

6. Fundamental Theorems in Theory of Linear Elasticity

Definition : State

Let $u_i(x_i, t)$, $e_{ij}(x_i, t)$, $\sigma_{ij}(x_i, t)$ with $\sigma_{ij} = \sigma_{ji}$ and $e_{ij} = e_{ji}$ be defined in a regular space $D + B$ for $0 \leq t < \infty$, the order array $\{u_1, u_2, u_3, e_{11}, \dots, e_{33}, \sigma_{11}, \dots, \sigma_{33}\}$ is called the state $S(x_i, t)$ in $D + B$ for $0 \leq t < \infty$.

Definition : Regular elastic state

Let $\rho(x_i)$, $C_{ijkl}(x_i)$, $f_i(x_i, t)$ be continuous in $D + B$, then $S(x_i, t)$ is a regular elastic state in $D + B$ for $0 \leq t < \infty$ if

(a) $u_i(x_i, t)$, $e_{ij}(x_i, t)$, $\sigma_{ij}(x_i, t)$ are regular

(b) $u_i(x_i, t)$, $e_{ij}(x_i, t)$, $\sigma_{ij}(x_i, t)$ satisfy fundamental system (5.1), (5.2), (5.3)

in D for $0 \leq t < \infty$

Note : the necessary conditions for the existence of a regular state $S(x_i, t)$ are

(a) $\rho(x_i)$, $f_i(x_i)$ continuous in $D + B$

(b) $u_i^*(x_i, t)$, $T_i^*(x_i, t)$ are C_1 with respect to x_i in every closed regular subregion of B_1, B_2 for $0 \leq t < \infty$

(c) $u_i^*(x_i, t)$ is C_2 with respect to t on B_1 for $0 \leq t < \infty$

Definition : Sum of two states

$$S(x_i, t) =$$

$$\text{means } u_i(x_i, t) =$$

$$e_{ij}(x_i, t) =$$

$$\sigma_{ij}(x_i, t) =$$

Theorem : Let $S(x_i)$ satisfy (5.1), (5.2') [eq. of equil.], (5.3) in D and be C_4 in D , and let the medium be homogeneous

1. if $\text{div}(\mathbf{f}) = f_{i,i} = 0$ in D

then $\nabla^2 I_1 = \nabla^2 J_1 = 0$

where $I_1 = \sigma_{ii}$ and $J_1 = e_{ii}$

then means I_1 and J_1 are harmonic functions

2. if f_i are constants, then

$$\nabla^4 u_i = \nabla^4 e_{ij} = \nabla^4 \sigma_{ij} = 0 \quad \text{in } D$$

where $\nabla^4(\) = (\)_{,iiij}$

note : a function of class C_4 and satisfying $\nabla^4(\) = 0$ is called biharmonic function

proof :

1. for linear, isotropic, homogeneous, elastic medium, the stress compatibility condition is

let $i = j$

$$\therefore \sigma_{ii,kk} = 0$$

$$\text{or} \quad \nabla^2 \sigma_{ii} = \nabla^2 I_1 = 0$$

$$\text{also} \quad e_{ii} =$$

$$=$$

$$\nabla^2 e_{ii} = \nabla^2 J_1 =$$

2. if $f_i = \text{constant}$

$$\text{then} \quad f_{i,i} = f_{i,j} = 0$$

equation of equilibrium in terms of u_i is

take div for each term

$$\therefore$$

$$\therefore$$

or

from the stress compatibility condition

take div for each term

$$\therefore f_{i,i} = 0 \quad \text{then}$$

$$\therefore$$

or

from the constitutive equation

$$e_{ij} =$$

$$\nabla^4 e_{ij} =$$

$$\therefore \nabla^4 e_{ij} =$$

Theorem : Betti's and Rayleigh's Reciprocal Theorem

Let $S'(x_i)$ and $S''(x_i)$ be regular equilibrium states in $D + B$ for $C_{ijkl}(x_i)$, and for $f'_i(x_i)$ and $f''_i(x_i)$, respectively, then

$$\int_B T'_i u''_i dA + \int_D f'_i u''_i dV = \int_B T''_i u'_i dA + \int_D f''_i u'_i dV$$

i.e.
$$\int_D \sigma_{ij}' e_{ij}'' dV = \int_D \sigma_{ij}'' e_{ij}' dV$$

proof
$$\begin{aligned} \int_B T_i' u_i'' dA + \int_D f_i' u_i'' dV \\ = \\ = \\ = \\ = \end{aligned}$$

