

2. Analysis of Strains

Theory of Deformation

when the relative position of points in a body is altered, then the body is strained, the change in relative of points is a deformation

for a body, every pair of its points remain invariant, it is called rigid body

configuration from $D \rightarrow D^*$

$$P(x_i) \rightarrow P^*(\xi_i)$$

$$\xi_i = \xi_i(x_i) \quad \text{and} \quad x_i = x_i(\xi_i) \quad \text{single-valued function}$$

Definition : Displacement transformation

the transformation T taken $P(x_i)$ into $P^*(\xi_i)$ is written

$$T: \quad \xi_i = \xi_i(x_1, x_2, x_3) = \xi_i(x_i) = x_i + u_i(x_i)$$

where u_i is defined as the displacement and

$$u_i = \xi_i - x_i$$

for real transformation, T is a single-valued continuous 1-1 mapping of x_i into ξ_i , thus a unique inverse T^{-1} exists, and

$$T^{-1}: \quad x_i = x_i(\xi_1, \xi_2, \xi_3) = x_i(\xi_i)$$

Rigid transformation : T is rigid if $|PQ| = |P^*Q^*|$ for arbitrary points P and Q in the body

Definition : Linear Displacement Transformation (Affine Transformation)

a linear displacement transformation L is one for which

$$L : \quad \xi_i =$$

$$\text{i.e.} \quad \xi_1 =$$

$$\xi_2 =$$

$$\xi_3 =$$

where the coefficients α_{ij} are constants, and α_i is a translation

$\therefore$ the transformation is one to one mapping, $\therefore x_j$ may solvable,
i.e. $x_j = x_j(\xi_i)$, for $\det(\alpha_{ij} + \delta_{ij}) \neq 0$, then L^{-1} exists

$$L^{-1} : \quad x_i =$$

Theorem : L carries surface of degree n into surface of degree n , in particular, plane are carried into plane, straight line into straight line

for a vector A_i , L takes A_i into A_i^* by

$$A_i^* = (\alpha_{ij} + \delta_{ij}) A_j$$

$$A_i = x_i^1 - x_i^0$$

$$A_i^* = \xi_i^1 - \xi_i^0$$

$$A_i^* =$$

$$=$$

in general, A_i and A_i^* differ in direction and magnitude

$$\delta A_i = A_i^* - A_i = \alpha_{ij} A_j$$

Theorem : Under L , two parallel straight line segments of equal length are carried into two parallel straight line segments of equal length

$$\mathbf{a} \parallel \mathbf{b}, \quad |\mathbf{a}| = |\mathbf{b}| \quad \Rightarrow \quad a_i = b_i$$

$$a_i^* =$$

$$b_i^* =$$

$$\therefore \quad \mathbf{a}^* \parallel \mathbf{b}^* \quad \text{and} \quad |\mathbf{a}^*| = |\mathbf{b}^*|$$

$$[\alpha_i \text{ and } \alpha_{ij} \text{ are independent of } x_i]$$

the transformation carries the same deformation, independently of the position of the parts of the body, is called homogeneous deformation

Theorem : Given two linear transformations L_1 and L_2 , where

$$L_1 : [\alpha_i^{(1)} \text{ and } \alpha_{ij}^{(1)}], \quad P(x_i) \Rightarrow P^{**}(\eta_i)$$

$$L_2 : [\alpha_i^{(1)} \text{ and } \alpha_{ij}^{(1)}], \quad P^{**}(\eta_i) \Rightarrow P^*(\xi_i)$$

then, there exist a linear transformation L , where $L = L_1 L_2$, such that

$$L : (\alpha_i \text{ and } \alpha_{ij}) \quad P(x_i) \Rightarrow P^*(\xi_i)$$

proof : we must find the α_i and α_{ij}

$$\therefore \quad L_1 : \quad \eta_i = \quad (a)$$

$$L_2 : \quad \xi_i = \quad (b)$$

substitute (a) into (b), then

$$\xi_i =$$

$$=$$

$$=$$

$$=$$

where $\alpha_i =$

$$\alpha_{ij} =$$

Theorem : the α_i and α_{ij} associated are the components of a vector and second order tensor, respectively

proof : for L $\xi_i =$ (a)

$$\xi_{i'} =$$
 (b)

