

8. Problem of Saint-Venant

Bending due to Transverse Loading

Let a cantilever beam of uniform cross section have one end ($x_3 = 0$) fixed and other end ($x_3 = l$) loaded by a force $L_1 = L$ at some point (x_1, x_2, l), the lateral surface of the beam is free from external forces, and the body forces are assumed to vanish.

$$L_1 = L$$

$$L_2 = L_3 = 0$$

$$M_1 = M_2 = M_3 = 0$$

$$f_i = 0$$

$$\begin{aligned} \text{on } B_L, T_i &= 0 & \sigma_{31} n_1 + \sigma_{32} n_2 &= 0 \\ & & \sigma_{21} n_1 + \sigma_{22} n_2 &= 0 \\ & & \sigma_{11} n_1 + \sigma_{12} n_2 &= 0 \end{aligned}$$

$$\begin{aligned} \text{on } x_3 = l & & \int_R \sigma_{32} dA &= L_2 = 0 & \int_R \sigma_{33} dA &= L_3 = 0 \\ & & \int_R x_2 \sigma_{32} dA &= M_1 = 0 \\ & & \int_R x_1 \sigma_{33} dA &= M_2 = 0 \\ & & \int_R (x_1 \sigma_{32} - x_2 \sigma_{31}) dA &= M_3 = 0 \\ & & \int_R \sigma_{31} dA &= L_1 = L \end{aligned}$$

from semi-inverse method of Saint-Venant, we get

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0$$

on the cross section AA

$$M_2 = L (l - x_3)$$

if x_1, x_2, x_3 are principal axes, from (B)

$$\sigma_{33} = -M_2 x_1 / I = -L (l - x_3) x_1 / I$$

where $I = \int_{\mathcal{R}} x_1^2 dA$ is the moment of inertia

equation of equilibrium with body force $f_i = 0$

$$\sigma_{13,3} = 0 \quad \rightarrow \quad \sigma_{13} = \sigma_{13}(x_1, x_2) \quad (\text{a1})$$

$$\sigma_{23,3} = 0 \quad \rightarrow \quad \sigma_{23} = \sigma_{23}(x_1, x_2) \quad (\text{a2})$$

i.e. the shear components σ_{13}, σ_{23} have the same value in all section of the beam (independent of x_3)

the third equation of equilibrium

$$\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0$$

$$\text{or} \quad \sigma_{31,1} + \sigma_{32,2} = -L x_1 / I \quad (\text{a3})$$

stress-strain relation

$$e_{ij} = [(1 + \nu) \sigma_{ij} - \nu \delta_{ij} \sigma_{kk}] / E$$

$$e_{11} = e_{22} = \nu \sigma_{33} / E = \nu L (l - x_3) x_1 / EI$$

$$e_{33} = \sigma_{33} / E = -L (l - x_3) x_1 / EI$$

$$e_{12} = 0$$

$$e_{13} = \sigma_{13} / 2\mu$$

$$e_{23} = \sigma_{23} / 2\mu \quad (\text{b1 - b6})$$

compatibility conditions

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$

4 of these 6 equations are identically satisfied

the remaining two equations are

$$-\frac{\nu L}{EI} + \left(\frac{\sigma_{13}}{2\mu}\right)_{,22} = \frac{\sigma_{23,12}}{2\mu} \quad [e_{22,13} + e_{13,22} = e_{23,12} + e_{12,23}]$$

$$\frac{\sigma_{23,11}}{2\mu} = \frac{\sigma_{13,12}}{2\mu} \quad [e_{11,23} + e_{23,11} = e_{12,12} + e_{13,12}]$$

or
$$\sigma_{13,22} - \sigma_{23,12} = \frac{\nu L}{(1 + \nu) I} \quad (c1)$$

$$\sigma_{23,11} = \sigma_{13,12} \quad (c2)$$

so that σ_{13} and σ_{23} must satisfy equations (a3), (c1) and (c2)

[3 equations for 2 unknowns]

integration w. r. t. x_2 for equation (c1)

or
$$\sigma_{13,2} - \sigma_{23,1} = \frac{\nu L x_2}{(1 + \nu) I} + h(x_1) \quad (c3)$$

$$d(c3) / dx_1 \quad \sigma_{13,12} - \sigma_{23,11} = h'(x_1) = 0$$

thus $h(x_1)$ is a constant set $h = -2\mu a$

the stress equations are now

$$\sigma_{13,1} - \sigma_{23,2} = -\frac{L x_1}{I} \quad (d1)$$

$$\sigma_{13,2} - \sigma_{23,1} = \frac{\nu L x_2}{(1 + \nu) I} - 2\mu a \quad (d2)$$

let
$$\sigma_{13} = \psi_2 - \frac{L x_1^2}{2I} + \frac{\nu L}{1 + \nu} \frac{x_2^2}{2I} - \mu a x_2 + \frac{L(2 + \nu)}{1 + \nu} \frac{(x_1^2 - x_2^2)}{4I}$$

$$\sigma_{23} = -\psi_1 + \mu a x_1 - \frac{L(2 + \nu)}{1 + \nu} \frac{x_1 x_2}{2I} \quad (e1, e2)$$

where ψ_1 and ψ_2 are functions of x_1 and x_2

from (d1)

$$\psi_{2,1} - \frac{L x_1}{I} + \frac{L(2+\nu)}{1+\nu} \frac{x_1}{2I} - \psi_{1,2} - \frac{L(2+\nu)}{1+\nu} \frac{x_1}{2I} = -\frac{L x_1}{I}$$

or $\psi_{2,1} - \psi_{1,2} = 0 \quad (\text{f1})$

from (d2)

$$\begin{aligned} \psi_{2,2} + \frac{\nu L x_2}{(1+\nu)I} - \mu a - \frac{L(2+\nu)}{(1+\nu)} \frac{x_2}{2I} + \psi_{1,1} - \mu a \\ + \frac{L(2+\nu)}{(1+\nu)} \frac{x_2}{2I} = \frac{\nu L x_2}{(1+\nu)I} - 2\mu a \end{aligned}$$

