

9. Two Dimensional Elasticity

This chapter provided a general method of solution of certain broad classes of two-dimensional boundary-value problem in elasticity.

The method is based on a reduction of the boundary value problem in elasticity to the solution of certain functional equation in a complex domain.

The two-dimensional problems with which we shall be concerned fall into two physically distinct types :

1. To study the deformation of large cylindrical bodies acted on by the external forces so distributed that the component of deformation in the direction of the axis of the cylinder vanishes and the remaining components do not vary along the length of the cylinder. This is the class problem in *plane deformation*, or *plane strain*.

2. To study the deformation of thin plates, the state of stress in which is characterized by the vanishing of the stress components in the direction of the thickness of the plate. This is the class of problem of *plane stress*.

A. Plane Strain Problem

A body is said to be in the state of plane strain parallel to the x_1 - x_2 plane, if the displacement component u_3 vanishes and the components u_1 and u_2 are functions of the coordinates x_1 and x_2 , but not of x_3

$$\begin{aligned}
u_1 &= u_1(x_1, x_2) \\
u_2 &= u_2(x_1, x_2) \\
u_3 &= 0
\end{aligned} \tag{a1}$$

the strain and rotation tensors are

$$\begin{aligned}
e_{\alpha\beta} &= \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha}) \\
\omega_{\alpha\beta} &= \frac{1}{2} (u_{\alpha,\beta} - u_{\beta,\alpha}) \quad \alpha, \beta = 1, 2
\end{aligned}$$

the stress components are

$$\begin{aligned}
\sigma_{ij} &= \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \\
\sigma_{\alpha\beta} &= \lambda \delta_{\alpha\beta} \theta + \mu (u_{\alpha,\beta} + u_{\beta,\alpha}) \\
\sigma_{33} &= \lambda \theta \\
\sigma_{13} &= \sigma_{23} = 0
\end{aligned} \tag{a2}$$

$$\text{where} \quad \theta = u_{1,1} + u_{2,2}$$

now, we have

$$\begin{aligned}
\sigma_{11} &= \lambda (u_{1,1} + u_{2,2}) + 2\mu u_{1,1} \\
\sigma_{22} &= \lambda (u_{1,1} + u_{2,2}) + 2\mu u_{2,2} \\
\sigma_{12} &= \mu (u_{1,2} + u_{2,1})
\end{aligned}$$

$$\begin{aligned}
\text{and} \quad \sigma_{11} + \sigma_{22} &= (\lambda \theta + 2\mu u_{1,1}) + (\lambda \theta + 2\mu u_{2,2}) \\
&= 2(\lambda + \mu) \theta
\end{aligned}$$

$$\begin{aligned}
\text{or} \quad \sigma_{33} &= \lambda \theta = \frac{\lambda}{2(\lambda + \mu)} (\sigma_{11} + \sigma_{22}) \\
&= \nu (\sigma_{11} + \sigma_{22}) \neq 0
\end{aligned} \tag{a3}$$

this is not a plane stress problem

in general plane strain problem $\neq$ plane stress problem

equations of equilibrium

$$\begin{aligned}\sigma_{11,1} + \sigma_{12,2} + f_1 &= 0 \\ \sigma_{21,1} + \sigma_{22,2} + f_2 &= 0\end{aligned}\quad (\text{a4})$$

If the components $T_a(x_1, x_2)$ of external stress are specified, we have a two dimensional 1st boundary value problem

i.e. $\sigma_{a\beta} n_\beta = T_a$

equilibrium equations

$$\sigma_{a\beta,\beta} + f_a = 0 \quad 2 \text{ equations}$$

compatibility condition

$$\nabla^2 \Xi + \frac{2(\lambda + \mu)}{\lambda + 2\mu} f_{a,a} = 0 \quad 1 \text{ equation}$$

where $\Xi = \sigma_{11} + \sigma_{22}$

there are 3 equations for 3 unknowns σ_{11} , σ_{22} and σ_{12} to satisfy the boundary conditions

If the displacement components $u_a(x_1, x_2)$ on C are specified, we have a two dimensional 2nd boundary value problem

Navier's equations

$$\mu \nabla^2 u_a + (\lambda + \mu) \frac{\partial \theta}{\partial x_a} + f_a = 0 \quad 2 \text{ equations}$$

2 equations for two unknowns u_1 , u_2 to satisfy the boundary conditions
 $u_a = u_a^0$ on C

Consider a cylinder with plane ends $\perp x_3$ axis, with lateral surface subjected to surface traction $T_a(x_1, x_2)$, and $T_3 = 0$

if body forces are presented as $f_a = f_a(x_1, x_2)$ and $f_3 = 0$

$$\therefore \sigma_{33} = \nu (\sigma_{11} + \sigma_{22})$$

σ_{33} may produce a bending couple $\perp x_3$ axis

indeed, the longitudinal stresses σ_{33} are necessary to maintain the cylinder in the state of plane deformation

without σ_{33} , $u_3 \neq 0$ in general

if the ends are free in the given problem, the solution can be obtained by superposition

plane problem

plane strain
problem

Saint-Venant's
problem

S

S'

$\widehat{S}$

S' : plane strain problem

$\widehat{S}$: simple Saint-Venant's problem

[pure extension and pure bending $\implies$ plane remains plane]

$$S = S' + \widehat{S}$$

$$\widehat{S} = S - S'$$

$$\widehat{\mathbf{S}} : \widehat{\mathbf{f}} \equiv 0$$

$$\text{on } B_L \quad \widehat{T}_a = 0$$

$$\text{on } x_3 = \pm h \quad \widehat{\sigma}_{13} = \widehat{\sigma}_{23} = 0$$

$$\widehat{\sigma}_{33} = -\nu (\sigma_{11}' + \sigma_{22}')$$

$$L = \nu \int_R (\sigma_{11}' + \sigma_{22}') dA$$

$$M = L d$$

if the cylinder is long enough

$\sigma_{ij} = \sigma_{ij}' + \sigma_{ij}$ is valid for sufficiently far from the ends