(a) is referred to (x_1, x_2, x_3)

(b) is referred to (x_1', x_2', x_3')

$$\therefore x_{i'} =$$

$$(b) \Rightarrow \xi_{i'} =$$

$$=$$

multiply by λ_{ji}

$$\lambda_{rj} \xi_r' =$$

$$\xi_j =$$

$$\xi_i =$$

$$(a) - (c) \quad 0 =$$

for arbitrary x_j in $\underline{D}$, then $\alpha_i =$

$$\alpha_{ij} =$$

then α_i is a vector, α_{ij} is a second order tensor

Definition : Lagrangian components of finite strain L_{ij}

$$L_{ij} =$$

Lagrangian description (or material description) : whose independent variables are the position $\mathbf{a}$ of the particle and the time t (in elasticity, the reference configuration is usually chosen to be the natural or unstressed state)

$$x_i = x_i(a_i, t)$$

where a_1, a_2, a_3 are the material coordinate of the particle, and x_1, x_2, x_3 are the spatial coordinate

Eulerian description (or spatial description) : whose independent variables are the present position $\mathbf{x}$ occupied by the particle at time t and the present time t (it is the description most used in fluid mechanics)

$$a_i = a_i(x_i, t)$$

Theorem : Given an arbitrary vector A_i , whose image under $L(\alpha_i, \alpha_{ij})$ is A_i^* , then

$$\begin{aligned} & A_i^* A_i^* - A_i A_i = \\ \text{proof : } & A_i^* A_i^* = \\ & = \\ & = \end{aligned}$$

$$\therefore A_i^* A_i^* - A_i A_i =$$

$$=$$

Theorem : L is rigid if and only if $L_{ij} = 0$

proof : by definition, L is rigid if and only if $A_i^* A_i^* = A_i A_i$

(a) if L is rigid, then $2 L_{ij} A_i A_j = 0$

for arbitrary A_i and A_j , $\Rightarrow L_{ij} = 0$

(b) if $L_{ij} = 0$, then $A_i^* A_i^* - A_i A_i = 0$

thus $|\mathbf{A}| = |\mathbf{A}^*|$, $\Rightarrow L$ is rigid

Theorem : L_{ij} are the components of symmetric 2nd order tensor

proof : $L_{ij} =$

$\therefore \alpha_{ij}$ is a tensor, we only need to proof $\alpha_{ki} \alpha_{kj}$ is a tensor

let $C_{ij} =$

$C'_{\alpha\beta} =$

$=$

$=$

$=$

$\therefore C_{ij}$ is a 2nd order tensor, then L_{ij} is a symmetric 2nd order tensor

Physical Significance of L_{ij}

take three vectors A, B, C along
 x_1, x_2, x_3 axes, respectively

let A^*, B^*, C^* be their images
 under L

define $A^* = |A^*|$, $A = |A|$ etc.

$$\text{let } E_1 = \frac{A^* - A}{A} \quad A^* = \quad \quad \quad (a1)$$

$$E_2 = \frac{B^* - B}{B} \quad B^* = \quad \quad \quad (a2)$$

$$E_3 = \frac{C^* - C}{C} \quad C^* = \quad \quad \quad (a3)$$

so E_i is the percent change in length of the vectors in x_i directions

$$\text{by definition } A_i^* A_i^* - A_i A_i =$$

$$\text{then } A^* = \quad \quad \quad (b)$$

eliminate A^* from (a1) and (b)

