

5. Problem Classification of Linear Theory of Elasticity

Fundamental System of Equations

1. Strain-Displacement Relations

(5.1)

2. Equations of Motion

(5.2)

Equations of Equilibrium

(5.2')

3. Constitutive Equation (Stress-Strain Relation)

or

(5.3)

for isotropic medium

or

(5.3')

there are 6 equations in (5.1)

there are 3 equations in (5.2)

there are 6 equations in (5.3) or (5.3')

15 equations solve for 15 unknown functions σ_{ij} , e_{ij} , and u_i

Boundary relation associated with (5.1) ~ (5.3)

4. Traction - Stress relation

$$\text{on } B \quad (5.4)$$

5. Surface Traction - Displacement

$$\text{on } B \quad (5.5)$$

for isotropic medium

$$\text{on } B \quad (5.5')$$

Equations Implied by Fundamental System

assume medium is homogeneous

$$C_{ijkl} = \text{constants} \quad [\text{independent of position}]$$

1. Stress-Displacement Relation

$$(5.6)$$

for isotropic medium

$$(5.6')$$

2. Displacement Equation of Motion

$$(5.7)$$

for isotropic medium

thus the equation of motion becomes

$$(5.7')$$

this is called Navier equation, for $\ddot{u}_i = 0$, it becomes equation of equilibrium

3. Rotation-Dilatation Equation of Motion

$$\therefore \omega_i = \frac{1}{2} \varepsilon_{ijk} u_{k,j} \quad \text{and} \quad e_{ii} = u_{i,i} = \theta$$

take curl of the rotation vector

$$\begin{aligned} 2 \varepsilon_{mni} \omega_{i,n} &= \\ &= \\ &= \end{aligned}$$

$$\therefore u_{i, jj} =$$

substitute into equation (5.7'), it becomes

$$(5.7'')$$

4. Strain Compatibility Condition

$$\text{set } k = l$$

this six equations are the only independent equations for 81 compatibility equations

5. Stress Compatibility equations

for isotropic, homogeneous, linear elastic medium

substitute into the strain compatibility equations

$$\begin{aligned}
 (1 + \nu) (\sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik}) \\
 &= \\
 &= \quad \quad \quad (a)
 \end{aligned}$$

equation of equilibrium

$$\sigma_{ik,k} = -f_i$$

$$\sigma_{ik,jk} = -f_{i,j}$$

$$\sigma_{jk,ik} = -f_{j,i}$$

then equation (a) becomes

set $i = j$

or

$$\sigma_{ii,kk} + f_{i,i} =$$

then $\sigma_{ii,kk} =$

thus the stress compatibility equation becomes

this is called Beltrami-Mitchell equation (stress compatibility equation)

The Fundamental Problems

1. Elastodynamic Problem

Given $D + B$, $\rho(x_i)$, $C_{ijkl}(x_i)$, $f_i(x_i, t)$ for $0 \leq t < \infty$

Find $u_i(x_i, t)$, $e_{ij}(x_i, t)$, $\sigma_{ij}(x_i, t)$ satisfying (5.1), (5.2), (5.3) in D and

$0 \leq t < \infty$, and subjected to

i) initial conditions

$$u_i(x_i, 0) = h_i(x_i) \quad \text{in } D$$

$$\dot{u}_i(x_i, 0) = g_i(x_i) \quad \text{in } D$$

ii) boundary conditions

problem I : surface displacement

$$u_i = u_i^*(x_i, t) \quad \text{on } B, 0 \leq t < \infty$$

problem II : surface traction

$$T_i = T_i^*(x_i, t) \quad \text{on } B, 0 \leq t < \infty$$

problem III : mixed problem

$$u_i = u_i^*(x_i, t) \quad \text{on } B_1, 0 \leq t < \infty$$

$$T_i = T_i^*(x_i, t) \quad \text{on } B_2, 0 \leq t < \infty$$

where $B = B_1 + B_2$, and B_1, B_2 consists of at most a finite number of subregion of the boundary

problem IV : mixed problem

$$\begin{array}{ccccccc}
 B & = & B_1 & + & B_2 & + & B_3 & + & B_4 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & u_i & & T_i & & u_i^n, T_i^t & & u_i^t, T_i^n \\
 & & t : \text{tangent} & & & & n : \text{normal} & &
 \end{array}$$