interchange (') and ('') to get the result

consider two equilibrium state of a beam

$$\begin{array}{cc} p(x) & \underline{p}(x) \\ \hline & \hline \end{array}$$

by reciprocal theorem

assume body force vanish

$$\begin{array}{cc} P_1 & \underline{P}_2 \\ \hline & \hline \\ y_1 & y_2 & \underline{y}_1 & \underline{y}_2 \end{array}$$

$$y_1 = \underline{y}_1 =$$

$$y_2 = \underline{y}_2 =$$

a_{ij} : influence coefficients

reciprocal theorem

=

=

 $\therefore$

moreover

[Maxwell's Law]

Theorem : Let $S(x_i)$ be a regular state of equilibrium in $D + B$, then

$$\int_B T_i u_i dA + \int_D f_i u_i dV = 2 \varepsilon$$

where $\varepsilon = \int_D e dV$

proof L.H.S. =

consider now the effect of varying a force T_i applied at a point P

now, we choose

$$f'_i = f_i$$

$$T'_i = \begin{cases} T_i + \Delta T_i & \text{on } B^* \\ T_i & \text{on } B - B^* \end{cases}$$

by reciprocal theorem

$$\int_B T_i u'_i dA + \int_D f_i u'_i dV =$$

(a)

the strain energy ε associated with the original load T_i, f_i is

$$2 \varepsilon =$$

for the varied state T'_i, f'_i

$$2 \varepsilon' =$$

=

substitute 2ε and $2 \varepsilon'$ into equation (a)

$$2 \varepsilon' - \int_{B^*} \Delta T_i u_i' dA =$$

$$2 (\varepsilon' - \varepsilon) = 2 \Delta \varepsilon =$$

for B^* becomes small

$$2 \Delta \varepsilon =$$

the increment of force on B^* is

$$\Delta P_i =$$

i.e.
$$\frac{\Delta \varepsilon}{\Delta P_i} =$$

for $\Delta P_i \rightarrow 0 \quad u_i' \rightarrow u_i$

we get
$$d\varepsilon / dP_i = u_i$$

this is the theorem of Castigliano

Theorem : Uniqueness Theorem for Elastic-Kinetics

Let $S'(x_i, t)$ and $S''(x_i, t)$ be regular states in $D + B$ for $0 \leq t < \infty$, and let both be solution to same problem D-III (mixed type problem), let $C_{ijkl}(x_i)$ be such that e is positive definite, then $S'(x_i, t) = S''(x_i, t)$

proof : let $S(x_i, t) =$

we will show $S(x_i, t) =$

by hypothesis and superposition principle, we have $S(x_i, t)$ is regular

state with $f_i =$

$$u_i(x_i, 0) =$$

and $\dot{u}_i(x_i, 0) =$

also $u_i(x_i, t) =$ on B_1

$$T_i(x_i, t) =$$
 on B_2

$$\therefore \int_B T_i \dot{u}_i dA + \int_D f_i \dot{u}_i dV =$$

$$\int_{B_1} T_i \dot{u}_i dA + \int_{B_2} T_i \dot{u}_i dA =$$

i.e. $K + \varepsilon = \text{constant}$ at any time

$$\text{at } t = 0 \quad \dot{u}_i(x_i, t) = 0 \quad \text{in } D$$

then $K =$

and $e_{ij}(x_i, 0) =$

$$\therefore u_i(x_i, 0) = \quad \text{in } D$$

$$\therefore u_{i,j}(x_i, 0) = \quad \text{in } D$$

and then $e_{ij}(x_i, 0) = \quad \text{in } D$

thus $\varepsilon(e_{ij}) =$

$$\therefore K + \varepsilon = \quad \text{at time} = 0$$

$$\therefore K + \varepsilon =$$

therefore $K + \varepsilon = \quad \text{at all time}$

now we have $K \geq 0$ and $\varepsilon \geq 0$

$\therefore$ at all time

$$K \equiv 0 \quad \rightarrow$$

but on B_1

$\therefore$

and then

thus

that is

the solution is unique.