$$E_1 =$$

$$\text{similarly } E_2 =$$

$$E_3 =$$

as L_{11} becomes small,

$$\therefore E_1 \doteq$$

$$E_2 \doteq$$

$$E_3 \doteq$$

let $\cos \theta_{12} =$

θ_{12} is the angle between **A** and **B**

after deformation

and let $\varphi_{12} =$

by definition $\cos \theta_{12} =$

$$\therefore A_i^* =$$

then $A_1^* =$

$$A_2^* =$$

$$A_3^* =$$

similarly $B_1^* =$

$$B_2^* =$$

$$B_3^* =$$

$$\therefore \cos \theta =$$

$$=$$

for infinitesimal case $E_1 \ll 1$, $E_2 \ll 1$, $\sin \phi_{12} \doteq \phi_{12}$

$$\therefore \phi_{12} = 2 L_{12} \quad \text{angle change of } \mathbf{A} \text{ and } \mathbf{B}$$

$$\text{similarly } \phi_{13} = 2 L_{13} \quad \text{and} \quad \phi_{23} = 2 L_{23}$$

for rigid rotation, $L_{12} = L_{13} = L_{23} = 0$

Principal Strains

Theorem : For every L , there exists at least one set of three mutually perpendicular straight line whose images under L are mutually perpendicular

proof : $\because L_{ij}$ is a symmetric 2nd order tensor, $\therefore$ it may be diagonalized by a transformation tensor, such that

$$L_{ij} = \begin{pmatrix} L_I & 0 & 0 \\ 0 & L_{II} & 0 \\ 0 & 0 & L_{III} \end{pmatrix}$$

where eigenvectors are along $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$

$$\therefore \begin{aligned} L'_{12} &= 0 & \phi_{12} &= 0 \\ L'_{13} &= 0 & \phi_{13} &= 0 \\ L'_{23} &= 0 & \phi_{23} &= 0 \end{aligned}$$

so that the eigenvector directions
(principal directions) are the set of
3 mutually perpendicular vectors

Rigid Body Motion

$$\therefore \delta A_i =$$

$$\text{let } A = |\mathbf{A}| \quad \delta A = \text{change in length of } A$$

then $A \delta A =$

$=$

$=$

since for rigid body transformation, δA vanishes for all value of A_1, A_2, A_3 , then

$$\alpha_{11} = \alpha_{22} = \alpha_{33} = 0$$

and $\alpha_{12} + \alpha_{21} =$

thus the transformation represent a rigid body motion if and only if

$$\alpha_{ij} = -\alpha_{ji} \quad \text{i.e.} \quad \alpha_{ij} \text{ is a skew-symmetric tensor}$$

Decompoite of tensor α_{ij}

$$\alpha_{ij} =$$

where $e_{ij} =$ is a symmetric tensor

$\omega_{ij} =$ is a skew-symmetric tensor

e_{ij} is called strain tensor

ω_{ij} is called rotation tensor (rigid body motion)

Definition : Infinitesimal Linear Displacement Transformation $L [\alpha_i, \alpha_{ij}]$ is said to be infinitesimal (designated by $\mathcal{L}$) if non-linear terms in the coefficients are negligible compared to linear terms

Theorem : Given $\mathcal{L}_1[\alpha_i^{(1)}, \alpha_{ij}^{(1)}]$ and $\mathcal{L}_2[\alpha_i^{(2)}, \alpha_{ij}^{(2)}]$, then

$$\mathcal{L}_1 \mathcal{L}_2 = \mathcal{L}_2 \mathcal{L}_1 =$$

where $\alpha_i =$

$$\alpha_{ij} =$$

Theorem : Given $\mathcal{L}[\alpha_i, \alpha_{ij}]$, then $e_{ij} = L_{ij}$

i.e. $e_{11} = E_1$ $e_{12} = \phi_{12}$ etc.

