

1. Mathematical Preliminaries

Vector Notation

Let x_1, x_2, x_3 be a right handed coordinate system, and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors along the x_1, x_2, x_3 axes, respectively

vector from A to B is written $\mathbf{AB}$

in component form

$$\mathbf{V} = \mathbf{AB} =$$

the magnitude of $\mathbf{V}$ is written as $|\mathbf{V}|$

$$|\mathbf{V}| =$$

scalar (dot) product of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} =$$

vector (cross) product of $\mathbf{a}$ and $\mathbf{b}$ is

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \end{aligned}$$

scalar triple product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

vector triple product

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} =$$

gradient of a function

$$\text{grad} (u) = \nabla u =$$

divergence of a vector

$$\text{div} (\mathbf{v}) = \nabla \cdot \mathbf{v} =$$

curl of a vector

$$\text{curl} (\mathbf{v}) = \nabla \times \mathbf{v} =$$

$$=$$

Laplacian of a function

$$\nabla^2 \varphi = \nabla \cdot (\nabla \varphi) =$$

Identities :

$$\nabla (u v) = u \nabla v + v \nabla u$$

$$\nabla \cdot (u \mathbf{v}) = \mathbf{v} \cdot \nabla u + u \cdot (\nabla \mathbf{v})$$

$$\nabla \times (u \mathbf{v}) = \nabla u \times \mathbf{v} + u (\nabla \times \mathbf{v})$$

$$\nabla^2 (u \mathbf{v}) = u \nabla^2 \mathbf{v} + \mathbf{v} \nabla^2 u + 2 (\nabla u) \cdot (\nabla \mathbf{v})$$

$$\text{div} (\text{curl } \mathbf{v}) = \nabla \cdot (\nabla \times \mathbf{v}) = 0$$

$$\text{curl} (\text{grad } u) = \nabla \times (\nabla u) = 0$$

$$\text{curl} (\text{curl } \mathbf{v}) = \nabla \times (\nabla \times \mathbf{v}) = \nabla (\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla) \mathbf{v}$$

Index Notation

The coordinates x_1, x_2, x_3 may be expressed, in general, by x_i , where i is understood to take on value 1, 2 and 3

Instead of writing $\mathbf{v}$ in component form, just write v_i as i^{th} ($i = 1, 2, 3$) component of the vector $\mathbf{v}$

Definition : Summation Convention

In any single term, wherever a subscript is repeated, this indicates summation over this subscript from 1 to 3

e.g. $\mathbf{v} =$

$$|\mathbf{a}| =$$

$$\mathbf{a} \cdot \mathbf{b} =$$

$$|\mathbf{a}| |\mathbf{b}| =$$

Note : no subscript may be repeated more than once in a single term, whenever a subscript is repeated, it is a dummy subscript

e.g. $a_i a_i = a_j a_j$

Definition : Differential Convention

Subscripts attached to a function $f(x_i)$ be preceded by comma indicate differentiation w. r. t. the variables

e.g. $f_{,i} =$

$$v_{i,jk} =$$

$$v_{i,3} =$$

$$v_{i,jj} =$$

$$=$$

Definition : Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

i.e. $\delta_{11} = \delta_{22} = \delta_{33} = 1$

$$\delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0$$

$$\delta_{ii} =$$

$$\delta_{ij} v_j =$$

$$a_{ijk} \delta_{ip} =$$

(this is called contraction)

$$\delta_{ij} \delta_{ij} =$$

$$\delta_{ij} \delta_{jk} =$$

$$=$$

$$=$$

$$\therefore \delta_{ij} \delta_{jk} =$$

and

$$x_{i,j} = \delta_{ij}$$

$$\delta_{ij} \delta_{jk} \delta_{ki} = \delta_{ik} \delta_{ki} = \delta_{ii} = 3$$

Definition : Permutation Symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if the value of } i, j, k \text{ from an even permutation of } 1, 2, 3 \\ -1 & \text{if the value of } i, j, k \text{ from an odd permutation of } 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{i.e.} \quad \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

$$\text{others} = 0$$

$$\text{Identities} \quad \varepsilon_{ijk} \varepsilon_{ijk} =$$

$$\varepsilon_{ijk} \varepsilon_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}$$