B. Generalized Plane Stress Problem

let $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ in D

$$\therefore \sigma_{ij} = \lambda \theta \delta_{ij} + \mu (u_{i,j} + u_{j,i}) \quad [\theta = u_{k,k}]$$

$$\text{for } i=j=3 \quad 0 = \lambda \theta + 2\mu u_{3,3}$$

$$\text{or} \quad u_{3,3} = -\frac{\lambda}{\lambda + 2\mu} (u_{1,1} + u_{2,2})$$

$$\text{then} \quad \theta = u_{k,k} = \frac{2\mu}{\lambda + 2\mu} (u_{1,1} + u_{2,2})$$

$$\therefore \sigma_{11} = \frac{2\lambda\mu}{\lambda + 2\mu} (u_{1,1} + u_{2,2}) + 2\mu u_{1,1}$$

$$\therefore \sigma_{22} = \frac{2\lambda\mu}{\lambda + 2\mu} (u_{1,1} + u_{2,2}) + 2\mu u_{2,2}$$

$$\sigma_{12} = \mu (u_{1,2} + u_{2,1})$$

$$\sigma_{13} = \mu (u_{1,3} + u_{3,1}) = 0$$

$$\sigma_{23} = \mu (u_{2,3} + u_{3,2}) = 0$$

$$\begin{aligned}\sigma_{33} &= \lambda \theta + 2\mu u_{3,3} \\ &= \lambda (u_{1,1} + u_{2,2}) + (2\mu + \lambda) u_{3,3} = 0\end{aligned}$$

equations of equilibrium

$$\sigma_{11,1} + \sigma_{12,2} + f_1 = 0$$

$$\sigma_{21,1} + \sigma_{22,2} + f_2 = 0$$

substitute σ_{ij} into the equations of equilibrium, we have

$$\left(\frac{2\lambda\mu}{\lambda + 2\mu} + \mu \right) \frac{\partial}{\partial x_a} (u_{1,1} + u_{2,2}) + \mu \nabla^2 u_a + f_a = 0$$

these 2 equations are same as in plane strain problem if we replace

$$\frac{2\lambda\mu}{\lambda + 2\mu} \equiv \lambda \text{ by } \lambda \quad \text{in the equation on p. 9-3}$$

for the plane strain problem, u_a , $\sigma_{a\beta}$ are independent of x_3

but in the plane stress problem, u_a , $\sigma_{a\beta}$ may depend on x_3 , because x_3 may appear as a parameter in all equations, thus this is not truly two dimensional problem

Filon (1903) modify the system of equations such that the resulting two dimensional system corresponds to a physical problem of great practical interest

the field equations must also satisfy the boundary conditions

$$\left. \begin{aligned} T_a &= T_a^*(x_1, x_2) \\ T_3 &= 0 \end{aligned} \right\} \quad \text{on } B_L$$

$$T_i = 0 \quad \text{on } x_3 = \pm h$$

if $2h$ is small compared with the linear dimensions of the cross section, external forces act on the edge of the plate and lie in the planes parallel to the middle plane and are symmetrically distributed with respect to it

and the force act on the element are $2h T_a dS$, where T_a is the stress vector on C of the middle plane

assume $f_3 = 0$, f_1, f_2 are symmetrical with respect to the middle plane

then the points of the middle plane will undergo no displacement in x_3 direction

because the plate is very thin, the displacement u_3 is small

define u_i^* , σ_{ij}^* by

$$u_i^*(x_1, x_2) = \frac{1}{2h} \int_{-h}^h u_i(x_1, x_2, x_3) dx_3$$

$$\sigma_{ij}^*(x_1, x_2) = \frac{1}{2h} \int_{-h}^h \sigma_{ij}(x_1, x_2, x_3) dx_3$$

$\therefore u_3$ is odd in x_3

then $u_3^* = 0$

also $\sigma_{13}^* = \sigma_{23}^* = \sigma_{33}^* = 0$

apply averaging process

$$\sigma_{11,1}^* + \sigma_{12,2}^* + f_1 = 0 \quad (a1)$$

$$\sigma_{21,1}^* + \sigma_{22,2}^* + f_2 = 0 \quad (\text{a2})$$

$$\sigma_{11}^* = \lambda^*(u_{1,1}^* + u_{2,2}^*) + 2\mu u_{1,1}^* \quad (\text{b1})$$

$$\sigma_{22}^* = \lambda^*(u_{1,1}^* + u_{2,2}^*) + 2\mu u_{2,2}^* \quad (\text{b2})$$

$$\sigma_{12}^* = \mu (u_{1,2}^* + u_{2,1}^*) \quad (\text{b3})$$

$$\text{where} \quad \lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu}$$

the plane stress solution S^* approximates the plane solution S if h is sufficiently small

if we no longer assume $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$, instead we assume h is very small, equations (a) and (b) are still valid

boundary conditions on B_L are

$$\sigma_{11}^* n_1 + \sigma_{12}^* n_2 = T_1^*$$

$$\sigma_{12}^* n_1 + \sigma_{22}^* n_2 = T_2^*$$

$$\sigma_{13}^* n_1 + \sigma_{23}^* n_2 = 0$$

$$\text{on } x_3 = \pm h \quad \sigma_{33}^* = 0$$

consider the stress component σ_{33}

$$\therefore \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0 \quad [f_3 = 0]$$

evaluate on $x_3 = \pm h$,

$$\sigma_{31} = \sigma_{32} = 0 \quad [\text{independent of } x_1, x_2]$$

$$\therefore \sigma_{31,1} + \sigma_{32,2} = 0$$

$$\text{then } \sigma_{33,3}(x_1, x_2, \pm h) = 0$$

$$\text{but } \sigma_{33} = \sigma_{33}(x_1, x_2, \pm h) + (\pm h) \sigma_{33,3}(x_1, x_2, \pm h) + O(h^2)$$

$$= O(h^2)$$

$$\therefore \sigma_{33} \simeq 0 \quad \text{over the thickness } \pm h$$

$$\text{also } 2h \sigma_{i3}^* = \int_{-h}^h \sigma_{i3,3} dx_3 = \sigma_{13}(x_1, x_2, h) - \sigma_{13}(x_1, x_2, -h) = 0$$

thus the equations of equilibrium are

$$\sigma_{11,1}^* + \sigma_{12,2}^* + f_1 = 0$$

$$\sigma_{21,1}^* + \sigma_{22,2}^* + f_2 = 0$$

same as in equation (a)