2. Elastostatic Problem

Given $D + B$, $C_{ijkl}(x_i)$, $f_i(x_i)$

Find $u_i(x_i)$, $e_{ij}(x_i)$, $\sigma_{ij}(x_i)$ such that (5.1), (5.2), (5.3) are satisfied in D
and subjected to boundary conditions

$$\begin{array}{lll}
 \text{I.} & u_i = u_i^*(x_i) & \text{on } B \\
 \text{II.} & T_i = T_i^*(x_i) & \text{on } B \\
 \text{III.} & u_i = u_i^*(x_i) & \text{on } B_1 \text{ and} \\
 & T_i = T_i^*(x_i) & \text{on } B_2
 \end{array}$$

Type I : Displacement Formulation

f_i is given

$$u_i = u_i^*(x_i, t) \quad \text{on } B, 0 \leq t < \infty$$

to find $u_i(x_i, t)$, $e_{ij}(x_i, t)$, $\sigma_{ij}(x_i, t)$ in D for $0 \leq t < \infty$ as functions of
time and position.

equation of motion

for isotropic, homogeneous, linear elastic medium

$$\sigma_{ij} =$$

$$\text{then } \rho \ddot{u}_i =$$

$$\text{but } e_{kk} = u_{k,k} \quad 2 e_{ij} = u_{i,j} + u_{j,i}$$

thus equation of motion becomes

$$\rho \ddot{u}_i =$$

this is called displacement equation of motion.

there are 3 scalar partial differential equations for 3 unknown functions u_1, u_2, u_3 . A solution of these equations satisfying the initial and boundary conditions yields the u_i for every point in D . (for elastostatic problem, $\ddot{u}_i = 0$, drop this term from the equations)

With u_i known, e_{ij} can be obtained by strain-displacement relations, and then σ_{ij} can be calculated from the constitutive equations.

$$\text{i.e.} \quad u_i \rightarrow e_{ij} \rightarrow \sigma_{ij}$$

Type II : Stress formulation

f_i is given

$$T_i = T_i^*(x_i, t) \quad \text{on } B, 0 \leq t < \infty$$

to find $\sigma_{ij}(x_i, t), e_{ij}(x_i, t), u_i(x_i, t)$ in D for $0 \leq t < \infty$ as functions of time and position.

it is more complex in the elastodynamic case, now consider the elastostatic case below

equation of equilibrium

3 equations for 6 unknown functions σ_{ij}

a set of stresses satisfying the equation of equilibrium may not be correct one unless the compatibility condition are also satisfied.

the Beltrami-Mitchell equation of compatibility

a set of stresses satisfying the above two sets of equations and the boundary conditions are the solutions of the given problem

with σ_{ij} known, e_{ij} can be obtained by constitutive equations, and the displacement u_i can be computed by strain-displacement relations

$$\text{i.e.} \quad \sigma_{ij} \rightarrow e_{ij} \rightarrow u_i$$

9 equations to solve for 6 unknowns, is the solution unique ?

(we will present a uniqueness theorem which state that there is only one solution satisfying a given set of boundary condition for a region of finite extent)

for elastodynamic problem, Beltrami-Michell equation can not be used, because the equilibrium is needed to derive these equation

Type III : Mixed Formulation

f_i is given

$$u_i = u_i^*(x_i, t) \quad \text{on } B_1, 0 \leq t < \infty$$

$$T_i = T_i^*(x_i, t) \quad \text{on } B_2, 0 \leq t < \infty$$

$$B = B_1 + B_2$$

to find $\sigma_{ij}(x_i, t)$, $e_{ij}(x_i, t)$, $u_i(x_i, t)$ in D for $0 \leq t < \infty$ as functions of time and position.

to solve this type of problem, we must use

equation of motion 3 equations

strain-displacement relation 6 equations

constitutive equation 6 equations

(or use compatibility condition in terms of stress or strain instead of strain-displacement relation)

15 equations to solve for 15 unknown functions to satisfy the initial and boundary conditions are the solutions of this problem

no general short-cut method of solution is available for this type of problem

Principle of superposition

since the governing equations in linear elasticity are linear equations, the solution of these equations may be superimposed to yield a new solution

Can a mixed problem solved by superimposed of a traction B.V.P. and a displacement B.V.P.

the answer is “ ”