$$\therefore L_{ij} =$$

$$\therefore E_1 =$$

$$\phi_{12} =$$

Theorem : $\mathcal{L}[\alpha_i, \alpha_{ij}]$ is rigid if and only if $e_{ij} = 0$

i.e. $\alpha_{ij} = -\alpha_{ji}$

$$\omega_{ij} = \alpha_{ij} \quad \text{skew-symmetric, rigid body motion}$$

Definition : Infinitesimal rotation vector

$$\omega_i =$$

Theorem : An infinitesimal rigid displacement transformation $\mathcal{L}$ can be written

as $\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2$

where $\mathcal{L}_1 = \mathcal{L}_1[\alpha_i, 0]$ pure translation

$$\mathcal{L}_2 = \mathcal{L}_2[0, \omega_{ij}] \quad \text{pure rotation}$$

proof : $\mathcal{L}_1[\alpha_i, 0] : \quad \xi_i = x_i + \alpha_i \quad (a)$

$$\mathcal{L}_2 [0, \omega_{ij}] : \quad \xi_i = (\omega_{ij} + \delta_{ij}) x_j \quad (b)$$

$$\begin{aligned} \therefore \quad \omega_i &= \\ \text{multiply } \varepsilon_{ipq} \quad \varepsilon_{ipq} \omega_i &= \\ &= \\ &= \\ &= \end{aligned} \quad (c)$$

substitute (c) into (b)

$$\begin{aligned} \mathcal{L}_2 [0, \omega_{ij}] : \quad \xi_i &= \\ &= \\ \xi &= \end{aligned}$$

$\omega \times \mathbf{x} \perp$ the plane containing
 ω and $\mathbf{x}$

$$\begin{aligned} \therefore \quad \mathcal{L}_1 \mathcal{L}_2 : \quad \xi_I &= \\ \text{is a translation + rotation} \end{aligned}$$

Definition : Pure Deformation

$$\begin{aligned} \mathcal{L}[\alpha_i, \alpha_{ij}] \text{ is pure deformation is } \alpha_i = 0, \alpha_{ij} = \alpha_{ji} [\text{i.e. } \alpha_i = \omega_{ij} = 0] \\ \text{i.e. } \mathcal{L} = \mathcal{L}[0, e_{ij}] \end{aligned}$$

Theorem : Every $\mathcal{L}[\alpha_i, \alpha_{ij}]$ may be written as $\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3$

$$\begin{aligned} \text{where } \mathcal{L}_1[\alpha_i, 0] & \quad \text{pure translation} \\ \mathcal{L}_2[0, \omega_{ij}] & \quad \text{pure rotation} \\ \mathcal{L}_1[0, e_{ij}] & \quad \text{pure deformation} \end{aligned}$$

moreover, this decomposition is unique

Strain Quadric

for each point $P_0(x_i^0)$, we shall associate a quadric surface, which enables one to determine the elongation of any vector

for a vector $\mathbf{A} (P_0 P) \quad \mathbf{A} = (x_i - x_i^0) \mathbf{e}_i$

then the extension of $\mathbf{A}$ is defined by

$$e = \frac{1}{a} \sqrt{\mathbf{A} \cdot \mathbf{A}} \quad \text{where } a \text{ is the length of vector } \mathbf{A}$$

$$\therefore A_i \delta A_i = \frac{1}{a^2} \left[(x_i - x_i^0) \delta (x_i - x_i^0) \right] \quad [P_0(x_i^0) \text{ is selected as origin}]$$

$$\therefore A_i \delta A_i =$$

$$\therefore a^2 e =$$

consider a quadratic function

$$2 g(x_i) =$$

where k is a constant, $\pm$ sign is chosen to make the surface real, then

$$e =$$

the extension of any line through $P_0(x_i^0)$ is inversely proportional to the square of the radius vector that runs along the line from $P_0(x_i^0)$ to a point $P(x_i)$ on the quadric, then the maximum and minimum elongation will be in the directions of the axes of the quadric.

