma_{31}^2 + \sigma_{32}^2)^{1/2} = \mu a (2 a \cos \theta - b)$$

$$\text{for } \theta = 0 \quad S = S_{\max} = \mu a (2 a - b)$$

$$\text{if } a \gg b \quad S_{\max} = 2 \mu a a$$

twice greater than circular cross section

with $\psi(x_1, x_2)$ known, $\phi(x_1, x_2)$ can be obtained, and the displacement $u_3(x_1, x_2)$ can be calculated.

3. Direct Method

for a member of rectangular cross section under twisting moment

$$\nabla^2 \phi = 0 \quad \text{in } R \quad (\text{a})$$

$$\phi_{,n} = x_2 n_1 - x_1 n_2 \quad \text{on } C \quad (\text{b})$$

$$\text{or } \nabla^2 \psi = 0 \quad \text{in } R \quad (\text{a}')$$

$$\psi = \frac{1}{2} (x_1^2 + x_2^2) \quad \text{on } C \quad (\text{b}')$$

from (b) on $x_1 = \pm a$

$$\psi = \frac{1}{2} (a^2 + x_2^2) \quad \psi_{,22} = 1$$

$$\begin{aligned} &\text{on } x_2 = \pm b \\ \psi &= \frac{1}{2} (x_1^2 + b^2) \quad \psi_{,11} = 1 \end{aligned}$$

it will be supposed that $b \geq a$ and that the x_3 -axis passes through O
the boundary conditions are complicated, so introduce a function $f(x_1, x_2)$
such that

$$\begin{aligned} f(x_1, x_2) &= \psi_{,11} + 1 = -\psi_{,22} + 1 \\ \text{then } f_{,22} &= -\psi_{,1122} \\ f_{,11} &= \psi_{,2211} \\ \therefore f_{,11} + f_{,22} &= f_{,ii} = 0 \end{aligned} \quad (c)$$

f is also harmonic, and the boundary conditions for $f(x_1, x_2)$ is

$$\begin{aligned} f(x_1, x_2) &= 0 && \text{on } x_1 = \pm a \\ f(x_1, x_2) &= 2 && \text{on } x_2 = \pm b \end{aligned} \quad (d)$$

assume $f(x_1, x_2)$ in the form

$$\begin{aligned} f(x_1, x_2) &= X(x_1) Y(x_2) \\ \nabla^2 f &= 0 \quad \frac{X''}{X} + \frac{Y''}{Y} = 0 \\ \text{or} \quad \frac{X''}{X} &= -\frac{Y''}{Y} = -k^2 \end{aligned}$$

$$\begin{aligned} \text{we have } X(x_1) &= A \cos k x_1 + B \sin k x_1 \\ Y(x_2) &= A' \cosh k x_2 + B' \sinh k x_2 \end{aligned}$$

$\therefore$ the solution must satisfy the boundary conditions, so we reject the odd functions, i.e. $\sin k x_1$ and $\sinh k x_2$

$$\therefore f(x_1, x_2) = C \cos k x_1 \cosh k x_2$$

$$\text{on } x_1 = \pm a$$

$$f(x_1, x_2) = C \cos k a \cosh k x_2 = 0$$

$$\therefore \cos k a = 0 \quad [\because \cosh k x_2 \neq 0]$$

$$\text{then } k a = [(2 n + 1) \pi] / 2 \quad n = 0, 1, \dots$$

$$\text{or } k_n = [(2 n + 1) \pi] / 2 a \quad \text{eigenvalues}$$

$$f(x_1, x_2) = \sum_{n=0}^{\infty} C_n \cos k_n x_1 \cosh k_n x_2 \quad \text{eigen-functions}$$

to determine C_n , use the remaining boundary conditions on $x_2 = \pm b$

$$\begin{aligned} f(x_1, x_2) &= \sum_{n=0}^{\infty} C_n \cos k_n x_1 \cosh k_n b \\ &= \sum_{n=0}^{\infty} C_n \cos k_n x_1 \cosh [(2 n + 1) \pi b] / 2 a \end{aligned}$$

multiple $\cosh [(2 m + 1) \pi x_1] / 2 a$ in both side and integrate term by term

$$\therefore \int_{-a}^a 2 C_n \cos k_m x_1 \cos k_n x_1 dx_1 = \begin{cases} 0 & m \neq n \\ a & m = n \end{cases}$$

$$\therefore \int_{-a}^a 2 \cos k_m x_1 dx_1 = a C_m \cosh k_m b$$

$$\frac{(-1)^m 8 a}{\pi (2 m + 1)} = a C_m \cosh k_m b$$

$$\therefore C_m = \frac{(-1)^m 8}{\pi (2 m + 1)} \frac{1}{\cosh k_m b}$$

then the solution of $f(x_1, x_2)$ is

$$f(x_1, x_2) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2 n + 1)} \frac{\cosh k_n x_2}{\cosh k_n b} \cos k_n x_1$$