or $\psi_{1,1} + \psi_{2,2} = 0 \quad (\text{f2})$

$$\therefore \quad \sigma_{23,11} = -\psi_{1,11} \quad \sigma_{13,12} = \psi_{2,12}$$

equation (c2) $\sigma_{23,11} - \sigma_{13,12} = -\psi_{1,11} - \psi_{2,12}$
 $= -(\psi_{1,1} + \psi_{2,2})_{,1} = 0 \quad (\text{O.K.})$

equation (f2) $\nabla \cdot \psi = 0$

then $\psi_a = -\mu \varepsilon_{3a\beta} H_{,\beta}$

$$\therefore \quad \psi_1 = -\mu H_{,2}$$

$$\psi_2 = \mu H_{,1}$$

then eq. (f1) becomes

$$\nabla^2 H = 0$$

Definition : Flexure function $F(x_1, x_2)$

$$F(x_1, x_2) = \frac{EI}{a} (\phi - H)$$

$$L$$

where ϕ is the warping function for the cross section R

$$\nabla^2 \phi = 0 \quad \text{in } R$$

$$\phi_{,n} = x_2 n_1 - x_1 n_2 \quad \text{on } C$$

$$\text{then} \quad \nabla^2 F = 0 \quad \text{in } R$$

from equation (e)

$$\sigma_{13} = \mu H_{,1} - \frac{L x_1^2}{2I} + \frac{\nu L x_2^2}{(1+\nu) 2I} - a \mu x_2 + \frac{L(2+\nu)}{(1+\nu)} \frac{(x_1^2 - x_2^2)}{4I}$$

$$\sigma_{23} = \mu H_{,2} + a \mu x_1 - \frac{L(2+\nu)}{(1+\nu)} \frac{(x_1^2 - x_2^2)}{2I}$$

H may be written as

$$H = a \phi - \frac{FL}{EI}$$

$$\begin{aligned} \therefore \mu H_{,1} &= \mu a \phi_{,1} - \frac{\mu L}{EI} F_{,1} \\ &= \mu a \phi_{,1} - \frac{L}{2(1+\nu)I} F_{,1} \end{aligned}$$

thus

$$\sigma_{13} = \mu a (\phi_{,1} - x_2) - \frac{L}{2(1+\nu)I} [F_{,1} + \frac{\nu x_1^2}{2} + \frac{2-\nu}{2} x_2^2]$$

similarly (g1)

$$\sigma_{23} = \mu a (\phi_{,2} + x_1) - \frac{L}{2(1+\nu)I} [F_{,2} + (2+\nu) x_1 x_2] \quad \text{(g2)}$$

on B_L

$$\sigma_{13} n_1 + \sigma_{23} n_2 = 0$$

$$a \mu \left(\frac{d\phi}{dn} - x_2 n_1 + x_1 n_2 \right) - \frac{L}{2(1+\nu)I} \left[\frac{dF}{dn} + \frac{\nu x_1^2 + (2-\nu)x_2^2}{2} n_1 + (2+\nu)x_1 x_2 n_2 \right] = 0$$

i.e.
$$\frac{dF}{dn} = - \frac{\nu x_1^2 + (2-\nu)x_2^2}{2} n_1 - (2+\nu)x_1 x_2 n_2 \quad \text{on } C$$

boundary conditions on R

$$\begin{aligned} L_1 &= \int_R \sigma_{31} dA = \int_R [\sigma_{31} + x_1 (\sigma_{31,1} + \sigma_{32,2}) + \frac{L x_1^2}{I}] dA \\ &= (L/I) \int_R x_1^2 dA + \int_R [(x_1 \sigma_{31})_{,1} + (x_1 \sigma_{32})_{,2}] dA \\ &= L + \oint_C x_1 (\sigma_{31} n_1 + \sigma_{32} n_2) dS \\ &= L \end{aligned} \quad (\text{O. K.})$$

$$\begin{aligned} L_2 &= \int_R \sigma_{32} dA = \int_R [\sigma_{32} + x_2 (\sigma_{31,1} + \sigma_{32,2}) + \frac{L x_1 x_2}{I}] dA \\ &= \int_R [(x_2 \sigma_{32})_{,2} + (x_2 \sigma_{31})_{,1}] dA \\ &= \oint_C x_2 (\sigma_{32} n_2 + \sigma_{31} n_1) dS \\ &= 0 \end{aligned} \quad (\text{O. K.})$$

$$L_3 = \int_R \sigma_{33} dA = - \frac{L(1-\nu_3)}{I} \int_R x_1 dA = 0 \quad (\text{O. K.})$$

on $x_3 = l$, $\sigma_{33} = 0$

$$M_1 = \int_R x_2 \sigma_{33} dA = 0 \quad (\text{O. K.})$$

$$M_2 = \int_R x_1 \sigma_{33} dA = 0 \quad (\text{O. K.})$$

$$\begin{aligned} M_3 &= \int_R (x_1 \sigma_{32} - x_2 \sigma_{31}) dA \quad [\text{from (g1) and (g2)}] \\ &= \mu a \int_R (x_1^2 + x_2^2 + x_1 \phi_{,2} - x_2 \phi_{,1}) dA \end{aligned}$$

$$+ \frac{L}{2(1+\nu)I} \int_R [x_2 F_{,1} - x_1 F_{,2} - \frac{4+\nu}{2} x_1^2 x_2 + \frac{2-\nu}{2} x_2^3] dA$$