C. Airy Stress Function

in the absence of body force, the equations of equilibrium are

$$\sigma_{a\beta,\beta} = 0$$

the compatibility conditions are

$$\nabla^2 (\sigma_{11} + \sigma_{22}) = 0$$

the boundary conditions are

$$\sigma_{a\beta} n_\beta = T_a$$

Airy proposed a function $\Phi(x_1, x_2)$ such that

$$\sigma_{11} = \Phi_{,22}$$

$$\sigma_{22} = \Phi_{,11}$$

$$\sigma_{12} = -\Phi_{,12}$$

then Φ satisfy the equations of equilibrium

$$\therefore \nabla^2 (\sigma_{11} + \sigma_{22}) = 0$$

$$\implies \nabla^4 \Phi = 0 \quad \text{in } R$$

Φ is a biharmonic function

the boundary conditions for Φ are

$$\left. \begin{aligned} \Phi_{,22} n_1 - \Phi_{,12} n_2 &= T_1 \\ -\Phi_{,12} n_1 + \Phi_{,22} n_2 &= T_2 \end{aligned} \right\} \quad \text{on } C$$

$$n_1 = \cos \theta = dx_2 / dS$$

$$n_2 = -\sin \theta = -dx_1 / dS$$

boundary conditions

$$\begin{aligned} \Phi_{,22} n_1 - \Phi_{,12} n_2 &= \Phi_{,22} dx_2 / dS + \Phi_{,12} dx_1 / dS \\ &= d(\Phi_{,2}) / dS = T_1 \end{aligned}$$

$$\text{similarly} \quad -d(\Phi_{,1}) / dS = T_2 \quad \text{on } C$$

$$\text{i.e.} \quad \Phi_{,1} = - \int_{S_0}^S T_2 dS = f_1(S) + C_1$$

$$\Phi_{,2} = \int_{S_0}^S T_1 dS = f_2(S) + C_2$$

$$\text{thus, we have} \quad \nabla^4 \Phi = 0 \quad \text{in } R$$

$$\Phi_{,a} = f_a(S) \quad \text{on } C$$

$$\text{since} \quad d\Phi = \Phi_{,a} dx_a = f_a dx_a$$

$$\Phi(S) = \int^S f_a \frac{dx_a}{dS} dS = f(S) + \text{constant}$$

$$S_0 \quad dS$$

$$\frac{d\Phi}{dn} = \Phi_{,a} n_a = \Phi_{,1} \frac{dx_2}{dS} - \Phi_{,2} \frac{dx_1}{dS} = g(S)$$

the function $\Phi(x_1, x_2)$ can be solved by

$$\nabla^4 \Phi = 0 \quad \text{in } R$$

$$\Phi = f(S) + \text{constant}$$

$$\text{or} \quad \Phi_{,n} = g(S) \quad \text{on } C$$

if the body force f_a are considered

$$\text{set} \quad f_a = -V_{,a}$$

$$\text{we can find} \quad \sigma_{11} = \Phi_{,22} + V$$

$$\sigma_{12} = -\Phi_{,12}$$

$$\sigma_{22} = \Phi_{,11} + V$$

the compatibility conditions are

$$\nabla^4 \Phi = -\frac{2\mu}{\lambda + 2\mu} \nabla^2 V$$

if V is harmonic, then Φ is biharmonic, otherwise

$$\nabla^4 \Phi = F(x_1, x_2) \quad \text{in } R$$

Theorem : every solution to (a) can be expressed by (b)

$$\text{proof} \quad \therefore \quad \sigma_{a\beta,\beta} = 0$$

there exists functions ϕ and ψ such that

$$\sigma_{11} = \phi_{,2} \quad \sigma_{12} = -\phi_{,1}$$

$$\sigma_{21} = \psi_{,2} \quad \sigma_{22} = -\psi_{,1}$$

$$\text{let } \Phi^* = \int_0^{x_2} \phi(x_1, \xi_2) d\xi_2$$

$$\begin{aligned} \text{then } \sigma_{11} &= \Phi^*_{,22} \\ \sigma_{12} &= -\Phi^*_{,12} = \psi_{,2} \end{aligned}$$

$$\therefore \psi = -\Phi^*_{,1} + f(x_1)$$

$$\text{and } \sigma_{22} = \Phi^*_{,11} - f'(x_1)$$

$$\therefore \nabla^2 \sigma_{aa} = 0 \quad \Rightarrow \quad \nabla^4 \Phi^* = f''(x_1)$$

$$\text{let } \Phi = \Phi^* - \int_0^{x_1} f(\xi) d\xi$$

$$\text{then } \Phi_{,11} = \Phi^*_{,11} - f'(x_1) = \sigma_{22}$$

$$\Phi_{,22} = \Phi^*_{,22} = \sigma_{11}$$

$$\Phi_{,12} = \Phi^*_{,12} = -\sigma_{12}$$

General Solution of the Biharmonic Equation

$$\nabla^4 \Phi = \nabla^2 \nabla^2 \Phi = 0 \quad \text{in } R$$

[a polynomial of highest order less than 3 is automatic the solution of the biharmonic equation]