$$\therefore \psi_{,11} = f(x_1, x_2) - 1$$

$$\therefore \psi_{,1} = -x_1 + \int f(x_1, x_2) dx_1$$

$$\begin{aligned}
\text{then } \sigma_{32} &= \mu a (-\psi_{,1} + x_1) \\
&= \mu a [2x_1 - \int f(x_1, x_2) dx_1] \\
&= \mu a [2x_1 - \frac{16a}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\cosh k_n x_2}{\cosh k_n b} \sin k_n x_1]
\end{aligned}$$

the constant of the integral must satisfy $\sigma_{32} = 0$ on $x_2 = \pm b$

$$\begin{aligned}
\text{and } \sigma_{31} &= \mu a (\psi_{,1} + x_2) \\
&= - \frac{16a\mu a}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\sinh k_n x_2}{\cosh k_n b} \cos k_n x_1
\end{aligned}$$

which satisfy $\sigma_{31} = 0$ on $x_1 = \pm a$

it is also seen that $\sigma_{31} = 0$ when $x_2 = 0$, while the shear σ_{32} has a maximum at the midpoint of the longer side is equal to

$$\tau_{\max} = \sigma_{32} \Big|_{(a,0)} = 2a\mu a \left[1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \operatorname{sech} k_n b \right]$$

the series converge so rapidly

for $a = b$

$$\tau_{\max} = 2a\mu a \left\{ 1 - \frac{8}{\pi^2} \left[\operatorname{sech} \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \operatorname{sech} \frac{(2n+1)\pi}{2} \right] \right\}$$

hence, for practical calculations, the value of $\tau_{\max}$ can be assumed to be given by the formula

$$\tau_{\max} \simeq 2 a \mu a \left(1 - \frac{8}{\pi^2} \operatorname{sech} \frac{\pi b}{2 a} \right)$$

and the twisting moment

$$\begin{aligned} M &= \int_{-b}^b \int_{-a}^a (x_1 \sigma_{32} - x_2 \sigma_{31}) dx_1 dx_2 \\ &\simeq \frac{\mu a (2b) (2a)^3}{3} - \frac{64 \mu a (2a)^4}{\pi^5} \tanh \frac{\pi b}{2 a} \end{aligned}$$

and the warping function $\phi(x_1, x_2)$ can be found

$$\phi(x_1, x_2) = x_1 x_2 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \frac{\sinh k_n x_2}{\cosh k_n b} \sin k_n x_1$$

the displacement u_3 is given by $a \phi(x_1, x_2)$, the contour lines for the case $b = a$ are shown as follows

4. Membrane analogy

consider a thin homogeneous membrane under a uniform tension T per unit length, if p is the pressure per unit area of the membrane

equation of equilibrium for the membrane

$$\begin{aligned} u_{3,11} + u_{3,22} &= -p/T && \text{in } R \\ u_3 &= 0 && \text{on } C \end{aligned}$$

for torsional problem, we have

$$\begin{aligned} \Psi_{,ii} &= -2 && \text{in } R \\ \Psi &= 0 && \text{on } C \end{aligned}$$

and

$$\begin{aligned} \sigma_{31} &= \mu a \Psi_{,2} \\ \sigma_{32} &= -\mu a \Psi_{,1} \end{aligned}$$

if we let

$$u_3 = (p/2T) \Psi$$

then two problems has the same form

and

$$\begin{aligned} \sigma_{31} &= \mu a \Psi_{,2} = \mu a (p/2T) u_{3,2} \\ \sigma_{32} &= -\mu a \Psi_{,1} = -\mu a (p/2T) u_{3,1} \end{aligned}$$

where $u_{3,1}$ and $u_{3,2}$ are the slope of the membrane

$$M = 2\mu a \int \int \Psi dA = \mu a (p/T) \int \int u_3 dA$$

where $\int \int u_3 dA$ is the volume between the membrane and the x_1 - x_2 plane

for the same a

$$\begin{aligned} \frac{(\tau_{\max})_c}{(\tau_{\max})_{nc}} &= \frac{(\text{slope})_c}{(\text{slope})_{nc}} \\ \frac{(M)_c}{(M)_{nc}} &= \frac{(V)_c}{(V)_{nc}} \end{aligned}$$

i.e. for a given a , M and $\tau_{\max}$ for circular cross section are exactly known, then M and $\tau_{\max}$ for noncircular cross section can be found.