M_3 must be equal to zero

recall the torsional stiffness K

$$K = \mu \int_R (x_1^2 + x_2^2 + x_1 \phi_{,2} - x_2 \phi_{,1}) dA$$

then the first term of M_3 is equal $a K$

$$\therefore a = \frac{L}{2(1+\nu)IK} \int_R (x_1 F_{,2} - x_2 F_{,1} - \frac{2-\nu}{2} x_2^3 + \frac{4+\nu}{2} x_1^2 x_2) dA \quad (h)$$

so for a given R , find ϕ , F , then K can be obtained, and compute a from (h), also use (g1) and (g2) to find the stresses

the strains can be calculated by the constitutive equations, and the displacement components are also obtained by the strain-displacement relations

the displacement components are

$$\begin{aligned} u_1 &= -a x_2 x_3 + \frac{L}{EI} \left[\frac{\nu}{2} (l - x_3) (x_1^2 - x_2^2) + \frac{l}{2} x_3^3 - \frac{x_3^2}{6} \right] \\ &\quad + a_1 + \omega_2 x_3 - \omega_3 x_2 \\ u_2 &= a x_1 x_3 + \frac{L \nu}{EI} [(l - x_3) x_1 x_2 + a_2 + \omega_3 x_1 - \omega_1 x_3] \\ u_3 &= a \phi - \frac{L}{EI} \left[(l - \frac{x_3}{2}) x_1 x_3 + x_1 x_2^2 + F(x_1, x_2) \right] \\ &\quad + a_3 + \omega_1 x_2 - \omega_2 x_1 \end{aligned}$$

to eliminate the rigid body motion, set $a_i = \omega_i = 0$, then

$$u_1(0, 0, x_3) = \frac{L}{2 E I} x_3^2 \left(l - \frac{x_3}{3} \right)$$

at $x_3 = l$

$$u_1(0, 0, l) = \frac{L l^3}{3 E I}$$

the result is the same as calculated in strength of materials, however, in general cross-section, plane will not remain plane

Summary :

given R as a simply connected region

$$\nabla^2 \phi = 0 \quad \text{in } R$$

$$\phi_{,n} = x_2 n_1 - x_1 n_2 \quad \text{on } C$$

$$\nabla^2 F = 0 \quad \text{in } R$$

$$F_{,n} = -\frac{1}{2} [\nu x_1^2 + (2 - \nu) x_2^2] n_1 - (2 + \nu) x_1 x_2 n_2 \quad \text{on } C$$

$$K = \mu \int_R (x_1^2 + x_2^2 + x_1 \phi_{,2} - x_2 \phi_{,1}) dA$$

$$a = \frac{L}{2(1 + \nu) I K} \int_R \left(x_1 F_{,2} - x_2 F_{,1} - \frac{2 - \nu}{2} x_2^3 + \frac{4 + \nu}{2} x_1^2 x_2 \right) dA$$

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0$$

$$\sigma_{13} = \mu a (\phi_{,1} - x_2) - \frac{L}{2(1 + \nu) I} \left[F_{,1} + \frac{\nu x_1^2}{2} + \frac{2 - \nu}{2} x_2^2 \right]$$

$$\sigma_{23} = \mu a (\phi_{,2} + x_1) - \frac{L}{2(1+\nu)I} [F_{,2} + (2+\nu)x_1 x_2]$$

$$\sigma_{33} = -L(l - x_3)x_1 / I$$

$$e_{ij} = e_{ij}(\sigma_{ij})$$

$$u_i = u_i(e_{ij})$$

the rotation vector is defined

$$\omega_i = \frac{1}{2} \varepsilon_{ijk} u_{k,j}$$

then $\omega_3 = \frac{1}{2} (u_{2,1} - u_{1,2})$

thus $\omega_{3,3}$ is the rate of twist along the x_3 -axis

Theorem : a is the average rate of twist over the cross the cross section

$$\begin{aligned} \therefore \omega_{3,3} &= \frac{1}{2} (u_{2,13} - u_{1,23}) \\ &= a - \frac{L\nu}{EI} x_2 \end{aligned}$$

$$\omega_{3,3} = a \quad \text{at } x_2 = 0$$

or $\int_R \omega_{3,3} dA = a \int dA - \frac{L\nu}{EI} \int_R x_2 dA$

$$\therefore a = \frac{\int_R \omega_{3,3} dA}{\int dA} \quad \text{average twist}$$

- Consider a cantilever beam of uniform cross section, a transverse loading $L \mathbf{e}_1$ pass through point $Q(x_1^0, x_2^0, l)$

[A]

[B]

[C]

problem [A] is equivalent to superimposing of [B] and [C]

problem [B] : bending due to transverse load

get $\phi(x_1, x_2)$, $F(x_1, x_2)$, $K(x_1, x_2)$ as before

$$a^B = \frac{L}{2(1+\nu)IK} \int_R (x_1 F_{,2} - x_2 F_{,1} - \frac{2-\nu}{2} x_2^3 + \frac{4+\nu}{2} x_1^2 x_2) dA$$

$$\sigma_{11}^B = \sigma_{22}^B = \sigma_{12}^B = 0$$

σ_{31}^B and σ_{32}^B can be calculated from equations (g1) and (g2)

$$\sigma_{33}^B = -L(l - x_3)x_1 / I$$

problem [C] : pure torsional problem

same $\phi(x_1, x_2)$, $K(x_1, x_2)$ as before

$$a^C = -L x_2^0 / K$$

$$\sigma_{11}^C = \sigma_{22}^C = \sigma_{33}^C = \sigma_{12}^C = 0$$

$$\sigma_{31}^C = \mu a^C (\phi_{,1} - x_2)$$

$$\sigma_{32}^C = \mu a^C (\phi_{,2} + x_1)$$

by superposition principle

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0$$

$$\sigma_{33} = \sigma_{33}^B \quad \text{independent of } a^B, a^C \text{ and } x_2$$

$$\sigma_{31} = \sigma_{31}^B + \sigma_{31}^C$$

$$\sigma_{32} = \sigma_{32}^B + \sigma_{32}^C$$

$$a = a^B + a^C$$

Definition : Center of flexure

The location of point $Q^*(x_1^*, x_2^*, l)$ for which L passes through such that $a = 0$ is called the center of flexure

$$\therefore a = a^B + a^C \quad \text{for } L \text{ located at center of flexure}$$

then

$$x_2^* = \frac{1}{2(1+\nu)I} \int_R (x_1 F_{,2} - x_2 F_{,1} - \frac{2-\nu}{2} x_2^3 + \frac{4+\nu}{2} x_1^2 x_2) dA$$

similarly

$$x_1^* = \frac{1}{2(1+\nu)I} \int_R (x_2 F_{,1} - x_1 F_{,2} - \frac{2-\nu}{2} x_1^3 + \frac{4+\nu}{2} x_1 x_2^2) dA$$

