

3. Analysis of Stress

Definition : Body Force Density (specific body force)

consider a point $P(x_i) \in D$, and a sequence of closed subregions D_j with boundary B_j , ($j = 1, 2, \dots$). Let V_j be volume of D_j , δ_j be maximum diameter of D_j , $P(x_i) \in D_j$. Let $\mathbf{R}_j$ is the resultant body force on the material occupying D_j . Let $\delta_j \rightarrow 0$, thus the body force density $\mathbf{f}(x_i)$ is

$$\mathbf{f}(x_i) = \lim (\mathbf{R}_j / V_j) \quad [j \text{ not sum}]$$

resultant force / unit volume

Definition : Traction $\mathbf{T}$

consider a closed subregion D^* of D with boundary B^* . Let $x_i \in B^*$, $\mathbf{n}$ be the normal unit vector to B^* at x_i . Let Σ_j ($j = 1, 2, \dots$) be a sequence of surface elements on B^* . Such that $x_i \in \Sigma_j$ for all j . Let A_j be the surface area of Σ_j , δ_j be maximum diameter of Σ_j , $\delta_j \rightarrow 0$ as $j \rightarrow \infty$. Let $\mathbf{P}_j$ be the resultant surface force on Σ_j , the traction $\mathbf{T}$ at x_i on the plane with outer normal is

$$\mathbf{T}(\mathbf{x}, \mathbf{n}) = \lim_{j \rightarrow \infty} (\mathbf{P}_j / A_j) \quad [j \text{ not sum}]$$

stress vector force / unit area

$$\therefore \quad \mathbf{f} =$$

$$\mathbf{T}(\mathbf{x}, \mathbf{n}) =$$

Definition : Normal and Shear Components of Traction

a. scalar normal component

$$N =$$

b. vector normal component

$$\mathbf{N} =$$

c. vector shear component (tangent to the plane)

$$\mathbf{S} = \quad S_i =$$

4. scalar shear component

$$S =$$

if $N > 0 \rightarrow$ tension

$N < 0 \rightarrow$ compression

Definition : Cartesian Components of stress σ_{ij}

$$\sigma_{ij} =$$

where $\mathbf{T}_i(\mathbf{x}, \mathbf{e}_i)$ is the traction action on the plane $\perp \mathbf{e}_i$ at x_i

$$\therefore \quad \sigma_{11} =$$

$$\sigma_{12} =$$

$$\sigma_{13} =$$

similarly for σ_{21} etc.

in general, σ_{ij} is the j^{th} component of the traction vector associated with the direction $\mathbf{e}_i$.

Theorem : Given $T_i(\mathbf{x}, \mathbf{n})$, the T_i is completely characterized by σ_{ij} at x_i , moreover

$$T_i(\mathbf{x}, \mathbf{n}) =$$

consider an infinitesimal
tetrahedron $PABC$

$$dS = \triangle ABC$$

$$dS_i = \text{area of the surface} \perp \mathbf{e}_i \text{ direction}$$

let $\mathbf{n}$ be the unit normal of dS , $\mathbf{T}(\mathbf{n})$ be surface traction on dS , and $\mathbf{f}$ be body force per unit volume

$$\therefore dS_1 =$$

$$\therefore dS_i =$$

let T_1, T_2, T_3 be the average value of the surface traction per unit area on the areas dS_1, dS_2, dS_3 respectively

if we reduce the dimensions of the tetrahedron, the resultant of the body forces tends more rapidly to zero than that of the surface traction, the equilibrium of the tetrahedron requires that

$$\mathbf{T}^{(n)} dS =$$

$$\text{i.e. } \mathbf{T}^{(n)} dS =$$

$$\therefore \sigma_{1i}, \sigma_{2i}, \sigma_{3i} \text{ be components of vector } \mathbf{T}_i$$