$$\text{if we let } P(x_1, x_2) = \nabla^2 \Phi$$

$$\Rightarrow \nabla^2 P = 0 \quad [P \text{ is harmonic}]$$

in general, stress function may be expressed in terms of two analytic functions

$$\text{let } F(z) = P + iQ \quad \text{be analytic}$$

where $z = x_1 + i x_2$, and Q is harmonic conjugate of P

define a function $\psi(z) = \frac{1}{4} \int F(z) dz = p + i q$

clearly, integral for an analytic function is still analytic, thus $\psi(z)$ is analytic

$$\text{then} \quad \psi'(z) = \frac{\partial p}{\partial x_1} + i \frac{\partial q}{\partial x_1} = \frac{1}{4} (P + i Q)$$

$$\begin{aligned} \text{i.e.} \quad \frac{\partial p}{\partial x_1} &= \frac{1}{4} P = \frac{\partial q}{\partial x_2} & [\text{Cauchy-Riemann}] \\ \frac{\partial q}{\partial x_1} &= \frac{1}{4} Q = - \frac{\partial p}{\partial x_2} & \text{condition} \end{aligned}$$

consider the terms

$$\frac{\partial^2}{\partial x_1^2} (x_1 p) = \frac{\partial}{\partial x_1} (p + x_1 \frac{\partial p}{\partial x_1}) = x_1 \frac{\partial^2 p}{\partial x_1^2} + 2 \frac{\partial p}{\partial x_1}$$

$$\frac{\partial^2}{\partial x_2^2} (x_1 p) = x_1 \frac{\partial^2 p}{\partial x_2^2}$$

$$\therefore \nabla^2 (x_1 p) = x_1 \left(\frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2} \right) + 2 \frac{\partial p}{\partial x_1}$$

$$\therefore \frac{1}{4} P = \frac{\partial p}{\partial x_1} \quad - \frac{1}{4} Q = \frac{\partial p}{\partial x_2}$$

$$\therefore \nabla^2 p = \frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2} = \frac{1}{4} \left(\frac{\partial P}{\partial x_1} - \frac{\partial Q}{\partial x_2} \right) = 0$$

$$\text{thus} \quad \nabla^2 (x_1 p) = 2 \frac{\partial p}{\partial x_1} = P / 2$$

$$\text{similarly} \quad \nabla^2 (x_2 q) = 2 \frac{\partial q}{\partial x_2} = P / 2$$

$$\begin{aligned}\text{then} \quad \nabla^2 (\Phi - x_1 p - x_2 q) &= \nabla^2 \Phi - \nabla^2 (x_1 p) - \nabla^2 (x_2 q) \\ &= P - \frac{1}{2} P - \frac{1}{2} P = 0\end{aligned}$$

$\therefore \Phi - x_1 p - x_2 q$ is also harmonic

$$\text{let} \quad p_1 = \Phi - x_1 p - x_2 q \quad [p_1 \text{ is harmonic}]$$

$$\text{then} \quad \Phi = p_1 + x_1 p + x_2 q$$

thus, any stress function can be formed from suitably chosen conjugate harmonic function p, q and a harmonic function p_1

let q_1 be harmonic conjugate of p_1 , and an analytic function χ is defined

$$\chi(z) = p_1 + i q_1$$

consider the function

$$\begin{aligned}z \psi(z) + \chi(z) &= (x_1 - i x_2) (p + i q) + (p_1 + i q_1) \\ &= (x_1 p + x_2 q + p_1) + i (x_1 q - x_2 p + q_1)\end{aligned}$$

$$\text{then} \quad \Phi(x_1, x_2) = \text{Re} [z \psi(z) + \chi(z)]$$

$$\Rightarrow \nabla^4 \Phi = 0 \quad \text{for any choice of analytic functions } \psi(z) \text{ and } \chi(z)$$

$$\therefore \Phi = \text{Re} [\bar{z} \psi(z) + \chi(z)]$$

$$\text{then} \quad 2 \Phi = \bar{z} \psi(z) + z \overline{\psi(z)} + \chi(z) + \overline{\chi(z)}$$

$$2 \Phi_{,1} = \bar{z} \psi'(z) + \psi(z) + z \overline{\psi'(z)} + \overline{\psi(z)} + \chi'(z) + \overline{\chi'(z)}$$

$$2 \Phi_{,2} = i [\bar{z} \psi'(z) - \psi(z) - z \overline{\psi'(z)} + \overline{\psi(z)} + \chi'(z) - \overline{\chi'(z)}]$$

$$\therefore \quad \sigma_{11} = \Phi_{,22} \quad \sigma_{22} = \Phi_{,11} \quad \sigma_{12} = -\Phi_{,12}$$

$$\begin{aligned} \text{then} \quad \sigma_{11} + i \sigma_{12} &= \Phi_{,22} - i \Phi_{,12} = -i (\Phi_{,1} + i \Phi_{,2})_{,2} \\ &= -i [\psi(z) + z \overline{\psi'}(\bar{z}) + \overline{\chi'}(\bar{z})]_{,2} \\ &= \psi'(z) + \overline{\psi'}(\bar{z}) - z \overline{\psi''}(\bar{z}) - \overline{\chi''}(\bar{z}) \end{aligned} \quad (a)$$

$$\text{similarly} \quad \sigma_{22} - i \sigma_{12} = \psi'(z) + \overline{\psi'}(\bar{z}) + z \overline{\psi''}(\bar{z}) + \overline{\chi''}(\bar{z}) \quad (b)$$

$$(a) + (b) \quad \sigma_{11} + \sigma_{22} = 2 \psi'(z) + 2 \overline{\psi'}(\bar{z}) = 4 \operatorname{Re} [\psi'(z)] \quad (A)$$

$$(b) - (a) \quad \sigma_{22} - \sigma_{11} - 2i \sigma_{12} = 2 [z \overline{\psi''}(\bar{z}) + \overline{\chi''}(\bar{z})]$$

change i to $-i$

$$\sigma_{22} - \sigma_{11} + 2i \sigma_{12} = 2 [z \overline{\psi''}(\bar{z}) + \overline{\chi''}(\bar{z})] \quad (B)$$

with ψ and χ known, σ_{11} , σ_{22} and σ_{12} can be calculated from (A) and (B)