Theorem : If R has an axis of symmetry, then Q^* is on this axis, if R has two axes of symmetry, then Q^* is the centroid of the area

proof if R is symmetry with respect to x_1 axis, then

$$\int (x_2, x_2^3, \text{odd function of } x_2) dA = 0$$

$$\text{then } x_2^* = \frac{1}{2(1+\nu)I} \int_R (x_1 F_{,2} - x_2 F_{,1}) dA$$

$$\text{but } \nabla^2 F = 0 \quad \text{in } R$$

$$F_{,n} = -\frac{1}{2} [\nu x_1^2 + (2-\nu) x_2^2] n_1 - (2+\nu) x_1 x_2 n_2 \quad \text{on } C$$

$$\text{at } A(x_1, x_2) \quad \mathbf{n} = n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2$$

$$\text{at } A'(x_1, -x_2) \quad \mathbf{n} = n_1 \mathbf{e}_1 - n_2 \mathbf{e}_2$$

and $x_2 n_2$ is the same at A and A'

i.e. $F_{,n}$ is symmetric w. r. t. x_1 axis on C

$\therefore F(x_1, x_2)$ is an even function of x_2

$F_{,1}$ is also an even function of x_2

$F_{,2}$ is an odd function of x_2

i.e. $x_1 F_{,2}$ and $x_2 F_{,1}$ are odd functions of x_2

then $\int_R (x_1 F_{,2} - x_2 F_{,1}) dA = 0$

$$\therefore x_2^* = 0$$

- Consider R is a circular section of radius a subjected to a transverse L at the end

$$\phi = 0 \quad \mathbf{n} = \mathbf{e}_\rho$$

$$n_1 = \cos \theta \quad x_1 = \rho \cos \theta$$

$$n_2 = \sin \theta \quad x_2 = \rho \sin \theta$$

flexure function $F(x_1, x_2)$ satisfies

$$\nabla^2 F = F_{,\rho\rho} + \frac{1}{\rho} F_{,\rho} + \frac{1}{\rho} F_{,\theta\theta} = 0 \quad \text{in } R \quad (\text{a})$$

$$\frac{\partial F}{\partial n} = \frac{\partial F}{\partial \rho} \Big|_{\rho=a} = -\frac{1}{2} [\nu x_1^2 + (2 - \nu) x_2^2] n_1 - (2 + \nu) x_1 x_2 n_2$$

on C

$$= -\frac{3+2\nu}{4} a^2 \cos \theta + \frac{3}{4} a^2 \cos 3\theta \quad (b)$$

general solution for the harmonic function in polar coordinate is

$$F(\rho, \theta) = \sum_{n=0}^{\infty} \rho^n (A_n \cos n\theta + B_n \sin n\theta) \quad (c)$$

$$\left. \frac{\partial F}{\partial \rho} \right|_{\rho=a} = \sum_{n=0}^{\infty} n a^{n-1} (A_n \cos n\theta + B_n \sin n\theta) \quad (d)$$

from (b) and (d), we have

$$B_n = 0 \quad \text{and} \quad A_n = 0 \quad \text{for } n \neq 1 \text{ and } 3$$

$$\left. \frac{\partial F}{\partial \rho} \right|_{\rho=a} = A_1 \cos \theta + 3 a^2 A_3 \cos 3\theta \quad (d)$$

$$\text{i.e.} \quad A_1 = -\frac{3+2\nu}{4} a^2 \quad A_3 = \frac{1}{4}$$

substitute A_1 and A_3 into (c), we have

$$F(\rho, \theta) = -\frac{3+2\nu}{4} a^2 \rho \cos \theta + \frac{1}{4} \rho^3 \cos 3\theta$$

or in x_1, x_2 coordinates

$$F(x_1, x_2) = -\frac{3+2\nu}{4} a^2 x_1 + \frac{1}{4} (x_1^2 - 3 x_2^2)$$

$$a = \frac{L}{2(1+\nu)IK} \int_R (x_1 F_{,2} - x_2 F_{,1} - \frac{2-\nu}{2} x_2^3 + \frac{4+\nu}{2} x_1^2 x_2) dA$$

$$\begin{aligned}
&= \frac{L}{2(1+\nu)IK} \int_R \left\{ x_1 \left(-\frac{3}{2} x_1 x_2 - x_2 \left[-\frac{3+2\nu}{4} a^2 + \frac{3}{4} (x_1^2 - x_2^2) \right] \right. \right. \\
&\quad \left. \left. - \frac{2-\nu}{2} x_2^3 + \frac{4+\nu}{2} x_1^2 x_2 \right) \right\} dA \\
&= \frac{L}{2(1+\nu)IK} \int_R \left(\frac{2\nu-1}{4} x_1^2 x_2 + \frac{2\nu-1}{4} x_2^3 + \frac{3+2\nu}{4} a^2 x_2 \right) dA \\
&= \frac{L}{2(1+\nu)IK} \frac{2\nu-1}{4} \int_R (x_1^2 + x_2) x_2 dA \\
&= \frac{L}{2(1+\nu)IK} \frac{2\nu-1}{4} \int_R \rho^2 \rho \sin \theta dA
\end{aligned}$$