(this is called ε - δ identity)

The vector notation can be expressed in index notation

$$\mathbf{a} \cdot \mathbf{b} =$$

$$\mathbf{a} \times \mathbf{b} =$$

$$\mathbf{e}_i \cdot \mathbf{e}_j =$$

$$\mathbf{e}_j \times \mathbf{e}_k =$$

$$\nabla u =$$

$$\nabla \cdot \mathbf{v} =$$

$$\nabla \times \mathbf{v} =$$

$$\nabla^2 u =$$

$$\nabla^2 \mathbf{v} =$$

use index notation to prove the vector identities

$$\text{e.g. } (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$\text{div} (\text{curl } \mathbf{v}) =$$

$$=$$

$$=$$

$$\therefore \varepsilon_{ijk} v_{k,ij} =$$

Determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

=

=

$$\begin{array}{llll}
 \text{let} & a_1 = a_{11} & a_2 = a_{12} & a_3 = a_{13} & a_i = a_{1i} \\
 & b_1 = a_{21} & b_2 = a_{22} & b_3 = a_{23} & \text{i.e. } b_i = a_{2i} \\
 & c_1 = a_{31} & c_2 = a_{32} & c_3 = a_{33} & c_i = a_{3i}
 \end{array}$$

then

$$\det(a_{ij}) =$$

consider the sum

$$\begin{aligned}
 \varepsilon_{ijk} a_{ir} a_{js} a_{kp} &= \\
 &=
 \end{aligned}$$

the sum is skew symmetric w. r. t. the index r and s , similarly it is skew symmetric w. r. t. the index r and p , and also to the index s and p

this is complete skew symmetric in the index r, s , and p

for $r, s, p = 1, 2, 3$

$$\varepsilon_{ijk} a_{ir} a_{js} a_{kp} = \varepsilon_{ijk} a_{i1} a_{j2} a_{k3}$$

for $r, s, p = 2, 1, 3$

$$\begin{aligned}
 \varepsilon_{ijk} a_{ir} a_{js} a_{kp} &= \varepsilon_{ijk} a_{i2} a_{j1} a_{k3} \\
 &= \\
 &=
 \end{aligned}$$

$$\therefore \quad \varepsilon_{ijk} a_{ir} a_{js} a_{kp} =$$

$$\text{in general} \quad \varepsilon_{rsp} \det(a_{ij}) =$$

$$=$$

Coordinate Rotation

let $\mathbf{e}_i, x_i$ be original coordinate

$\mathbf{e}_i', x_i'$ be new coordinate

Definition : Direction cosines

The direction cosines corresponding to a rotation of a right-handed coordinate system x_i into the right-handed coordinate system x_α' are denoted by $\lambda_{\alpha i}$, i.e.

$$\lambda_{\alpha i} = \cos(x_\alpha', x_i)$$

$$=$$

if we know the coordinate in one system, what are they in other

$$\mathbf{x} =$$

dot with vector $\mathbf{e}_j$ in both side

$$x_\alpha' \mathbf{e}_\alpha' \cdot \mathbf{e}_j =$$

$$\lambda_{\alpha j} x_\alpha' =$$

similarly $x_{\alpha}' =$

The direction cosines are orthonormal

$$\text{i.e.} \quad \lambda_{\alpha i} \lambda_{\alpha j} =$$

$$\text{or} \quad \lambda_{i\alpha} \lambda_{j\alpha} =$$

$$\begin{aligned} \text{proof :} \quad & \therefore x_j = \\ & \delta_{ij} x_i = \\ & x_i (\lambda_{\alpha j} \lambda_{\alpha i} - \delta_{ij}) = \\ & \therefore \lambda_{\alpha j} \lambda_{\alpha i} = \end{aligned}$$