Displacements in terms of ψ and χ

for plane strain problem

$$\sigma_{11} = \Phi_{,22} = (\lambda + 2\mu) u_{1,1} + \lambda u_{2,2} \quad (1)$$

$$\sigma_{22} = \Phi_{,11} = (\lambda + 2\mu) u_{2,2} + \lambda u_{1,1} \quad (2)$$

$$\sigma_{12} = -\Phi_{,12} = \mu (u_{1,2} + u_{2,1}) \quad (3)$$

solve the first two equations, we can get

$$2\mu u_{1,1} = -\Phi_{,11} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} \nabla^2 \Phi = -\Phi_{,11} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} p_{,1}$$

$$2 \mu u_{2,2} = -\Phi_{,22} + \frac{\lambda + 2 \mu}{2 (\lambda + \mu)} \nabla^2 \Phi = -\Phi_{,22} + \frac{2 (\lambda + 2 \mu)}{\lambda + \mu} q_{,2}$$

integration and get

$$2 \mu u_1 = -\Phi_{,1} + \frac{2 (\lambda + 2 \mu)}{\lambda + \mu} p + f(x_2)$$

$$2 \mu u_2 = -\Phi_{,2} + \frac{2 (\lambda + 2 \mu)}{\lambda + \mu} q + g(x_1)$$

$$2 \mu u_{1,2} = -\Phi_{,12} + \frac{2 (\lambda + 2 \mu)}{\lambda + \mu} p_{,2} + f'(x_2)$$

$$2 \mu u_{2,1} = -\Phi_{,12} + \frac{2 (\lambda + 2 \mu)}{\lambda + \mu} q_{,1} + g'(x_1)$$

$$2 \mu (u_{1,2} + u_{2,1}) = -2 \Phi_{,12} + f'(x_2) + g'(x_1) \quad [\because p_{,2} = -q_{,1}]$$

compared to equation (3), we have

$$f'(x_2) + g'(x_1) = 0$$

$$\therefore f'(x_2) = -g'(x_1) = A = \text{constant}$$

$$\text{i.e. } f(x_2) = A x_2 + C_1$$

$$g(x_1) = -A x_1 + C_2$$

$f(x_2)$ and $g(x_1)$ are rigid body motion [neglected]

thus, the displacement components are

$$2 \mu u_1 = -\Phi_{,1} + \frac{2 (\lambda + 2 \mu)}{\lambda + \mu} p$$

$$2 \mu u_2 = -\Phi_{,2} + \frac{2 (\lambda + 2 \mu)}{\lambda + \mu} q$$

substitute $\Phi_{,1}$ and $\Phi_{,2}$ into these two equations

$$\begin{aligned}
 2\mu(u_1 + i u_2) &= -(\Phi_{,1} + i \Phi_{,2}) + \frac{2(\lambda + 2\mu)}{\lambda + \mu}(p + i q) \\
 &= \left(\frac{\lambda + 3\mu}{\lambda + \mu}\right) \psi(z) - z \overline{\psi'(z)} - \overline{\chi'(z)} \\
 &= (3 - 4\nu) \psi(z) - z \overline{\psi'(z)} - \overline{\chi'(z)} \quad (C)
 \end{aligned}$$

for plane stress problem

$$\mu \rightarrow \frac{2\lambda\mu}{\lambda + 2\mu} \quad \text{i.e.} \quad 3 - 4\nu \rightarrow \frac{3 - \nu}{1 + \nu}$$

$$\text{then} \quad 2\mu(u_1 + i u_2) = \frac{3 - \nu}{1 + \nu} \psi(z) - z \overline{\psi'(z)} - \overline{\chi'(z)} \quad (C')$$

with ψ and χ known, u_1 and u_2 can be calculated from (C) and (C')

Summary :

$$\nabla^4 \Phi = 0 \quad \text{in } R$$

Φ may be constructed by a polynomial in x_1 and x_2

$$\text{let } P = \nabla^2 \Phi \Rightarrow \nabla^2 P = 0 \quad \text{in } R$$

and let Q be harmonic conjugate of P

then $F = P + i Q$ is analytic

$$\psi(z) = p + i q \quad \text{with } P = 4 p / x_1$$

$$Q = 4 q / x_2$$

p, q are harmonic, and ψ is analytic

$$\chi(z) = p_1 + i q_1 \quad \begin{array}{l} p_1 \text{ is harmonic} \\ q_1 \text{ is harmonic conjugate of } p_1 \end{array}$$

χ is also analytic

$$\Phi = \operatorname{Re} [\bar{z} \psi(z) + \chi(z)]$$

$$\sigma_{11} = \Phi_{,22} \quad \sigma_{22} = \Phi_{,11} \quad \sigma_{12} = -\Phi_{,12}$$

$$\sigma_{11} + \sigma_{22} = 4 \operatorname{Re} [\psi'(z)]$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z} \psi''(z) + \chi''(z)]$$

$$2\mu(u_1 + iu_2) = \kappa \psi(z) - z \overline{\psi'(z)} - \overline{\chi'(z)}$$

$$\kappa = 3 - 4\nu \quad \text{for plane strain}$$

$$\kappa = \frac{3 - \nu}{1 + \nu} \quad \text{for plane stress}$$

Consider Φ be a polynomial of 2nd order

$$\Phi = \frac{a_2}{2} x_1^2 + b_2 x_1 x_2 + \frac{c_2}{2} x_2^2$$

$$\text{then } \sigma_{11} = c_2 \quad \sigma_{22} = a_2 \quad \sigma_{12} = -b_2$$

all stresses are constant, Φ represents a combination of uniform tension and compression in two perpendicular direction and uniform shear