$$\text{i.e. } T_1 = \quad \text{or} \quad T_i =$$

$$\text{then } T^{(n)} =$$

$$=$$

$$=$$

$$=$$

$$\therefore T_i^{(n)} =$$

the normal stress on the surface element with the unit vector n is given by

$$| \mathbf{N} | =$$

$$=$$

Theorem : Equilibrium

Given D (region), B (boundary), f_i (body force) continuous, and $\sigma_{ij} \in C_1$,
then necessary and sufficient condition for equilibrium are

$$\sigma_{ji,j} + f_i = 0$$

$$\text{and } \sigma_{ij} = \sigma_{ji} \quad \text{for no body moment}$$

proof : force balance of D

$$\therefore T_i^{(n)} =$$

$$\therefore$$

i.e.

these volume integral with continuous integrands must vanish for any region D , thus

$$\sigma_{ji,j} + f_i = 0 \quad [\text{equation of equilibrium}]$$

moment balance about O

where m_i is the body moment per unit volume

$$\int_B \varepsilon_{ijk} x_j T_k^{(n)} dS =$$

$$=$$

$\therefore$

for arbitrary dV $\varepsilon_{ijk} \sigma_{jk} + m_i = 0$

if no body moment acting on the body, then $m_i = 0$

i.e. $\varepsilon_{ijk} \sigma_{jk} = 0 \implies \sigma$ is a symmetric tensor

i.e. $\sigma_{ij} = \sigma_{ji}$

the shearing components of the stress tensor are symmetric, the stress field is specified by six rather than nine components

Theorem : Let the surface element dS and dS' with unit normal $\mathbf{n}$ and $\mathbf{n}'$ pass through P , then the component of stress vector $\mathbf{T}^{(n)}$ in the direction of $\mathbf{n}'$ is equal to the component of stress vector $\mathbf{T}^{(n')}$ in the direction of $\mathbf{n}$

$$\text{i.e.} \quad T_i^{(n)} n_i' =$$

$$\begin{aligned} \therefore T_i^{(n)} n_i' &= \\ &= \\ &= \\ &= \end{aligned}$$

Theorem : The Cartesian components of stress σ_{ij} are in fact of a components of second order tensor

consider a stress component in x_i'
system $\sigma'_{\alpha\beta}$

$$\begin{aligned} \text{then } \sigma'_{\alpha\beta} &= T_{\beta}^{(\alpha)} \quad [\text{acting on the plane } \perp x_{\alpha}' \text{ in the direction of} \\ &\quad x_{\beta}' \text{ axis}] \\ &= \end{aligned}$$

where $\mathbf{n}'$ is the unit vector along x_{α}' axis
 $\mathbf{n}$ is the unit vector along x_{β}' axis

$$n_i' = \lambda_{\alpha i}$$

$$n_j = \lambda_{\beta j}$$

$$\begin{aligned} \therefore T_j^{(n')} &= \\ \therefore \sigma'_{\alpha\beta} &= \end{aligned}$$

then σ_{ij} is a second order tensor

Stress quadric of Cauchy

the normal stress on the surface element

with unit normal $\mathbf{n}$ is

$$\sigma_N =$$

let point P be chosen along the outer normal $\mathbf{n}$

multiple n_i by the length OP , then

$$x_i =$$

$$\therefore \sigma_N |\mathbf{OP}|^2 =$$

$$=$$

consider the quadratic surface

$$2f(x_1, x_2, x_3) =$$

$$\text{then } |\mathbf{OP}|^2 = \quad \text{or} \quad \sigma_N =$$

$\therefore$ the normal stress at the point O is inversely proportional to the square of the radius vector along the line from O to $P(x_i)$ on the quadric

$$2 \frac{f}{x_k} =$$

$$=$$

$$=$$

$$=$$

since f / x_k is the normal vector of the surface at point P , the T_k^n has the same direction of the normal vector

if the magnitude of N is known, it can readily determine the length of the vector $\mathbf{T}^{(n)}$

if $\mathbf{n}$ is taken along the axes of quadric, then $\mathbf{n}$ has the same direction as the normal vector at P , therefore $\mathbf{T}^{(n)}$ and $\mathbf{n}$ coincide in direction, i.e.

