

1. Mathematical Preliminaries

Vector Notation

Let x_1, x_2, x_3 be a right handed coordinate system, and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors along the x_1, x_2, x_3 axes, respectively

vector from A to B is written $\mathbf{AB}$

in component form

$$\mathbf{V} = \mathbf{AB} =$$

the magnitude of $\mathbf{V}$ is written as $|\mathbf{V}|$

$$|\mathbf{V}| =$$

scalar (dot) product of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} =$$

vector (cross) product of $\mathbf{a}$ and $\mathbf{b}$ is

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \end{aligned}$$

scalar triple product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

vector triple product

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} =$$

gradient of a function

$$\text{grad} (u) = \nabla u =$$

divergence of a vector

$$\text{div} (\mathbf{v}) = \nabla \cdot \mathbf{v} =$$

curl of a vector

$$\begin{aligned} \text{curl} (\mathbf{v}) &= \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \end{aligned}$$

Laplacian of a function

$$\nabla^2 \varphi = \nabla \cdot (\nabla \varphi) =$$

Identities :

$$\nabla (u v) =$$

$$\nabla \cdot (u \mathbf{v}) =$$

$$\nabla \times (u \mathbf{v}) =$$

$$\nabla^2 (u \mathbf{v}) =$$

$$\text{div} (\text{curl } \mathbf{v}) =$$

$$\text{curl} (\text{grad } u) =$$

$$\text{curl} (\text{curl } \mathbf{v}) =$$

Index Notation

The coordinates x_1, x_2, x_3 may be expressed, in general, by x_i , where i is understood to take on value 1, 2 and 3

Instead of writing $\mathbf{v}$ in component form, just write v_i as i^{th} ($i = 1, 2, 3$) component of the vector $\mathbf{v}$

Definition : Summation Convention

In any single term, wherever a subscript is repeated, this indicates summation over this subscript from 1 to 3

e.g. $\mathbf{v} =$

$$|\mathbf{a}| =$$

$$\mathbf{a} \cdot \mathbf{b} =$$

$$|\mathbf{a}| |\mathbf{b}| =$$

Note : no subscript may be repeated more than once in a single term, whenever a subscript is repeated, it is a dummy subscript

e.g. $a_i a_i = a_j a_j$

Definition : Differential Convention

Subscripts attached to a function $f(x_i)$ be preceded by comma indicate differentiation w. r. t. the variables

e.g. $f_{,i} =$

$$v_{i,jk} =$$

$$v_{i,3} =$$

$$v_{i,jj} =$$

$$=$$

Definition : Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

i.e. $\delta_{11} = \delta_{22} = \delta_{33} = 1$

$$\delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0$$

$$\delta_{ii} =$$

$$\delta_{ij} v_j =$$

$$a_{ijk} \delta_{ip} =$$

(this is called contraction)

$$\delta_{ij} \delta_{ij} =$$

$$\delta_{ij} \delta_{jk} =$$

$$=$$

$$=$$

$$\therefore \delta_{ij} \delta_{jk} =$$

and $x_{i,j} =$

$$\delta_{ij} \delta_{jk} \delta_{ki} =$$

Definition : Permutation Symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if the value of } i, j, k \text{ from an even permutation of } 1, 2, 3 \\ -1 & \text{if the value of } i, j, k \text{ from an odd permutation of } 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

i.e. $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

$$\text{others} = 0$$

Identities $\varepsilon_{ijk} \varepsilon_{ijk} = 6$

$$\varepsilon_{ijk} \varepsilon_{ipq} =$$

(this is called $\varepsilon - \delta$ identity)