$$a = 0 \quad (\text{no twist angle})$$

the stress components are

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0$$

$$\sigma_{33} = -L(l-x_3)x_1/I = -4L(l-x_3)x_1/\pi a^4 \quad [I = \pi a^4/4]$$

$$\sigma_{13} = \mu a(\phi_{,1} - x_2) - \frac{L}{2(1+\nu)I} \left[F_{,1} + \frac{\nu x_1^2}{2} + \frac{2-\nu}{2} x_2^2 \right]$$

$$\begin{aligned}
&= -\frac{4L}{2\pi a^4(1+\nu)} \left[\frac{3+2\nu}{4} a^2 - \frac{3}{4} (x_1^2 - x_2^2) - \frac{\nu x_1^2}{2} \right. \\
&\quad \left. - \frac{2-\nu}{2} x_2^2 \right]
\end{aligned}$$

$$= -\frac{4L}{2\pi a^4(1+\nu)} [(3+2\nu)(a^2 - x_1^2) - (1-2\nu)x_2^2]$$

similarly

$$(1+2\nu)Lx_1x_2$$

$$\sigma_{23} = - \frac{L}{\pi a^4 (1 + \nu)}$$

also we get maximum shear stress occurs at (0, 0)

$$\sigma_{31}(0, 0) = \frac{3 + 2 \nu}{2 (1 - \nu)} \frac{L}{A}$$

along $x_1 = 0$, $\sigma_{32} = 0$, but σ_{31} is not constant

if we use $\nu = 0.3$

we have $\sigma_{31} = 1.385 L / A$ at center

$\sigma_{31} = 1.231 L / A$ at $x_2 = \pm a$

from the elementary beam theory

$$\begin{aligned} \sigma_{31} &= 4 L / 3 A \\ &= \text{constant on } x_1 = 0 \\ &= \text{average value} \end{aligned}$$

Shear trajectories

$$\frac{d x_2}{d x_1} = \frac{\sigma_{32}}{\sigma_{31}} = \frac{2 (1 + 2 \nu) x_1 x_2}{(3 + 2 \nu) (x_1^2 - a^2) + (1 - 2 \nu) x_2^2}$$

or

$$2 (1 + 2 \nu) x_1 x_2 dx_1 - (3 + 2 \nu) (-a^2 + x_1^2 + \frac{1 - 2 \nu}{3 + 2 \nu} x_2^2) dx_2 = 0$$

the general solution is

$$x_1^2 + x_2^2 = a^2 - C x_2^{\frac{3+2\nu}{1+2\nu}} \quad C \text{ is a constant}$$

for $\nu = 0.3$, the numerical results
are indicated in figure

the distortion of the cross section at $x_3 = c$ can be obtained

$$\begin{aligned} u_3 &= -\frac{L}{EI} \left[\left(l - \frac{x_3}{2} \right) x_3 x_1 + x_1 x_2^2 + F(x_1, x_2) \right] \\ &= -\frac{L}{\pi a^4 E} [2c(2l - c)x_1 + 4x_1 x_2^2 - (3 + 2\nu)a^2 x_1 \\ &\quad + x_1(x_1^2 - 3x_2^2)] \\ &= -\frac{L}{\pi a^4 E} [- (3 + 2\nu)a^2 + 2c(2l - c)] x_1 - \frac{L}{\pi a^4 E} (x_1^2 + x_2^2) x_1 \\ &\quad \text{[linear term]} \quad \text{[nonlinear term]} \end{aligned}$$

the linear term represents the rotation of the cross section, and the
nonlinear term represents the distortion of the cross section

the contour lines of the distortion are given by

$$-\frac{L}{\pi a^4 E} (x_1^2 + x_2^2) x_1 = \text{constant}$$

some numerical results of the contour
lines are shown in the figure

plane will not remain plane

- Consider R of the beam as a rectangular cross section

$$\nabla^2 F = 0 \quad \text{in } R$$

$$F_{,n} = -\frac{1}{2} [\nu x_1^2 + (2 - \nu) x_2^2 n_1] \\ - (2 + \nu) x_1 x_2 n_2 \quad \text{on } C$$

$$\text{on } x_1 = \pm b \quad n_1 = \pm 1 \quad n_2 = 0$$

$$F_{,n} = F_{,1} n_1 = -\frac{1}{2} [\nu b_1^2 + (2 - \nu) x_2^2]$$

$$\text{on } x_2 = \pm a \quad n_1 = 0 \quad n_2 = \pm 1$$

$$F_{,n} = F_{,2} n_2 = - (2 + \nu) a x_1$$

$$\text{let } G = F + \frac{2 + \nu}{2} x_1 x_2^2 - \frac{2 + \nu}{2} \frac{x_1^3}{3}$$

$$\text{then } \nabla^2 G = \nabla^2 F + \frac{2 + \nu}{2} (2 x_1 - 2 x_1) = 0$$

$$G_{,1} = F_{,1} + \frac{1}{2} (2 + \nu) x_2^2 - \frac{1}{2} (2 + \nu) x_1^2$$

$$G_{,1}(\pm b, x_2) = -\frac{1}{2} [\nu b_1^2 + (2 - \nu) x_2^2] + \frac{1}{2} (2 + \nu) (x_2^2 - x_1^2) \\ = \nu x_2^2 - (1 + \nu) b^2 \quad (\text{a})$$

similarly

$$G_{,2}(x_2, \pm a) = 0 \quad (\text{b})$$

to solve the function $G(x_1, x_2)$, let

$$G(x_1, x_2) = X(x_1) Y(x_2)$$

$$\text{and} \quad X'(\pm b) = \nu x_2^2 - (1 + \nu) b^2 \quad (\text{a'})$$

$$Y'(\pm a) = 0 \quad (\text{b'})$$

$\therefore \nabla^2 G = 0$, then we have

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\text{let} \quad \frac{Y''}{Y} = -\lambda^2$$