Theorem : Elastostatic Uniqueness Theorem

Let $S'(x_i)$ and $S''(x_i)$ be regular equilibrium states in $D + B$, and let both be solution to same problem S-III (mixed type problem), then $S'(x_i, t) = S''(x_i, t)$

proof let $S(x_i) =$

we will show $S(x_i) =$

again S is regular with the following boundary conditions

$$f_i =$$

$$u_i(x_i) =$$

$$T_i(x_i) =$$

$$\therefore 2\varepsilon =$$

$$=$$

$$\therefore \varepsilon =$$

since ε positive definite of e_{ij}

$$\therefore \varepsilon =$$

and u_i at most a rigid body displacement $u_i = \text{constant}$

$$\text{but } u_i =$$

$$\therefore u_i =$$

$$\text{therefore } S(x_i) =$$

$$\text{that is } S'(x_i) =$$

the solution is unique.

Definition : Kinematically Admissible State

$S(x_i)$ is kinematically admissible state if

- (a) u_i, e_{ij}, σ_{ij} are C_1 in $D + B$
- (b) S satisfies strain-displacement relation and constitutive equation
[equilibrium not necessary to satisfy]

Theorem : Principle of Minimum Potential Energy

Given

- (a) $D + B$ with C_{ijkl} , and e is positive definite
- (b) let $S(x_i)$ be regular state which is unique solution to S-III problem
corresponding to $C_{ijkl}, f_i, u_i^*, T_i^*$
- (c) let $\tilde{S}(x_i)$ be class of kinematically admissible state in $D + B$ corresponding to C_{ijkl} and satisfying $u_i = u_i^*$ on B_1 [u_i are arbitrary on B_2]
- (d) let $\Phi(\tilde{S})$ be the function

$$\Phi(\tilde{S}) =$$

where $\varepsilon =$

then $\Phi(\tilde{S})$ is an absolute minimum for $\tilde{S} = S$ [regular elastic state]

$\Phi(S)$ is called potential energy

i.e. $\Phi(S) =$ minimum when the displacement of the body are those of the equilibrium configuration

Other statement : of all displacements satisfying the given boundary conditions those which satisfy the equilibrium equations makes the potential energy an absolute minimum

proof let $S' =$

then S' is also a kinematically admissible state

recall for $e_{ij} = e_{ij}' + e_{ij}''$ and $\sigma_{ij} = \sigma_{ij}' + \sigma_{ij}''$

then $e(e_{ij}) =$

$$\therefore \varepsilon(\tilde{S}) =$$

from statement (d)

$$\Phi(\tilde{S}) =$$

and $\Phi(S) =$

$$\therefore \Phi(\tilde{S}) - \Phi(S) =$$

$$\begin{aligned} \text{now} \quad \int_D \sigma_{ij} e_{ij}' dV &= \\ &= \\ &= \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

where $u_i' = \tilde{u}_i - u_i$ are called virtual displacements

and $\int_D \sigma_{ij} e_{ij}' dV$ is called virtual work

then $\Phi(\tilde{S}) - \Phi(S) = \varepsilon(S') \geq 0$ [$\varepsilon(S')$ is positive definite]

$$\therefore \Phi(\tilde{S}) \geq \Phi(S) \leftarrow \text{absolute minimum}$$

equality only when $\varepsilon(S') = 0$

$$\Rightarrow \quad \tilde{e}_{ij} = e_{ij} \quad \tilde{\sigma}_{ij} = \sigma_{ij}$$

and $\tilde{u}_i = u_i + \text{constant}$ [at most a rigid body disp.]

Definition : Statically Admissible State

S is a statically admissible state in $D + B$, corresponding the C_{ijkl} and f_i if

(a) e_{ij}, σ_{ij} are C_1 in $D + B$

(b) e_{ij}, σ_{ij} satisfies equilibrium equation and constitutive equation in D ,
[strain-displacement relation not necessary to satisfy]

Theorem : Castigliano's Theorem

Given

(a) S is a regular elastic state in $D + B$, and is the unique solution to S-III problem corresponding to $C_{ijkl}, f_i, u_i^*, T_i^*$

(b) e is positive definite

(c) $\tilde{S}$ is the class of statically admissible state satisfying $\tilde{T}_i = T_i^*$ on B_2 ($\tilde{T}_i$ are arbitrary on B_1)