The vector notation can be expressed in index notation

$$\mathbf{a} \cdot \mathbf{b} =$$

$$\mathbf{a} \times \mathbf{b} =$$

$$\mathbf{e}_i \cdot \mathbf{e}_j =$$

$$\mathbf{e}_j \times \mathbf{e}_k =$$

$$\nabla u =$$

$$\nabla \cdot \mathbf{v} =$$

$$\nabla \times \mathbf{v} =$$

$$\nabla^2 u =$$

$$\nabla^2 \mathbf{v} =$$

use index notation to prove the vector identities

$$\text{e.g. } (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$\text{div} (\text{curl } \mathbf{v}) =$$

$$=$$

$$=$$

$$\therefore \varepsilon_{ijk} v_{k,ij} =$$

Determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$=$$

$$=$$

$$\begin{array}{llll} \text{let} & a_1 = a_{11} & a_2 = a_{12} & a_3 = a_{13} & a_i = a_{1i} \\ & b_1 = a_{21} & b_2 = a_{22} & b_3 = a_{23} & \text{i.e. } b_i = a_{2i} \\ & c_1 = a_{31} & c_2 = a_{32} & c_3 = a_{33} & c_i = a_{3i} \end{array}$$

$$\begin{array}{l} \text{then} \\ \det(a_{ij}) = \end{array}
 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

consider the sum

$$\varepsilon_{ijk} a_{ir} a_{js} a_{kp} =$$

$$=$$

the sum is skew symmetric w. r. t. the index r and s , similarly it is skew symmetric w. r. t. the index r and p , and also to the index s and p

this is complete skew symmetric in the index r, s , and p

for $r, s, p = 1, 2, 3$

$$\varepsilon_{ijk} a_{ir} a_{js} a_{kp} =$$

for $r, s, p = 2, 1, 3$

$$\varepsilon_{ijk} a_{ir} a_{js} a_{kp} =$$

=

=

 $\det(a_{ij})$ when r, s, t are even permutation of 1, 2, 3

 $\therefore \varepsilon_{ijk} a_{ir} a_{js} a_{kp} = -\det(a_{ij})$ when r, s, t are odd permutation of 1, 2, 3

 0 when any two of r, s, t have same value

in general $\varepsilon_{rsp} \det(a_{ij}) =$
 $=$

Coordinate Rotation

let $\mathbf{e}_i, x_i$ be original coordinate

$\mathbf{e}'_i, x'_i$ be new coordinate

Definition : Direction cosines

The direction cosines corresponding to a rotation of a right-handed coordinate system x_i into the right-handed coordinate system x'_α are denoted by $\lambda_{\alpha i}$, i.e.

$$\lambda_{\alpha i} =$$

$$=$$

if we know the coordinate in one system, what are they in other

$$\mathbf{x} =$$

dot with vector $\mathbf{e}_j$ in both side

$$x_{\alpha}' \mathbf{e}_{\alpha}' \cdot \mathbf{e}_j =$$

$$\lambda_{\alpha j} x_{\alpha}' =$$

similarly $x_{\alpha}' =$

The direction cosines are orthonormal

i.e. $\lambda_{\alpha i} \lambda_{\alpha j} =$

or $\lambda_{i\alpha} \lambda_{j\alpha} =$

proof : $\therefore x_j =$

$$\delta_{ij} x_i =$$

Transformation Law for unit vector

$$\mathbf{x} =$$

for arbitrary $x_{\alpha}' \therefore \mathbf{e}_{\alpha}' =$

similarly $\mathbf{e}_i =$

we now shown that $\det (\lambda_{ij}) = 1$

scalar triple product

$$(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = 1$$

$$(\lambda_{\alpha 1} \mathbf{e}_{\alpha'} \times \lambda_{\beta 2} \mathbf{e}_{\beta'}) \cdot \lambda_{\gamma 3} \mathbf{e}_{\gamma'} = 1$$

$$\text{L.H.S.} =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$\therefore \det(\lambda_{ij}) =$$

Definition : Cartesian Tensor

The class of 3^N elements $\{a_{ij} \dots_{pq}\}$ associated with $Ox_1x_2x_3$ is called a Cartesian tensor of rank (or order) N , if under rotation, the elements transform according to the following :

$$a'_{\alpha \beta \dots \gamma \delta} =$$

$$\text{e.g.} \quad N = 1 \quad a'_{\alpha} =$$

$$N = 2 \quad a'_{\alpha \beta} =$$

Invariants : the following three functions of a tensor of rank 2 are invariants under rotation

$$\text{for a 2}^{\text{nd}} \text{ order tensor } a_{ij} \quad A_1 =$$

$$A_2 =$$

$$A_3 =$$

$$\text{proof : (a) to show } a_{ii} = a'_{\alpha \alpha}$$

$$\therefore a'_{\alpha \beta} =$$

set $\alpha = \beta$ then

$$a'_{\alpha\alpha} =$$

(b) to show $a'_{\alpha\beta} a'_{\beta\alpha} = a_{ij} a_{ji}$

$$\begin{aligned} a'_{\alpha\beta} a'_{\beta\alpha} &= \\ &= \\ &= \\ &= a_{ij} a_{ji} \\ \therefore A_2 &= \end{aligned}$$