$$\text{then} \quad Y(x_2) = \begin{cases} A \cos \lambda x_2 + B \sin \lambda x_2 & \text{if } \lambda \neq 0 \\ a_1 x_2 + a_2 & \text{if } \lambda = 0 \end{cases}$$

because $F(x_1, x_2)$ is a even function in x_2 , then $G(x_1, x_2)$ is also a even function in x_2 , therefore $B = a_1 = 0$

$$\text{from (b)} \quad Y'(\pm a) = 0$$

$$\sin \lambda a = 0 \quad \implies \quad \lambda_n = n \pi / a$$

$$\text{also} \quad \frac{X''}{X} = \lambda^2$$

$$\therefore X_n(x_1) = \begin{cases} C_n \cosh n \pi x_1 / a + D_n \sinh n \pi x_1 / a & \text{for } n \neq 0 \\ a_3 x_1 + a_4 & \text{for } n = 0 \end{cases}$$

$$\text{from (a)} \quad G_{,1}(\pm b, x_2) = \nu x_2^2 - (1 + \nu) b^2 \quad (\text{c})$$

$$\text{we have} \quad X'(b) = X'(-b) \quad \implies \quad C_n = 0$$

$$\therefore G(x_1, x_2) = a_3' x_1 + a_4' + \sum_{n=1}^{\infty} D_n' \sinh \frac{n \pi x_1}{a} \cos \frac{n \pi x_2}{a} \quad (d)$$

$$\text{then } G_{,1}(\pm b, x_2) = a_3' + \sum_{n=1}^{\infty} \frac{n \pi}{a} D_n' \cosh \frac{n \pi b}{a} \cos \frac{n \pi x_2}{a} \quad (e)$$

$$\text{but } x_2^2 = \frac{a^2}{3} + \frac{4 a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n \pi x_2}{a} \quad (f)$$

$$\begin{aligned} [\text{on } x_2^2 = \pm a] \quad a^2 &= \frac{a^2}{3} + \frac{4 a^2}{\pi^2} \left(1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2}\right) \\ &= \frac{a^2}{3} + \frac{4 a^2}{\pi^2} \frac{\pi^2}{6} = a^2 \quad (\text{O. K.}) \end{aligned}$$

from (c), (e) and (f)

$$\begin{aligned} \nu \left[\frac{a^2}{3} + \frac{4 a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n \pi x_2}{a} \right] - (1 + \nu) b^2 \\ = a_3' + \sum_{n=1}^{\infty} \frac{n \pi}{a} D_n' \cosh \frac{n \pi b}{a} \cos \frac{n \pi x_2}{a} \end{aligned}$$

thus, we have

$$a_3' = \frac{\nu a^2}{3} - (1 + \nu) b^2$$

$$D_n' = \frac{4 a^3 \nu (-1)^n}{\pi^3 n^3 \cosh (n \pi b / a)}$$

thus

$$G(x_1, x_2) = \left[\frac{\nu a^2}{3} - (1 + \nu) b^2 \right] x_1 + a_4$$

$$+ \sum_{n=1}^{\infty} \frac{4 a^3 \nu (-1)^n}{\pi^3 n^3 \cosh (n \pi b / a)} \sinh (n \pi x_1 / a) \cos (n \pi x_2 / a)$$

$$+ \frac{1}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{1}{n^3 \cosh(n\pi b/a)} \right]$$

and

$$F(x_1, x_2) = G(x_1, x_2) + \frac{1}{6} (2 + \nu) (x_1^3 - 3 x_1 x_2^2)$$

if the transverse load is applied at center, then no twisting angle

i.e. $\alpha = 0$

the stress components are

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0$$

$$\sigma_{33} = - \frac{L (l - x_3) x_1}{I} = - \frac{3 L (l - x_3) x_1}{4 a b^3}$$

$$\begin{aligned} \sigma_{31} &= - \frac{L}{2 (1 + \nu) I} \left[F_{,1} + \frac{\nu x_1^2}{2} + \frac{2 - \nu}{2} x_2^2 \right] \\ &= - \frac{L}{2 (1 + \nu) I} \left[G_{,1} + \frac{1}{2} (2 + \nu) (x_1^2 - x_2^2) + \frac{\nu x_1^2}{2} + \frac{2 - \nu}{2} x_2^2 \right] \\ &= - \frac{L}{2 (1 + \nu) I} \left[G_{,1} + (1 + \nu) x_1^2 - \nu x_2^2 \right] \end{aligned}$$

similarly

$$\sigma_{32} = - \frac{L}{2 (1 + \nu) I} G_{,2}$$

at $x_1 = 0$

$$\sigma_{31} = - \frac{L}{2 (1 + \nu) I} [G_{,1} - \nu x_2^2]$$

σ_{31} is not a constant, it vary along x_2 axis

$$G_{,1} = \frac{\nu a^2}{3} - (1 + \nu) b^2 + \frac{4 a^3 \nu}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos (n\pi x_2/a)}{n^3 \cosh (n\pi b/a)} \right]$$

for $a = b$ and $\nu = 0.3$

$$G_{,1} \simeq -1.2105 a^2$$

$$\begin{aligned} \sigma_{31} &= \frac{12 L}{2.6 (2a)^4} (1.2105 a^2 + \nu x_2^2) \\ &= 1.1538 \frac{L}{A} [1.2105 + 0.3 \left(\frac{x_2}{a}\right)^2] \end{aligned}$$

$$\sigma_{31} = 1.3967 L / A \quad \text{at } x_2 = 0$$

$$\sigma_{31} = 1.75 L / A \quad \text{at } x_2 = \pm a$$

from elementary strength of materials

$$\sigma_{31} = 1.5 L / A \quad \text{at } x_1 = 0$$

also u_3 may be calculated, it is seen that plane section do not remains plane, the contour line of the cross section as shown in the figure