$$T_i^{(n)} =$$

$$\sigma_{ij} n_j =$$

σ and n_j are the eigenvalues and eigenvectors of the second order tensor σ_{ij} are called principal stresses and principal direction of stress.

for nontrivial solution

$$| \sigma_{ij} - \sigma \delta_{ij} | = 0$$

$$\text{i.e.} \quad \sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

$$\text{where} \quad I_1 =$$

$$I_2 =$$

$$I_3 =$$

I_1, I_2 and I_3 are the stress invariants

The principal stresses are all real ($\because \sigma_{ij} = \sigma_{ji}$ is symmetric)

If σ_I, σ_{II} , and σ_{III} are all different, then the principal direction all orthogonal to each other.

If $\sigma_I = \sigma_{II} = \sigma_{III}$, an arbitrary rotation of the considered system about the principal axis corresponding to σ_I furnishes another system of principal axes.

If $\sigma_I = \sigma_{II} = \sigma_{III}$, any set of orthogonal axes can be regarded as a system of principal axes.

If the reference axes x_1, x_2, x_3 are chosen to coincide with the principal axes, then the matrix of the stress components becomes

$$\sigma_{ij} = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{pmatrix}$$

let x_1, x_2, x_3 be chosen as the principal axes, the Cauchy quadric surface takes the forms

for $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$, the equation represents

1. an ellipsoid if σ 's are all the same sign and all distinct

$$\sigma_I > \sigma_{II} > \sigma_{III} \quad \text{take } +k_0^2$$

$$\sigma_I < \sigma_{II} < \sigma_{III} \quad \text{take } -k_0^2$$

2. an ellipsoid of revolution if σ 's are all the same sign and either

$$\sigma_I = \sigma_{II} \quad \text{or} \quad \sigma_{II} = \sigma_{III} \quad \text{or} \quad \sigma_I = \sigma_{III}$$

3. a sphere if $\sigma_I = \sigma_{II} = \sigma_{III}$

4. an unparted hyperboloid if

$$\sigma_I \geq \sigma_{II} = 0 \quad \sigma_{III} < 0 \quad \text{take } +k_0^2$$

5. a biparted hyperboloid if

$$\sigma_I \geq \sigma_{II} = 0 \quad \sigma_{III} < 0 \quad \text{take } -k_0^2$$

the equation of (4) and (5) is

σ_N depends on the orientation of the surface, element at $P_0(x_i^0)$

$$\therefore \quad \sigma_{ij} x_i x_j =$$

$$\mathbf{P}_0 \mathbf{P} =$$

if $\mathbf{n}$ cut (4), then $\sigma_N =$ tension

if $\mathbf{n}$ cut (5), then $\sigma_N =$ compression

if $\mathbf{n}$ along the asymptotic cone, $k_0 = 0$, then $\sigma_N A^2 = 0$, i.e.

$$\sigma_N = 0 \quad \text{no normal stress, shear stress only}$$

Theorem : Mohr' s Circle

Given a state of stress at x_i , with $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$, then all point $Q(N, S)$ is the intersection of the sets C_1 , C_2 and C_3 , where

$$C_1 : \quad S^2 + [N - (\sigma_{II} + \sigma_{III}) / 2]^2 \geq [(\sigma_{II} - \sigma_{III}) / 2]^2$$

$$C_2 : \quad S^2 + [N - (\sigma_{III} + \sigma_I) / 2]^2 \leq [(\sigma_{III} - \sigma_I) / 2]^2$$

$$C_3 : \quad S^2 + [N - (\sigma_I + \sigma_{II}) / 2]^2 \geq [(\sigma_I - \sigma_{II}) / 2]^2$$

i.e.