Transformation Law for unit vector

$$\mathbf{x} =$$

$$\therefore x_{\alpha}' (\lambda_{\alpha i} \mathbf{e}_i - \mathbf{e}_{\alpha}') = 0$$

$$\text{for arbitrary } x_{\alpha}' \quad \therefore \mathbf{e}_{\alpha}' =$$

$$\text{similarly} \quad \mathbf{e}_i =$$

$$\text{we now shown that} \quad \det (\lambda_{ij}) = 1$$

scalar triple product

$$(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = 1$$

$$(\lambda_{\alpha 1} \mathbf{e}_{\alpha}' \times \lambda_{\beta 2} \mathbf{e}_{\beta}') \cdot \lambda_{\gamma 3} \mathbf{e}_{\gamma}' = 1$$

$$\text{L.H.S.} =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$\therefore \det(\lambda_{ij}) = 1$$

Definition : Cartesian Tensor

The class of 3^N elements $\{a_{ij} \dots p_q\}$ associated with $Ox_1x_2x_3$ is called a Cartesian tensor of rank (or order) N , if under rotation, the elements transform according to the following :

$$a'_{\alpha \beta \dots \gamma \delta} =$$

e.g. $N = 1 \quad a'_{\alpha} =$

$N = 2 \quad a'_{\alpha \beta} =$

Invariants : the following three functions of a tensor of rank 2 are invariants under rotation

for a 2nd order tensor $a_{ij} \quad A_1 =$

$$A_2 =$$

$$A_3 =$$

proof : (a) to show $a_{ii} = a'_{\alpha \alpha}$

$$\therefore a'_{\alpha \beta} =$$

set $\alpha = \beta$ then

$$a'_{\alpha \alpha} =$$

(b) to show $a'_{\alpha\beta} a'_{\beta\alpha} = a_{ij} a_{ji}$

$$a'_{\alpha\beta} a'_{\beta\alpha} =$$

$$=$$

$$=$$

$$=$$

$$\therefore A_2 =$$

(c) $\varepsilon_{\mu\nu\delta} \det(a'_{\alpha\beta}) =$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

set $\mu, \nu, \delta = 1, 2, 3$

then $\det(a'_{\alpha\beta}) = \det(a_{ij})$

Definition : Symmetric and skew-symmetric tensors

1. a tensor $a_{ij\cdots k\cdots l\cdots pq}$ is symmetric if

$$a_{ij\cdots k\cdots l\cdots pq} = a_{ij\cdots l\cdots k\cdots pq} \quad \text{for each choice of } k \text{ and } l$$

2. a tensor $a_{ij\cdots k\cdots l\cdots pq}$ is skew-symmetric if

$$a_{ij\cdots k\cdots l\cdots pq} = -a_{ij\cdots l\cdots k\cdots pq} \quad \text{for each choice of } k \text{ and } l$$

symmetric and skew-symmetric are not affected by rotation

Decomposition : any tensor of rank 2 may be written as the sum of a symmetric and skew-symmetric tensors

$$a_{ij} =$$

Isotropic : a tensor is said to be isotropic if its components do not change under rotation

zero order : all scalars are, of course, isotropic $a' = a$

1st order : $a'_i = a_i$

$$\lambda_{ij} a_j = \delta_{ij} a_j$$

$$(\lambda_{ij} - \delta_{ij}) a_j = 0$$

for arbitrary a_j , $\lambda_{ij} - \delta_{ij} = 0$

this is identical transformation (no rotation)

if $\lambda_{ij} - \delta_{ij} \neq 0$ then $a_j = 0$

i.e. no isotropic tensor of order 1

2nd order : an isotropic 2nd tensor may be reduced to the form

$$a_{ij} = a \delta_{ij}$$

check $a'_{ij} =$

3rd order : every isotropic tensor of rank 3 may be reduced to the form

$$a_{ijk} = a \varepsilon_{ijk}$$

check $a'_{ijk} =$

=

4th order : if a_{ijkl} is an isotropic tensor of order 4, then it is of the form

$$a_{ijkl} =$$

where λ, μ, ν are constants

$$\begin{aligned}
 \text{check } a'_{ijkl} &= \\
 &= \\
 &= \\
 &= \\
 &=
 \end{aligned}$$