Differentiation of $2g(x_i)$

$$\frac{[2g(x_i)]}{x_i} =$$

g / x_i is the normal direction on the quadric surface at point $P(x_i)$

i.e. the vector δA is directed along the normal to the plane tangent to the surface $\alpha_{ij} x_i x_j = \pm k^2$

in the axes of the quadric, the extension along the vector direction, i.e. principal direction

Principal Strain : the direction of A is not altered by strain, i.e. A and δA in same direction

$$\delta A_i =$$

$$e_{ij} x_j =$$

eigenvalue problem

for nontrivial solution

$$| e_{ij} - e \delta_{ij} | = 0$$

i.e.

where J_i are strain invariants, and e_{ij} are all real

$$J_1 =$$

$$J_2 =$$

$$J_3 =$$

Definition : Volume Dilatation

$$D =$$

where V is infinitesimal volume, and V^* is its image under $\mathcal{L}[0, e_{ij}]$

Theorem : The Volume Dilatation $D = J_1$

proof : take $\mathbf{e}_i$ to be principal strain directions

$$V =$$

$$V^* =$$

$$a^* =$$

$$b^* =$$

$$c^* =$$

$$V^* =$$

$$V^* / V =$$

$$\therefore D =$$

Displacement :

$$P(x_i^0) \rightarrow P^*(\xi_i^0)$$

$$Q(x_i) \rightarrow Q^*(\xi_i)$$

the transformation T takes

$$T: \quad \xi_i^0 = x_i^0 + u_i^0(x_i^0)$$

$$u_i^0(x_i^0) = \xi_i^0 - x_i^0 \quad \text{is the displacement of } x_i^0$$

$\therefore u_i$ is single-valued and continuous throughout the region, it will be assumed u_i of class C_3 (the function of u_i is continuous after differentiation for three times), i.e. $u_{i,jkl}$ are continuous

$$Q \rightarrow Q^* \quad u_i(x_i) =$$

$$=$$

the deformed vector A^* has the component

$$A_i^* =$$

$$\text{and} \quad \delta A_i =$$

$$=$$

$$=$$

$$=$$

$$[\therefore u_i(x_i^0 + A_i) = \quad]$$

if A is sufficiently small, then $\delta A_i =$

$$\text{but} \quad \delta A_i =$$

$$\therefore \quad \alpha_{ij} =$$

i.e. $e_{ij} =$ strain-disp. relation

$$\omega_{ij} =$$

$$L_{ij} =$$

and $\omega_i =$

Question : Given $e_{ij}(x_i)$, do single-valued u_i exists? [six eq. for 3 unknowns]

Let $u_i^{(1)}$ and $u_i^{(2)}$ exist and both satisfy

$$e_{ij} = \quad \text{for the same } e_{ij}$$

let $u_i =$

$$\begin{aligned} \text{then } u_{i,j} + u_{j,i} &= \\ &= \end{aligned}$$

$$\therefore u_{i,jk} + u_{j,ik} =$$

and $u_{i,jk} =$

$$= -$$

$$\therefore u_{i,jk} = 0$$

integration $\therefore u_{i,j} = \text{constant} = C_{ij}$

$$u_{i,j} + u_{j,i} =$$

$$\therefore C_{ij} =$$

integration again

$$u_i =$$

$$=$$

$$\omega_{ij} =$$

$$\omega_{ij} x_j =$$

$$\therefore \quad \mathbf{u} =$$

i.e. the solution is unique within a rigid body displacement

Theorem : Given $e_{ij}(x_i)$ in arbitrary D , then a necessary condition for the existence of u_i , such that strain-displacement hold is that e_{ij} satisfy the compatibility condition, i.e.

$$e_{ij,kl} + e_{kl,ij} - e_{il,jk} - e_{jk,il} = 0$$

proof : $e_{ij} =$

$$e_{ij,kl} =$$

$$e_{kl,ij} =$$

$$- e_{il,jk} =$$

$$- e_{jk,il} =$$

$$\Rightarrow e_{ij,kl} + e_{kl,ij} - e_{il,jk} - e_{jk,il} = 0$$

the system consists $3^4 = 81$ equations, but some of this are identically satisfied, and some are repetitions because of the symmetry in indices ij and kl , it may be reduced to 6 linear independent equations, there are

$$i = j = 1 \quad k = l = 2$$

$$i = j = 1 \quad k = l = 3$$

$$i = j = 2 \quad k = l = 3$$

$$i = j = 1 \quad k = 2, l = 3$$

$$i = j = 2 \quad k = 1, l = 2$$

$$i = j = 3 \quad k = 1, l = 2$$

since the compatibility conditions is a necessary condition, we may have strain fields satisfying them, but for which we can not find displacement field