(d) let $\Psi(\tilde{S})$ be the function

$$\Psi(\tilde{S}) = \varepsilon(\tilde{S}) - \int_{B_1} \tilde{T}_i u_i^* dA$$

Ψ is called complimentary energy

then $\Psi(\tilde{S})$ is an absolute minimum for $\tilde{S} = S$, that is $\Psi(\tilde{S}) = \text{minimum}$ when the traction of the body those of the equilibrium state

proof let $S' =$

then S' is also a statically admissible state

recall $\varepsilon(\tilde{S}) =$

$$\therefore \Psi(\tilde{S}) =$$

$$\text{and } \Psi(S) =$$

$$\Psi(\tilde{S}) - \Psi(S) = \sim$$

$\therefore S$ and $\tilde{S}$ are corresponding to the same f_i

$$\therefore \sigma_{ij,j}' = 0 \quad [f_i = 0 \text{ for } S']$$

$$\text{then } \int_D \sigma_{ij}' e_{ij} dV =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$\therefore \Psi(\tilde{S}) - \Psi(S) =$$

$$\text{i.e. } \Psi(\tilde{S}) \geq \Psi(S) \leftarrow \text{absolute minimum}$$

$$\text{equality only when } \varepsilon(S') = 0$$

$$\Rightarrow \tilde{e}_{ij} = e_{ij} \quad \tilde{\sigma}_{ij} = \sigma_{ij}$$

$$\text{and } \tilde{u}_i = u_i + \text{constant} \quad [\text{at most a rigid body disp.}]$$

Field equation Potential Energy Thm. Complimentary Energy Thm

satisfied by $\tilde{S}$ strain-displacement
stress-strain equilibrium equation
stress-strain

not satisfied	equilibrium equation	strain-displacement
Boundary Condition		
satisfied by $\tilde{S}$	u_i^* on B_1 (variation of displacement)	T_i^* on B_2 (variation of stress)
not satisfied	u_i on B_2	T_i on B_1

To seek a general solution of the field equation of elasticity is not easy. Thus, one can construct a solution of Navier's equation and Beltrami-Mitchell equations, containing arbitrary harmonic functions that enter in particular combinations with certain known functions (by try). Another general solution of Navier's equation and Beltrami-Mitchell equations can be constructed with the aid of the biharmonic functions. The criterion of the generality of a given form of solution lies in the possibility of determining the arbitrary functions so that the boundary conditions are fulfilled.

Stress Function of Equilibrium in Homogeneous, Isotropic, Linear Elastic Medium

Potential Theory :

Example : suppose we want to solve

$$\nabla^2 \mathbf{u} =$$

i.e.

let $u_i =$

then

it satisfies the above equation for all $\phi \in C_3$

question : are any other solution that cannot be represent by $u_i = \phi_{,i}$

e.g. $u_a =$

then $u_1 = x_2, \quad u_2 = -x_1, \quad u_3 = 0$

clearly satisfies $u_{i,jj} = u_{j,ji}$

does there exist $\phi_{,i}$ such that $u_i = \phi_{,i}$?

let $\phi_{,1} = x_2, \quad \phi_{,2} = -x_1, \quad \phi_{,3} = 0$

then $\phi =$

$$\phi =$$

$$\phi =$$

from the first two equations, we get

$$2 x_1 x_2 =$$

it is impossible,

so $u_i = \phi_{,i}$ is not the most general solution for $u_{i,jj} = u_{j,ji}$

Definition : Irrotational vector field

The set of vectors $\{\mathbf{b}\} \in C_1$ in D with vanishing curl is called an irrotational vector field

$$\text{i.e.} \quad \text{curl } \mathbf{b} = 0 \quad \text{or} \quad \varepsilon_{ijk} b_{k,j} = 0$$

Definition : Solenoidal vector field

The set of vectors $\{\mathbf{b}\} \in C_1$ in D with vanishing div is called an solenoidal vector field

$$\text{i.e.} \quad \text{div } \mathbf{b} = 0 \quad \text{or} \quad b_{i,i} = 0$$

Theorem : Let b_i be irrotational in a simple-connected region D , then there exists a scalar ϕ , such that $\mathbf{b} = \nabla \phi$ or $b_i = \phi_{,i}$

proof recall Stokes Theorem

$\oint_{\Gamma} b_i dx_i$ is independent of path

$\therefore b_i dx_i$ is an exact differential, say $d\phi$

$$\text{i.e.} \quad b_i dx_i =$$

since dx_i is arbitrary, thus

Theorem : Let $\mathbf{B}$ be solenoidal, then there exists a vector $\mathbf{A}$, such that

$$\mathbf{B} = \text{curl } \mathbf{A} \quad \text{or} \quad B_i = \varepsilon_{ijk} A_{k,j}$$