If Φ is taken as polynomial of 3rd order

$$\Phi = \frac{a_3}{6} x_1^3 + \frac{b_3}{2} x_1^2 x_2 + \frac{c_3}{2} x_1 x_2^2 + \frac{d_3}{6} x_2^3$$

$$3 \cdot 2 \qquad 2 \qquad 2 \qquad 3 \cdot 2$$

then

$$\begin{aligned}\sigma_{11} &= c_3 x_1 + d_3 x_2 \\ \sigma_{22} &= a_3 x_1 + b_3 x_2 \\ \sigma_{12} &= -b_3 x_1 - c_3 x_2\end{aligned}$$

for a rectangular plate

a. if $d_3 \neq 0$, others = 0

=> pure bending

b. if $b_3 \neq 0$, others = 0

=> normal + shear force

c. if $a_3 \neq 0$, others = 0

=> normal force + pure bending

If Φ is taken as polynomial of 4th order

$$\Phi = \frac{a_4}{4 \cdot 3} x_1^4 + \frac{b_4}{3 \cdot 2} x_1^3 x_2 + \frac{c_4}{2} x_1^2 x_2^2 + \frac{d_4}{3 \cdot 2} x_1 x_2^3 + \frac{e_4}{4 \cdot 3} x_2^4$$

$$\therefore \nabla^4 \Phi = 0 \Rightarrow e_4 = -(2c_4 + a_4)$$

then

$$\begin{aligned}\sigma_{11} &= c_4 x_1^2 + d_4 x_1 x_2 - (2c_4 + a_4) x_2^2 \\ \sigma_{22} &= a_4 x_1^2 + b_4 x_1 x_2 + c_4 x_2^2 \\ \sigma_{12} &= -\frac{1}{2} b_4 x_1^2 - 2c_4 x_1 x_2 - \frac{1}{2} d_4 x_2^2\end{aligned}$$

consider $d_4 \neq 0$, others = 0

then $\sigma_{11} = d_4 x_1 x_2 \quad \sigma_{22} = 0 \quad \sigma_{12} = -\frac{1}{2} d_4 x_2^2$

moment due to shear force is

$$M = \left(\frac{1}{2} d_4 c^2 l\right) 2c - \frac{1}{3} \left(\frac{1}{2} d_4 c^2\right) 2cl = \frac{2}{3} d_4 c^3 l$$

this moment must be balanced by normal force

If Φ is taken as polynomial of 5th order

$$\Phi = \frac{a_5}{5 \cdot 4} x_1^5 + \frac{b_5}{4 \cdot 3} x_1^4 x_2 + \frac{c_5}{3 \cdot 2} x_1^3 x_2^2 + \frac{d_5}{4 \cdot 3} x_1^2 x_2^3 + \frac{e_5}{4 \cdot 3} x_1 x_2^4 + \frac{f_5}{5 \cdot 4} x_2^5$$

$$\therefore \nabla^4 \Phi = 0 \Rightarrow e_5 = -(2c_5 + 3a_5)$$

$$f_5 = -\frac{1}{3}(b_5 + 2d_5)$$

and the stress components are

$$\sigma_{11} = \frac{1}{3} c_5 x_1^3 + d_5 x_1^2 x_2 - (2c_5 + 3a_5) x_1 x_2^2 - \frac{1}{3} (b_5 + 2d_5) x_2^3$$

$$\sigma_{22} = a_5 x_1^3 + b_5 x_1^2 x_2 + c_5 x_1 x_2^2 + \frac{1}{3} d_5 x_2^3$$

$$\sigma_{12} = -\frac{1}{3} b_5 x_1^3 - c_5 x_1^2 x_2 - d_5 x_1 x_2^2 + \frac{1}{3} (2c_5 + 3a_5) x_2^3$$

for $d_5 \neq 0$, others = 0

$$\sigma_{11} = d_5 (x_1^2 x_2 - \frac{2}{3} x_2^3)$$

$$\sigma_{22} = \frac{1}{3} d_5 x_2^3$$

$$\sigma_{12} = -d_5 x_1 x_2^2$$

the normal force are uniform distributed along the longitudinal sides, along $x_1 = l$, the normal force are of a cubic parabola, the shear forces are proportional to x_1 on the longitudinal sides, and follow a parabolic law along $x_2 = l$

Bending of Cantilever Beam Loaded at the End

consider the cross section is a narrow rectangular section, it is a plane stress case

taken Φ as a 4th order polynomial with $d_4 \neq 0$, others = 0, and add a term $b_2 x_1 x_2$

$$\text{i.e.} \quad \Phi = \frac{d_4}{3 \cdot 2} x_1 x_2^3 + b_2 x_1 x_2$$

$$\text{then} \quad \sigma_{11} = d_4 x_1 x_2$$

$$\sigma_{22} = 0$$

$$\sigma_{12} = -b_2 - \frac{1}{2} d_4 x_2^2$$

$$\sigma_{12} :$$

$$\sigma_{12} = 0 \text{ on } x_2 = \pm c$$

$$\therefore -b_2 - \frac{1}{2} d_4 c^2 = 0$$

$$\text{i.e. } d_4 = -2 b_2 / c^2$$

$$\text{and } \sigma_{12} = -b_2 + b_2 x_2^2 / c^2$$

$\therefore$ the resultant of the shear stress over the section is equal P , then

$$- \int_{-c}^c \sigma_{12} dx_2 = - \int_{-c}^c (-b_2 + b_2 x_2^2 / c^2) dx_2 = P$$