proof Let x_1, x_2, x_3 be principal stress axes

$$\mathbf{T} =$$

$$T_i =$$

$$\therefore \quad \mathbf{T} =$$

$$\mathbf{T} \cdot \mathbf{T} =$$

$$\mathbf{T} \cdot \mathbf{n} =$$

$$\text{and} \quad n_1^2 + n_2^2 + n_3^2 = 1$$

case 1, $\sigma_I > \sigma_{II} > \sigma_{III}$, solve equation (a) for n_1^2, n_2^2, n_3^2

$$n_1^2 = \underline{\hspace{2cm}}$$

$$n_2^2 = \underline{\hspace{2cm}} \quad (b)$$

$$n_3^2 = \underline{\hspace{2cm}}$$

$$\because n_1^2 \geq 0 \quad \therefore$$

this is equal to

[C₁]

similarly for $n_2^2 \geq 0$ and $n_3^2 \geq 0$

all points lie within this region

case 2, $\sigma_I = \sigma_{II} > \sigma_{III}$, all points lie on the circle

case 3, $\sigma_I = \sigma_{II} = \sigma_{III}$, equation (a) becomes

the Mohr's circle reduces to
a point in S-N plane

The maximum shear stress is

occurs on the plane given by

$$n_1 = \quad n_2 = \quad n_3 =$$

normal stress on this plane is

$$N =$$

Definition : Pure Shear

σ_{ij} is a state of pure shear if there exists at least one coordinate system such that $\sigma_{ij} = 0$ for $i = j$

Theorem : A necessary and sufficient condition that σ_{ij} is a state of pure shear is that $I_1 = \sigma_{ii} = 0$

proof necessary : if σ_{ij} is a pure shear, then

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0 \quad \rightarrow \quad I_1 = 0$$

sufficiency : assume $I_1 = \sigma_I + \sigma_{II} + \sigma_{III} = 0$

let $Ox_1x_2x_3$ be principal stress axes, and assume no eigenvalues vanishes without loss in generality, say $\sigma_I > 0$, $\sigma_{II} > 0$, and $\sigma_{III} < 0$,

$$\therefore \quad \sigma'_{\alpha\beta} =$$

$$\therefore \quad \sigma'_{11} =$$

$$\sigma'_{22} =$$

$$\sigma'_{33} =$$

we want to find $\lambda_{\alpha i}$ so that $\sigma'_{ij} = 0$ for $i \neq j$

for rotational propose, let $e_1' = n^1$ $e_2' = n^2$ $e_3' = n^3$

$$\text{i.e.} \quad \lambda_{ij} =$$

$$\text{so that} \quad \sigma'_{11} =$$

$$\sigma'_{22} =$$

$$\sigma'_{33} =$$

$$\text{now, we have} \quad \sigma'_{11} = \sigma'_{22} = \sigma'_{33} = 0 \quad 3 \text{ eq.}$$

$$|n^i| = 1 \quad 3 \text{ eq.}$$

$$n^i \cdot n^j = \delta_{ij} \quad 3 \text{ eq.}$$

9 equations to solve for n_j^i and show that n_j^i exists

kept x_1 fixed, rotate x_2 - x_3 axes to x_2'' and x_3'' axes with an angle θ'' , such that $\sigma''_{33} = 0$, and then $\sigma''_{22} = -\sigma_I$

$$\text{i.e.} \quad \lambda_{ij}'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta'' & \sin \theta'' \\ 0 & -\sin \theta'' & \cos \theta'' \end{pmatrix}$$

$$\text{then} \quad \sigma''_{ij} = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma''_{22} & \sigma''_{23} \\ 0 & \sigma''_{32} & 0 \end{pmatrix}$$

and then kept x_3'' fixed, rotate x_1'' , x_2'' to x_1' , x_2' with an angle θ' (45°) such that

$$\cos \theta' \quad \sin \theta' \quad 0$$

$$\lambda'_{ij} = \begin{pmatrix} -\sin \theta' & \cos \theta' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the stress tensor becomes

$$\text{then } \sigma'_{ij} = \begin{pmatrix} 0 & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{12} & 0 & \sigma'_{23} \\ \sigma'_{13} & \sigma'_{23} & 0 \end{pmatrix} = \text{pure shear state}$$