Theorem : Let e_{ij} satisfy the compatibility conditions in a simple connected region D , then there exists a single valued u_i of $P'(\mathbf{x}')$ in D . Moreover, u_i is given by the line integrals

$$u_i(\mathbf{x}') =$$

$$\text{where} \quad U_{ij} =$$

and $P^0(\mathbf{x}^0)$ in D , the displacements $u_i^0(\mathbf{x}^0)$ and the components of rotation tensor ω_{ij}^0 are known

$$\therefore u_i(\mathbf{x}') =$$

$$=$$

$$=$$

$$\text{but} \quad \int_{P^0}^{P'} \omega_{ij} dx_j =$$

$$=$$

$$=$$

$$=$$

$$\therefore u_i(\mathbf{x}') =$$

$$\therefore \omega_{ik,j} =$$

$$=$$

$$=$$

$$=$$

$$\text{let } U_{ij} =$$

$$\text{then } u_i(\mathbf{x}') =$$

for u_i be single valued displacement components, we need to proof

1. the line integral is independent of path
2. u_i satisfy the compatibility conditions

Recall Stokes' Theorem

$$\int_B \mathbf{n} \cdot \text{curl } \mathbf{v} dA =$$

where B is any surface in D

if $\text{curl } \mathbf{v} = 0$, then $\oint \mathbf{v}_i dx_i = 0$, i.e. independent of path

for $\text{curl } \mathbf{v} = 0$, $\varepsilon_{ijk} v_{k,j} = 0 \rightarrow v_{k,j} = v_{j,k}$ is symmetric

consider the integral $\int_{P^0}^{P'} U_{ij} dx_j$

for $i = 1$ $\int_{P^0}^{P'} U_{ij} dx_j =$

if the integral is independent of path, then $U_{1j,l} = U_{1l,j}$

i.e. if $\int_{P^0}^{P'} U_{ij} dx_j$ is independent of path, then $U_{ij,l} = U_{il,j}$

$$\therefore U_{ij,l} =$$

$$U_{il,j} =$$

$$\therefore U_{ij,l} - U_{il,j} =$$

$$=$$

i.e. $\int_{P^0}^{P'} U_{ij} dx_j$ is independent of path

for $u_i(\mathbf{x}') =$

$$\frac{u_i}{x_l'} =$$

$$=$$

$$=$$

similarly $\frac{u_l}{x_i'} =$

$$\therefore \frac{u_i}{x_l'} + \frac{u_l}{x_i'} =$$

$$\text{i.e.} \quad e_{il}(\mathbf{x}') =$$

$\therefore$ the compatibility condition is satisfied

$$\text{e.g. given } e_{23} = a x_1 / 2 \quad e_{13} = -a x_2 / 2 \quad \text{others} = 0$$

$$\text{then } U_{11} =$$

$$U_{12} =$$

$$=$$

$$U_{13} =$$

$$U_{21} =$$

$$U_{22} =$$

$$U_{23} =$$

$$U_{31} =$$

$$U_{32} =$$

$$U_{33} =$$

let $P^0(\mathbf{x}^0)$ be the origin, and $\omega_{ij}^0 = 0$, $u_i^0 = 0$

$$\text{then } u_1 =$$

$$=$$

$$=$$

$$=$$

$$=$$

similarly $u_2 =$

$$u_3 =$$

check $e_{12} =$

$$=$$

$$e_{23} =$$

$$e_{13} =$$

others = 0

(O. K.)