If a_{ijkl} has the symmetric properties as $a_{ijkl} = a_{jikl} = a_{ijlk}$, then

$$a_{ijkl} =$$

for a 4th isotropic tensor

$$a_{ijkl} =$$

$$\begin{aligned}
 \text{if } a_{ijkl} &= a_{jikl} && \text{interchange } i \text{ and } j \\
 a_{ijkl} &= a_{ijlk} && \text{interchange } k \text{ and } l
 \end{aligned}$$

first two terms remain the same, and the third term becomes

$$\begin{aligned}
 \nu (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) &= \nu (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) \\
 2 \nu (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) &= 0
 \end{aligned}$$

$$\text{for } (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \neq 0 \quad \rightarrow \quad \nu = 0$$

$$\therefore a_{ijkl} =$$

Definition : Outer and inner product of tensors

outer product : $a_{ij} b_{kl}$ 4th order tensor

inner product : $a_{ij} b_{jk}$ 2nd order tensor

Theorem : A symmetric [skew-symmetric] tensor of rank 2 is symmetric [skew-symmetric] under rotation

let a_{ij} be symmetric [skew-symmetric] tensor

$$a_{ij} = a_{ji}$$

$$a'_{\alpha\beta} =$$

$$=$$

$$=$$

similarly for skew-symmetric tensor

Definition : Eigenvalues and Eigenvectors of 2nd order tensor

The solutions $A, \mathbf{n}$ (unit vector) of the equation

$$a_{ij} n_j = A n_i$$

are called the eigenvalues and eigenvectors of the tensor a_{ij} , respectively

the solution of the equation

$$(a_{ij} - A \delta_{ij}) n_j = 0$$

for nontrivial solution, we must have $\det (a_{ij} - A \delta_{ij}) = 0$

i.e.

or $A^3 - A_1 A^2 + A_2 A - A_3 = 0$

where A_i are invariants of a_{ij}

Theorem : If $a_{ij} = a_{ji}$, then the three eigenvalues A_I, A_{II}, A_{III} are real

proof : A cubic equation must at least one real root, say A_I . Let the corresponding eigenvector be $\mathbf{n}^I$, rotate the coordinate system, so that $\mathbf{n}^I = \mathbf{e}_1^I$

then $a'_{ij} \mathbf{n}_i^I = A_I \mathbf{n}_j^I \quad \mathbf{n}_i^I = (1, 0, 0)$

$\therefore a'_{11} = A_I$

$a'_{12} = a'_{13} = 0 \quad a'_{21} = a'_{31} = 0 \quad \text{and} \quad a'_{23} = a'_{32}$

$\therefore a'_{ij} =$

$\therefore a'_{ij} \mathbf{n}_j = A \mathbf{n}_i$

$\therefore$

examine the discriminate of quadratic

$\therefore$ the other two roots are also real for a symmetric second order tensor a_{ij}

Theorem : The principal direction (eigenvectors) $\mathbf{n}^I, \mathbf{n}^{II}, \mathbf{n}^{III}$ corresponding to A_I, A_{II}, A_{III} are mutually orthogonal if the eigenvalues are distinct.

recall the equation for eigenvalue

$$a_{ij} n_i^I = A_I n_j^I \quad (1)$$

$$a_{ij} n_i^{II} = A_{II} n_j^{II} \quad (2)$$

$$a_{ij} n_i^{III} = A_{III} n_j^{III} \quad (3)$$

$\therefore i$ and j are dummy index for $a_{ij} n_i^{II} n_j^I$

$$\therefore a_{ij} n_i^{II} n_j^I =$$

(b) becomes

(a) - (c)

$$\therefore A_I \neq A_{II} \quad \therefore$$

$$\text{i.e. } \mathbf{n}^I \perp \mathbf{n}^{II} \quad \text{similarly for others}$$