Definition : Hydrostatic state of stress

A hydrostatic (isotropic) state of stress at x_i is that $\mathbf{T}(x_i, n_i) = \sigma(x_i) \mathbf{n}$ for all $\mathbf{n}$

this implies $\sigma_{ij} n_j =$
i.e.

$$\text{or } \sigma_{ij} = \sigma \delta_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

Theorem : Given an arbitrary $\mathbf{T}(x_i, n_i)$, then $\mathbf{T}$ can be uniquely decomposed into a pure shear state plus a hydrostatic state of stress at x_i

i.e. $\sigma_{ij} =$

where $\sigma'_{ij} =$ is hydrostatic state

$\tilde{\sigma}_{ij} =$ is deviatoric state

and $\tilde{\sigma}_{ii} =$ is pure shear state

for uniqueness, suppose there exists another decomposition σ'^*_{ij} and $\tilde{\sigma}^*_{ij}$

$$\begin{aligned}
 \text{where} \quad \sigma'_{ij} &= \\
 \tilde{\sigma}_{ij} &= \\
 \therefore \quad \tilde{\sigma}_{ii} &= \\
 \therefore \quad \sigma^* &=
 \end{aligned}$$

$$\text{and} \quad \sigma'_{ij} = \quad \text{and then} \quad \tilde{\sigma}_{ij}^* = \tilde{\sigma}_{ij}$$

Definition : Plane stress state

σ_{ij} is a plane state of stress at x_i parallel to plane Λ , if $T^{(n)}(x_i) = 0$ for all $\mathbf{n} \perp \Lambda$

suppose $\mathbf{e}_3 \perp \Lambda$, then $T(x_3, \mathbf{e}_3) = 0$

$$\text{i.e.} \quad \sigma_{i3} = 0 \quad [\because T_i = \sigma_{ji} n_j, \quad n_j = (0, 0, 1), \quad \therefore T_3 = \sigma_{i3} = 0]$$

$$\begin{aligned}
 \sigma_{ij} &= \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

Theorem : (a) A necessary and sufficient condition for plane stress is that $I_3 = 0$, (b) if $T(x_i, n_i)$ is a plane state of stress parallel to Λ , then $T(x_i, n_i)$ is parallel to Λ for all $\mathbf{n}$

(a) necessary : assume the plane state stress hold, for some coordinate system, then $I_3 = \det(\sigma_{ij}) = 0$

(b) sufficiency : assume $I_3 = 0$, refer to principal axes

$$\sigma'_{ij} = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{pmatrix}$$

$\therefore I_3 = \sigma_I \sigma_{II} \sigma_{III} = 0$, at least one of the principal stresses vanishes, say $\sigma_{III} = 0$, then σ'_{ij} is of the form

$$\sigma'_{ij} = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

kept x_3 axis fixed, rotate $x_1 x_2$ axes, to get

$$\sigma'_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{is a plane stress}$$

(b) choose $x_1 x_2$ plane parallel to Λ , $x_3 \perp \Lambda$, then

$$\mathbf{T}(x_i, n_i) \cdot \mathbf{e}_3 =$$

$\therefore$

$\therefore$

$$\text{i.e. } \mathbf{T}(x_i, n_i) \perp \mathbf{e}_3$$

Theorem: Given a plane state of stress parallel to Λ , then the principal stresses

are 0, $[I_1 \pm \sqrt{I_1^2 - 4I_2}] / 2$

proof : $\therefore$ the principal stresses are the roots of the characteristic equation

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

$\therefore$

$\therefore$

thus we have $\sigma = 0$ and $\sigma =$