Note : corresponding to a given $\mathbf{B}$, $\mathbf{A}$ is not unique

proof $\therefore$

$$\text{let } A_1 =$$

$$A_2 =$$

$$A_3 =$$

$$\text{then } A_{1,2} =$$

$$A_{2,1} =$$

$$A_{2,1} - A_{1,2} =$$

$$\therefore$$

$$\therefore$$

$$\text{and } \int_0^{x_3} B_{3,\xi}(x_1, x_2, \xi) d\xi =$$

$$=$$

$$=$$

$$\text{then } A_{2,1} - A_{1,2} =$$

$$=$$

$$\text{and } A_{1,3} =$$

$$=$$

$$A_{3,1} =$$

$$\therefore B_2 =$$

$$A_{2,3} =$$

$$A_{3,2} =$$

$$\therefore B_1 =$$

$$\text{then } B_i =$$

Theorem : Any vector G_i , continuous in $D + B$, class C_n in D , can be written

$$G_i = \varepsilon_{ijk} A_{k,j} + \phi_{,i}$$

$$\text{proof } \text{let } \phi \text{ satisfy } \phi_{,ii} = f$$

where f is a continuous function

and let $f =$

then

by the above theorem

$$B_{i,i} = 0 \quad \rightarrow \quad B_i =$$

$\therefore$

$$\text{i.e.} \quad G_i =$$

Theorem : Papkovitch-Neuber Stress Function

Let $\phi(x_i)$ and $\psi_i(x_i) \in C_3$ in D , and let

$$\phi_{,ii} = -x_i f_i / 2 (1 - \nu)$$

$$\psi_{i,jj} = f_i / 2 (1 - \nu)$$

$$[\text{Note : } \phi_{,ii} = -x_i \psi_{i,jj}]$$

with $f_i = 0$, $\phi(x_i)$ and $\psi_i(x_i)$ are 4 harmonic functions]

$$\text{then } u_i = \frac{1}{2\mu} [(\phi + x_j \psi_j)_{,i} - 4(1 - \nu) \psi_i]$$

satisfy the Navier's equation

$$\text{also } e_{ij} =$$

$$\text{and } \sigma_{ij} =$$

$$\text{proof } u_{i,kk} =$$

$$\phi_{,kk} =$$

$$\phi_{,ikk} =$$

$$\begin{aligned} (x_j \psi_j)_{,ikk} &= \\ &= \\ &= \end{aligned}$$

$$\psi_{i,kk} =$$

$$\begin{aligned} \therefore u_{i,kk} &= \\ &= \end{aligned} \quad (a)$$

$$\begin{aligned} u_{k,ki} &= \\ &= \\ &= \end{aligned}$$

$$\text{or} \quad (b)$$

(a) + (b) becomes

i.e. Navier's equation

$$\therefore e_{ij} =$$

$$\begin{aligned} u_{i,j} &= \\ &= \end{aligned}$$

$$u_{j,i} =$$

$$\therefore e_{ij} =$$

$$\text{and } \sigma_{ij} =$$

$$e_{kk} =$$

$$=$$

$$\therefore \sigma_{ij} =$$

$$\sigma_{ij} =$$

Navier's equation (in terms of Lamé's constants)

$$(\lambda + \mu) u_{j,ji} + \mu u_{i,jj} + f_i = \rho \ddot{u}_i$$

$$\text{let } u_i =$$

$$f_i =$$

then Navier's equation becomes

thus, we have

to solve the partial differential equations for ϕ and ψ_k , to form u_i
 where ϕ and ψ_k are called the scalar and vector potentials of u_i

for elastostatic problem with $f_i = 0$

we have

i.e.