● Inverse Method Using Curvilinear Orthogonal Coordinates

Let $z = f(\omega)$ be analytic

where $z = x_1 + i x_2$

$$\omega = y_1 + i y_2$$

and let $\phi = \text{Re } \{f\}$

$$\psi = \text{Im } \{f\}$$

ϕ and ψ satisfy Cauchy-Riemann condition

consider the curvilinear coordinates given by

$$\begin{aligned} x_1 &= \phi(y_1, y_2) \\ x_2 &= \psi(y_1, y_2) \end{aligned} \quad (\text{a})$$

if $y_1 = c$, (a) is a curve of $x_2 = x_2(x_1, c)$

similarly, if $y_2 = d$, (a) is a family of curves $x_2 = x_2(x_1, d)$

y_1 and y_2 are called curvilinear coordinates, the analytic function $f(\omega)$ is said to generate the curvilinear coordinates

$$\begin{aligned} \text{Notation} \quad \nabla_x(\) &= (\)_{,1} \mathbf{e}_1 + (\)_{,2} \mathbf{e}_2 \\ \nabla_y(\) &= (\)_{,y1} \mathbf{e}_{y1} + (\)_{,y2} \mathbf{e}_{y2} \end{aligned}$$

Theorem : The coordinates (y_1, y_2) generated by $f(\omega)$ are

(a) orthogonal

$$(b) \quad \frac{dS_1}{dy_1} = \frac{dS_2}{dy_2} = |\nabla_y \phi|$$

where S_1 and S_2 are the lengths along y_1 and y_2 , respectively

$$(c) \quad \nabla_x^2 = \frac{1}{|\nabla_y \phi|^2} \nabla_y^2$$

proof

$$\mathbf{r} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$$

$$\mathbf{t}_1 = \left. \frac{d\mathbf{r}}{dS_1} \right|_{y_2 = \text{const}} = \frac{dx_1}{dS_1} \mathbf{e}_1 + \frac{dx_2}{dS_1} \mathbf{e}_2$$

$$\text{but } \left. \frac{dx_1}{dy_1} \right|_{y_2 = \text{const}} = \frac{\partial \phi}{\partial y_1} \quad [x_1 = \phi(y_1, y_2)]$$

$$\left. \frac{dx_2}{dy_1} \right|_{y_2 = \text{const}} = \frac{\partial \psi}{\partial y_1} \frac{dy_1}{dy_1} = - \frac{\partial \phi}{\partial y_2} \frac{dy_1}{dy_2} \quad [x_2 = \psi(y_1, y_2)]$$

$$\therefore \quad \mathbf{t}_1 = \left(\frac{\partial \phi}{\partial y_1} \mathbf{e}_1 - \frac{\partial \phi}{\partial y_2} \mathbf{e}_2 \right) \frac{dy_1}{dS}$$

$$\text{similarly } \mathbf{t}_2 = \left(\frac{\partial \phi}{\partial y_2} \mathbf{e}_1 + \frac{\partial \phi}{\partial y_1} \mathbf{e}_2 \right) \frac{dy_2}{dS}$$

$$\text{let } \frac{dy_1}{dS_1} = \frac{1}{A} \quad \frac{dy_2}{dS_2} = \frac{1}{B}$$

$$\text{then } \mathbf{t}_1 \cdot \mathbf{t}_2 = \frac{1}{AB} \left(\frac{\partial \phi}{\partial y_1} \frac{\partial \phi}{\partial y_2} - \frac{\partial \phi}{\partial y_2} \frac{\partial \phi}{\partial y_1} \right) = 0$$

thus $\mathbf{t}_1 \perp \mathbf{t}_2$ i.e. the coordinates are orthogonal

$$\therefore \quad \mathbf{t}_1 \cdot \mathbf{t}_1 = 1 \quad A^2 = \left(\frac{\partial \phi}{\partial y_1} \right)^2 + \left(\frac{\partial \phi}{\partial y_2} \right)^2$$

$$\therefore \quad A = |\nabla_y \phi|$$

$$\text{similarly } \mathbf{t}_2 \cdot \mathbf{t}_2 = 1 \quad \Rightarrow \quad B = |\nabla_y \phi|$$

$$\therefore \quad \nabla_x^2 = \frac{\partial^2}{\partial x_i \partial x_i} = 4 \frac{\partial^2}{\partial z \partial z} = \frac{4}{f(\omega) f(\underline{\omega})} \frac{\partial^2}{\partial \omega \partial \underline{\omega}}$$

$$= \frac{1}{|f'(\omega)|^2} \frac{\partial^2}{\partial y_i \partial y_i} = \frac{1}{|f'(\omega)|^2} \nabla_y^2$$

$$f'(\omega) = \frac{\partial \phi}{\partial y_1} - i \frac{\partial \phi}{\partial y_2}$$

$$|f'(\omega)|^2 = \left(\frac{\partial \phi}{\partial y_1}\right)^2 - \left(\frac{\partial \phi}{\partial y_2}\right)^2 = A^2 = |\nabla_y \phi|^2$$

$$\therefore \nabla_x^2 = \frac{1}{|\nabla_y \phi|^2} \nabla_y^2$$

Consider the coordinates (y_1, y_2) generated by

$$f(\omega) = c \cosh \omega = c \cosh (y_1 + i y_2)$$

$$x_1 = \phi = c \cosh y_1 \cos y_2$$

$$x_2 = \psi = c \sinh y_1 \sin y_2$$

$$A^2 = \left(\frac{\partial \phi}{\partial y_1}\right)^2 - \left(\frac{\partial \phi}{\partial y_2}\right)^2 = A^2 = c^2 (\sin^2 y_2 + \sinh^2 y_1)$$