$$\therefore b_2 = \frac{3}{4} P / c \quad \text{and} \quad d_4 = -3 P / 2 c^3$$

$$\therefore \sigma_{11} = -\frac{3}{2} \frac{P x_1 x_2}{c^3} \quad \sigma_{22} = 0$$

$$\sigma_{12} = -\frac{3 P}{c} \left(1 - \frac{x_2^2}{c^2}\right)$$

$$\therefore I = 1 \cdot (2 c)^3 / 12 = \frac{2}{3} c^3$$

$$\therefore \sigma_{11} = -\frac{P x_1 x_2}{I} = -\frac{M x_2}{I}$$

$$\sigma_{12} = -\frac{P}{2 I} (c^2 - x_2^2)$$

and the strain components are

$$e_{11} = \frac{\partial u_1}{\partial x_1} = -\frac{P x_1 x_2}{E I}$$

$$e_{22} = \frac{\partial u_2}{\partial x_2} = -\frac{\nu \sigma_{11}}{E} = \frac{\nu P x_1 x_2}{E I}$$

$$e_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \frac{\sigma_{12}}{G} = -\frac{P}{2 I G} (c^2 - x_2^2)$$

also the displacement components are

$$u_1 = -\frac{P x_1^2 x_2}{2 E I} + g(x_2)$$

$$u_2 = \frac{\nu P x_1 x_2^2}{2 E I} + f(x_1)$$

substituting into e_{12}

$$-\frac{P x_1^2}{2 E I} + \frac{d g}{d x_2} + \frac{\nu P x_2^2}{2 E I} + \frac{d f}{d x_1} = -\frac{P}{2 I G} (c^2 - x_2^2)$$

let

$$F(x_1) = -\frac{P x_1^2}{2 E I} + \frac{d f}{d x_1} = d = \text{constant}$$

$$G(x_2) = -\frac{d g}{d x_2} + \frac{\nu P x_2^2}{2 E I} - \frac{P x_2^2}{2 I G} = e = \text{constant}$$

$$K = -\frac{P c^2}{2 I G}$$

thus, we have $F(x_1) + G(x_2) = K$

$$\text{or} \quad d = e = -\frac{P c^2}{2 I G}$$

and $d f \quad P x_1^2$

$$\frac{d}{dx_1} = \frac{d}{2EI} + d$$

$$\frac{dg}{dx_2} = -\frac{\nu P x_2^2}{2EI} + \frac{P x_2^2}{2IG} + e$$

thus

$$f(x_1) = \frac{P x_1^3}{6EI} + d x_1 + h$$

$$g(x_2) = -\frac{\nu P x_2^3}{6EI} + \frac{P x_2^3}{6IG} + e x_2 + g$$

$$\therefore u_1 = -\frac{P x_1^2 x_2}{2EI} - \frac{\nu P x_2^3}{6EI} + \frac{P x_2^3}{6IG} + e x_2 + g$$

$$u_2 = \frac{\nu P x_1 x_2^2}{2EI} + \frac{P x_1^3}{6EI} + d x_1 + h$$

$$\text{for } x_1 = l, \quad x_2 = 0 \quad u_1 = u_2 = 0$$

$$\text{then } g = 0 \quad \text{and} \quad h = -\frac{P l^3}{6EI} - d l$$

$$\text{and at } x_1 = l, \quad x_2 = 0 \quad u_{1,2} = u_{2,1} = 0$$

$$\text{we get } d = -\frac{P l^2}{2EI} \quad e = \frac{P l^2}{2EI} - \frac{P c^2}{2IG}$$

$$\therefore u_1 = -\frac{P x_1^2 x_2}{2EI} - \frac{\nu P x_2^3}{6EI} + \frac{P x_2^3}{6IG} + \left(\frac{P l^2}{2EI} - \frac{P c^2}{2IG}\right) x_2$$

$$u_2 = \frac{\nu P x_1 x_2^2}{2EI} + \frac{P x_1^3}{6EI} - \frac{P l^2}{2EI} x_1 + \frac{P l^3}{3EI}$$

the center line displacement (at $x_2 = 0$) is

$$P x_1^3 \quad P l^2 \quad P l^3$$

$$u_2 = \frac{P l^3}{6 E I} - \frac{P l^2}{2 E I} x_1 + \frac{P l x_1^2}{3 E I}$$

at the end of the beam ($x_1 = 0$)

$$u_2 = \frac{P l^3}{3 E I} \quad \text{same as in strength of materials}$$

but u_1 is not linear in x_2

warping effect may occurs

Airy Stress Function in Polar Coordinates (r, θ, z)

for plane stress problem

$$\sigma_{zz} = \sigma_{zx} = \sigma_{z\theta} = 0$$

for plane strain problem

$$u_z = 0 \quad \text{and all derivatives w.r.t. } z \text{ vanish}$$

$$e_{rr} = \frac{\partial u_r}{\partial r} \quad e_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$e_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)$$

stress-strain relations

a. plane stress case

$$e_{rr} = \frac{1}{E} (\sigma_{rr} - \nu \sigma_{\theta\theta})$$

$$e_{\theta\theta} = \frac{1}{E} (\sigma_{\theta\theta} - \nu \sigma_{rr})$$

$$e_{r\theta} = \frac{1 + \nu}{E} \sigma_{r\theta}$$

b. plane strain case

$$e_{rr} = \frac{1}{E} [(1 - \nu^2) \sigma_{rr} - \nu (1 + \nu) \sigma_{\theta\theta}]$$

$$e_{\theta\theta} = \frac{1}{E} [(1 - \nu^2) \sigma_{\theta\theta} - \nu (1 + \nu) \sigma_{rr}]$$

$$e_{r\theta} = \frac{1 + \nu}{E} \sigma_{r\theta}$$

equations of equilibrium

$$\frac{1}{r} \frac{\partial (r \sigma_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} - \frac{\partial \sigma_{\theta\theta}}{\partial r} + f_r = 0$$

$$\frac{1}{r^2} \frac{\partial (r^2 \sigma_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + f_\theta = 0$$

if $f_r = f_\theta = 0$, Airy stress function $\Phi(r, \theta)$ is proposed, such that