Theorem : $a_{ij} = A_I n_i^I n_j^I + A_{II} n_i^{II} n_j^{II} + A_{III} n_i^{III} n_j^{III}$

proof : use (1) ~ (3) of the above theorem

$$a_{ij} (n_i^I n_k^I + n_i^{II} n_k^{II} + n_i^{III} n_k^{III})$$

$$=$$

$$\text{let } \lambda_{1i} = \cos(\mathbf{e}_i, \mathbf{n}^I) = n_i^I$$

$$\text{similarly } \lambda_{2i} = n_i^{\text{II}}$$

$$\lambda_{3i} = n_i^{\text{III}}$$

substitute into (a)

$$a_{ij} (\lambda_{1i} \lambda_{1k} + \lambda_{2i} \lambda_{2k} + \lambda_{3i} \lambda_{3k})$$

$$=$$

$$a_{ij} \lambda_{pi} \lambda_{pk} =$$

$$a_{ij} \delta_{ik} =$$

$$\therefore a_{kj} =$$

so if the eigenvectors are chosen as coordinate axes

$$\text{i.e. } \mathbf{n}^I = (1, 0, 0)$$

$$\mathbf{n}^{\text{II}} = (0, 1, 0)$$

$$\mathbf{n}^{\text{III}} = (0, 0, 1)$$

then

$$a_{ij} =$$

that means if the coordinate rotate as $\mathbf{n}^I, \mathbf{n}^{\text{II}}, \mathbf{n}^{\text{III}}$, the matrix will be diagonal,
and

$$\lambda_{ij} = \begin{Bmatrix} \mathbf{n}^I \\ \mathbf{n}^{\text{II}} \\ \mathbf{n}^{\text{III}} \end{Bmatrix} =$$

so that λ_{ij} is the transformation matrix used to diagonalize the matrix

$$\text{i.e. } a'_{\alpha\beta} = \lambda_{\alpha i} \lambda_{\beta j} a_{ij} = \lambda_{\alpha i} a_{ij} \lambda^T_{j\beta} = [\setminus]$$

Divergence Theorem (Gauss' Theorem) :

$$\int_B a_{ij\cdots kl} n_r dA = \int_D a_{ij\cdots kl,r} dV$$

where $a_{ij\cdots kl}$ be a tensor which are function of x_i , and continuously differentiable in D , and D is a regular region bounded by the closed surface B , and $\mathbf{n}$ is outer normal to B

Corollary

$$\int_D \text{grad } \phi dV = \int_B \phi \mathbf{n} dA$$

$$\int_D \text{div } \mathbf{v} dV = \int_B \mathbf{v} \cdot \mathbf{n} dA$$

$$\int_D \text{curl } \mathbf{v} dV = \int_B \mathbf{n} \times \mathbf{v} dA$$

Divergence Theorem for plane

$$\int_B a_{ij\cdots kl,r} dV = \int a_{ij\cdots kl} n_r dS$$

where B is a regular region bounded by a close curve , $\mathbf{n}$ is the outer normal to

$$\int_B \text{grad } \phi dA = \int \phi \mathbf{n} dS$$

$$\int_B \text{div } \mathbf{v} dA = \int \mathbf{v} \cdot \mathbf{n} dS$$

$$\int_B \text{curl } \mathbf{v} dA = \int \mathbf{n} \times \mathbf{v} dS$$

Stokes' Theorem :

$$\int_B \mathbf{n} \cdot \text{curl } \mathbf{v} \, dA = \oint \mathbf{v} \cdot d\mathbf{x}$$

$$\int_B \mathbf{n} \times \text{grad } \phi \, dA = \oint \phi \, d\mathbf{x}$$

B and $\oint$ are defined as previous

Green's Theorem :

$$\int_D (\nabla \phi \nabla \psi + \phi \nabla^2 \psi) \, dV = \int_B \phi \frac{\psi}{n} \, dA \quad (1^{\text{st}} \text{ identity})$$

$$\int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dV = \int_B \left(\phi \frac{\psi}{n} - \psi \frac{\phi}{n} \right) \, dA \quad (2^{\text{nd}} \text{ identity})$$

i.e.

for plane

$$\int_B \left(\frac{v_2}{x_1} - \frac{v_1}{x_2} \right) \, dA = \oint \mathbf{v} \cdot d\mathbf{x}$$

i.e.