for $y_1 = \text{constant}$, we have

$$\frac{x_1^2}{c^2 \cosh^2 y_1} + \frac{x_2^2}{c^2 \sinh^2 y_1} = 1$$

this equation represents an ellipse with

$$a^2 = c^2 \cosh^2 y_1$$

$$b^2 = c^2 \sinh^2 y_1$$

for $y_2 = \text{constant}$, we have

$$\frac{x_1^2}{c^2 \cos^2 y_2} - \frac{x_2^2}{c^2 \sin^2 y_2} = 1$$

this equation represents a hyperbola

thus, (y_1, y_2) represent an elliptical coordinates

- consider the cross section of the beam is an ellipse

$$C: \quad \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

for this problem, elliptical coordinates will used

the elliptical coordinate $y_1 = \xi$ represents an ellipse

on C

$$\frac{\partial F}{\partial n} = -\frac{1}{2} [\nu x_1^2 + (2 - \nu) x_2^2] n_1 - (2 + \nu) x_1 x_2 n_2$$

$$\frac{\partial F}{\partial n} = \frac{\partial F}{\partial \xi} = \frac{\partial F}{\partial y_1} \frac{\partial y_1}{\partial \xi}$$

but
$$\frac{\partial y_1}{\partial \xi} = n = \frac{d y_1}{d S_2} = \frac{1}{A}$$

$$\begin{aligned} \therefore \frac{\partial F}{\partial \xi} &= \frac{1}{A} \frac{\partial F}{\partial y_1} \bigg|_{y_1 = \xi} \\ &= -\frac{1}{2A} [\nu \phi^2 + (2 - \nu) \psi^2] \frac{\partial \phi}{\partial y_1} + \frac{2 + \nu}{A} \phi \psi \frac{\partial \phi}{\partial y_2} \quad (a) \end{aligned}$$

$$\therefore \nabla_x^2 F = 0 \quad \text{in } R$$

by part (c) of the above theorem

$$\begin{aligned} (\nabla_y^2 F) / A^2 &= 0 && \text{in } R \\ \text{i.e. } \nabla_y^2 F &= 0 && \text{in } R \end{aligned} \quad (b)$$

$$\therefore x_1 = \phi = c \cosh y_1 \cos y_2$$

$$x_2 = \psi = c \sinh y_1 \sin y_2$$

$$a^2 = c^2 \cosh^2 y_1$$

$$b^2 = c^2 \sinh^2 y_1$$

$$c^2 = a^2 - b^2$$

then equation (a) becomes

$$\begin{aligned} 8 (\partial F / \partial y_1) |_{y_1=\xi} &= [-a^2 b (5\nu + 4) + b^3 (\nu - 2)] \cos y_2 \\ &+ [b^3 (2 - \nu) + a^2 b (4 + \nu)] \cos 3 y_2 \end{aligned} \quad (c)$$

solve (b) satisfy (c), try the solution of the form

$$8 F(y_1, y_2) = f(y_1) \cos y_2 + g(y_1) \cos 3 y_2$$

to satisfy the boundary condition on C , we get [equation (c)]

$$f(y_1) = a_1 \cosh y_1 + a_2 \sinh y_1$$

$$g(y_1) = \frac{a_3}{3} \cosh 3 y_1 + \frac{a_4}{4} \sinh 3 y_2$$

$$\therefore F(y_1, y_2) \text{ is an even function, then } a_2 = a_4 = 0$$

$$a_3$$

$$\therefore 8 F(y_1, y_2) = a_1 \cosh y_1 \cos y_2 + \frac{1}{3} \cosh 3 y_1 \cos 3 y_2 \quad (d)$$

from (c) and (d), we find

$$a_1 = -c [b^2 (2 - \nu) + a^2 (\nu + 4)]$$

$$a_3 = \frac{c^2}{3 a^2 + b^2} [b^2 (2 - \nu) + a^2 (\nu + 4)]$$

transform back into (x_1, x_2) coordinates

$$\begin{aligned} F(x_1, x_2) = & - \frac{a^2 [2 (1 + \nu) a^2 + b^2]}{3 a^2 + b^2} x_1 \\ & + \frac{2 a^2 + b^2 + \nu (a^2 - b^2) / 2}{3 a^2 + b^2} \frac{x_1^3 - 3 x_1 x_2^2}{3} \end{aligned}$$

$\therefore$ the section is symmetric in x_1, x_2 axes, $\therefore O$ is the shear center,

thus $a = 0$

$\therefore \phi(x_1, x_2)$ is not necessary to be used, so we don't want to find the warping function in this problem

from equations (g1) and (g2) in p. 8-5, σ_{31} and σ_{32} can be calculated

$$\begin{aligned} \sigma_{31} &= \frac{2 (1 + \nu) a^2 + b^2}{(1 + \nu) (3 a^2 + b^2)} \frac{L}{2 I} [a^2 - x_1^2 - \frac{(1 - 2 \nu) a^2}{2 (1 + \nu) a^2 + b^2} x_2^2] \\ \sigma_{32} &= - \frac{(1 + \nu) a^2 + \nu b^2}{(1 + \nu) (3 a^2 + b^2)} \frac{L x_1 x_2}{I} \end{aligned}$$

for $a = b$, the stress components are the same as in circular cross section which are calculated before

at $x_1 = 0$

$$\sigma_{31} = \frac{2(1+\nu)a^2 + b^2}{(1+\nu)(3a^2 + b^2)} \frac{L}{2I} \left[a^2 - \frac{(1-2\nu)a^2}{2(1+\nu)a^2 + b^2} x_2^2 \right]$$

$$\sigma_{32} = 0$$

maximum σ_{31} occurs at $(0, 0)$

$$(\sigma_{31})_{\max} = \frac{2(1+\nu)a^2 + b^2}{(1+\nu)(3a^2 + b^2)} \frac{L a^2}{2I}$$

if $b \gg a$, then

$$(\sigma_{31})_{\max} = \frac{2}{1+\nu} \frac{L}{A}$$

$$(\sigma_{31})_{x_2=\pm b} = \frac{4\nu}{1+\nu} \frac{L}{A}$$

take $\nu = 0.3$

$$(\sigma_{31})_{\max} = 1.54 L / A$$

$$(\sigma_{31})_{x_2=\pm b} = 0.92 L / A$$

also u_3 can be obtained