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right)$$

compatibility condition

$$\nabla^2 (\sigma_{11} + \sigma_{22}) = 0 \quad [\sigma_{11} + \sigma_{22} = \sigma_{rr} + \sigma_{\theta\theta}]$$

in polar coordinates, $\nabla^4 \Phi$ can be expressed

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right) = 0$$

for axial symmetric problem

$$\partial / \partial \theta = 0 \quad \text{then} \quad \Phi = \Phi(r)$$

thus the compatibility conditions becomes

$$\frac{d^4 \Phi}{dr^4} + \frac{2}{r} \frac{d^3 \Phi}{dr^3} - \frac{1}{r^2} \frac{d^2 \Phi}{dr^2} + \frac{1}{r^3} \frac{d\Phi}{dr} = 0$$

the general solution for this differential equation is

$$\Phi = A \log r + B r^2 \log r + C r^2 + D$$

thus the stress components are

$$\sigma_{rr} = A / r^2 + B (1 + 2 \log r) + 2 C$$

$$\sigma_{\theta\theta} = -A / r^2 + B (3 + 2 \log r) + 2 C$$

$$\sigma_{r\theta} = 0$$

the displacement components are [included rigid body motion]

a. plane stress case

$$u_r = \frac{1}{E} \left[-\frac{(1+\nu)}{r} A + 2 B (1-\nu) r \log r - B (1+\nu) r \right. \\ \left. + 2 C (1-\nu) r \right] + \alpha \sin \theta + \gamma \cos \theta$$

$$u_\theta = \frac{4 B r \theta}{E} + \alpha \cos \theta - \gamma \sin \theta + \beta r$$

b. plane strain case

$$u_r = \frac{1}{E} \left[-\frac{(1+\nu)}{r} A + 2B(1-\nu-2\nu^2)r \log r - B(1+\nu)r \right. \\ \left. + 2C(1-\nu-2\nu^2)r \right] + a \sin \theta + \gamma \cos \theta$$

$$u_\theta = \frac{4Br\theta}{E} (1-\nu^2) + a \cos \theta - \gamma \sin \theta + \beta r$$

Example : consider a thick walled cylinder with inner radius $2a$ and outer radius $2b$, subjected internal pressure p and outer pressure q

then $\sigma_{rr} = -p$ at $r = a$

$\sigma_{rr} = -q$ at $r = b$

$$u_\theta = \frac{4Br\theta}{E}$$

$$\text{but } u_\theta(0) = u_\theta(2\pi) = 0 \quad \Rightarrow \quad B = 0$$

$$\therefore \sigma_{rr} = \frac{A}{r^2} + 2C$$

use the boundary conditions to determine A and C , we get

$$\sigma_{rr} = -p \frac{(b^2/r^2 - 1)}{b^2/a^2 - 1} - q \frac{(1 - a^2/r^2)}{1 - a^2/b^2}$$

$$\sigma_{\theta\theta} = p \frac{(b^2/r^2 + 1)}{b^2/a^2 - 1} - q \frac{(1 + a^2/r^2)}{1 - a^2/b^2}$$

Example : pure bending in curved beam with narrow rectangular cross section [plane stress case]

$$\sigma_{rr} = 0 \quad \text{at } r = a, b \quad (1)$$

$$\int_a^b \sigma_{\theta\theta} dr = 0 \quad \int_a^b \sigma_{\theta\theta} r dr = -M \quad (2)$$

$$\sigma_{r\theta} = 0 \quad \text{at boundary} \quad (3)$$

from (1), we have

$$A/a^2 + B(1 + 2 \log a) + 2C = 0 \quad (a)$$

$$A/b^2 + B(1 + 2 \log b) + 2C = 0 \quad (b)$$

from (2), we have

$$\int_a^b \sigma_{\theta\theta} r dr = \int_a^b \frac{\partial^2 \Phi}{\partial r^2} r dr = -M$$

$$\int_a^b \frac{\partial^2 \Phi}{\partial r^2} r dr = \left| \frac{\partial \Phi}{\partial r} r \right|_a^b - \int \frac{\partial \Phi}{\partial r} dr = - \left| \Phi \right|_a^b = -M$$

$$[\because \sigma_{rr} = \frac{1}{r} \frac{\partial \Phi}{\partial r} \Rightarrow \frac{\partial \Phi}{\partial r} r = r^2 \sigma_{rr} = 0 \text{ at } r = a, b]$$

thus, we have $\left| \Phi \right|_a^b = M$

$$\text{i.e. } \left| \Phi \right|_a^b = A \log \frac{b}{a} + B(b^2 \log b - a^2 \log a) + C(b^2 - a^2) \quad (c)$$

from (a), (b) and (c), it is obtained

$$A = -\frac{4M}{a^2 b^2 \log \frac{b}{a}}$$

$$N = b^2 - a^2$$

$$B = -\frac{2M}{N}(b^2 - a^2)$$

$$C = \frac{M}{N}[b^2 - a^2 + 2(b^2 \log b - a^2 \log a)]$$

$$\text{where } N = (b^2 - a^2)^2 - 4a^2 b^2 \log(b/a)^2$$

the stress components are

$$\sigma_{rr} = -\frac{4M}{N}\left(\frac{a^2 b^2}{r^2} \log \frac{b}{a} + b^2 \log \frac{r}{b} + a^2 \log \frac{a}{r}\right)$$

$$\sigma_{\theta\theta} = -\frac{4M}{N}\left(-\frac{a^2 b^2}{r^2} \log \frac{b}{a} + b^2 \log \frac{r}{b} + a^2 \log \frac{a}{r} + b^2 - a^2\right)$$

$$\sigma_{r\theta} = 0$$