to solve the 4 harmonic function to form the displacement components u_i

Theorem : Completeness Theorem

Let u_i be satisfy the Navier' s equation, then there exists $\phi(x_i)$, $\psi(x_i)$
 such that

$$\phi_{,ii} = -x_i f_i / 2 (1 - \nu)$$

$$\psi_{i,jj} = f_i / 2 (1 - \nu)$$

$$\text{and } u_i = \frac{1}{2\mu} [(\phi + x_j \psi_j)_{,i} - 4(1 - \nu) \psi_i]$$

proof : for a vector u_i , there exists $F(x_i)$, $A_i(x_i) \in C_3$, such that

(a)

substitute into Navier' s equation

then

or

(b)

define (c)

then $\psi_i \in C_2$

from (b) and (c), it is obtained

$$\psi_{i,jj} = \quad (d)$$

$$\text{from (c)} \quad \psi_{i,i} = \quad (e)$$

use the identity

$$\begin{aligned} (x_i \psi_i)_{,kk} &= \\ &= \end{aligned}$$

$$\text{i.e.} \quad 2 \psi_{k,k} =$$

$$(e) \rightarrow \quad = \quad (f)$$

from equations (d) and (f)

$$(g)$$

$$\text{define} \quad \phi = \quad (h)$$

from equations (g) and (h), we get

$$\phi_{,ii} = \quad (i)$$

from equations (d) and (i), it is obtained

$$\phi_{,ii} =$$

$$\psi_{i,jj} =$$

as the theorem prescribed

$$\text{also, from (h)} \quad F_{,i} = \quad (j)$$

from (c) and (j) to eliminate $F_{,i}$

$$\varepsilon_{ijk} A_{k,j} = \quad (k)$$

substitute (j), (k) into (a)

$$\begin{aligned} 2 \mu u_i &= \\ &= \\ &= \end{aligned}$$

Theorem : Galerkin Stress Function

Let $G_i(x_i)$ be C_4 in D , and satisfy

$$\nabla^4 G_i = G_{i,jjkk} = -f_i / (1 - \nu)$$

then
$$u_i = \frac{1}{2\mu} [2(1 - \nu) G_{i,jj} - G_{j,ji}]$$

satisfy the Navier's equation

proof
$$u_{i,kk} = \tag{a}$$

$$\begin{aligned} \frac{1}{1 - 2\nu} u_{k,ki} &= \\ &= \end{aligned} \tag{b}$$

$$(a) + (b)$$

$$\begin{aligned} u_{i,kk} + u_{k,ki} / (1 - 2\nu) &= \\ &= \end{aligned}$$

the Navier's equation is satisfy

Lamma : The dual equation

$$B_{i,jj} = 0 \tag{a}$$

$$B_{i,i} = \theta \tag{b} \quad \text{where } \theta \text{ is harmonic}$$

always have a solution

proof let $B_1 =$

$$B_2 =$$

$$B_3 =$$

then $B_{1,1} =$

and $B_{2,2} =$ $B_{3,3} =$

$\therefore$ $B_{i,i} =$ satisfy (b)

$\therefore$ B_i is harmonic $[B_{i,jj} = 0]$

$\therefore$ $B_{1,jj} =$

also θ is harmonic $\theta_{,22} + \theta_{,33} =$

$\therefore$

$$=$$

$$=$$

$$=$$

$$=$$

$\therefore$ $B_{1,jj} =$

thus $a_{1,jj} =$

a solution exists such that $B_{1,jj} = 0$

similarly for $B_{2,jj} = 0$ and $B_{3,jj} = 0$

Theorem : The Galarkin Stress Function is Complete

proof let $\phi_{,ii} = -x_i f_i / 2 (1 - \nu)$

$$\psi_{i,jj} = f_i / 2 (1 - \nu)$$

then

$$=$$

$$=$$

(a)

let

(b)

a solution of F_i exists

(a) and (b)

(c)

for which a general solution is

← harmonic

or

(d)

(b) →

and

(e)

for complete

$$2 \mu u_i =$$

$$=$$

$$=$$

(f)

define $G_i =$

with B_i satisfies $B_{i,jj} = 0$ and $B_{i,i} = \theta$

B_i exists → G_i exists

$$G_{i,jj} =$$

also

$$G_{j,ji} =$$

$$= \quad (g)$$

then (e) $G_{i,jjkk} =$

$$G_{i,jj} = \quad (h)$$

substitute (g) and (h) into (f)

$$2 \mu u_i =$$

u_i may be expressed by a bihamonic function G_i

Theorem : In the x_1, x_2 plane, the stress functions ϕ, ψ_i, G_i become

$$\phi = \phi(x_1, x_2)$$

$$\psi_a = \psi_a(x_1, x_2)$$

$$\psi_a = G_a(x_1, x_2) \quad a = 1, 2$$

$$\psi_3 = 0$$

provided $u_a = u_a(x_1, x_2)$

$$u_3 = f_3 = 0$$