(c) $\varepsilon_{\mu\nu\delta} \det(a'_{\alpha\beta}) =$

$$\begin{aligned} &= \\ &= \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

set $\mu, \nu, \delta = 1, 2, 3$

then $\det(a'_{\alpha\beta}) = \det(a_{ij})$

Definition : Symmetric and skew-symmetric tensors

1. a tensor $a_{ij\cdots k\cdots l\cdots pq}$ is symmetric if

$$a_{ij\cdots k\cdots l\cdots pq} = a_{ij\cdots l\cdots k\cdots pq} \quad \text{for each choice of } k \text{ and } l$$

2. a tensor $a_{ij\cdots k\cdots l\cdots pq}$ is skew-symmetric if

$$a_{ij\cdots k\cdots l\cdots pq} = -a_{ij\cdots l\cdots k\cdots pq} \quad \text{for each choice of } k \text{ and } l$$

symmetric and skew-symmetric are not affected by rotation

Decomposition : any tensor of rank 2 may be written as the sum of a symmetric and skew-symmetric tensors

$$a_{ij} =$$

Isotropic : a tensor is said to be isotropic if its components do not change under rotation

zero order : all scalars are, of course, isotropic $a' = a$

1st order :

for arbitrary a_j ,

this is identical transformation (no rotation)

if

i.e. no isotropic tensor of order 1

2nd order : an isotropic 2nd tensor may be reduced to the form

$$a_{ij} = a \delta_{ij}$$

check $a'_{ij} =$

3rd order : every isotropic tensor of rank 3 may be reduced to the form

$$a_{ijk} = a \varepsilon_{ijk}$$

check $a'_{ijk} =$

=

4th order : if a_{ijkl} is an isotropic tensor of order 4, then it is of the form

$$a_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \nu (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$$

where λ, μ, ν are constants

check $a'_{ijkl} =$

$$=$$

$$=$$

$$=$$

$$=$$

If a_{ijkl} has the symmetric properties as $a_{ijkl} = a_{jikl} = a_{ijlk}$, then

$$a_{ijkl} =$$

for a 4th isotropic tensor

$$a_{ijkl} =$$

if $a_{ijkl} = a_{jikl}$ interchange i and j

$a_{ijkl} = a_{ijlk}$ interchange k and l

first two terms remain the same, and the third term becomes

$$\nu (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) =$$

$$2 \nu (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) =$$

for $(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) = 0 \quad \rightarrow \quad \nu = 0$

$\therefore a_{ijkl} =$

Definition : Outer and inner product of tensors

outer product : $a_{ij} b_{kl}$ 4th order tensor

inner product : $a_{ij} b_{jk}$ 2nd order tensor

Theorem : A symmetric [skew-symmetric] tensor of rank 2 is symmetric [skew-symmetric] under rotation

let a_{ij} be symmetric [skew-symmetric] tensor

$$\begin{aligned} a_{ij} &= \\ a'_{\alpha\beta} &= \\ &= \\ &= \end{aligned}$$

similarly for skew-symmetric tensor

Definition : Eigenvalues and Eigenvectors of 2nd order tensor

The solutions $A, \mathbf{n}$ (unit vector) of the equation

$$a_{ij} n_j =$$

are called the eigenvalues and eigenvectors of the tensor a_{ij} , respectively

the solution of the equation

$$(a_{ij} - A \delta_{ij}) n_j = 0$$

for nontrivial solution, we must have $\det (a_{ij} - A \delta_{ij}) = 0$

$$\text{i.e.} \quad \begin{vmatrix} a_{11} - A & a_{12} & a_{13} \\ a_{21} & a_{22} - A & a_{23} \\ a_{31} & a_{32} & a_{33} - A \end{vmatrix} = 0$$

or

where A_i are invariants of a_{ij}

Theorem : If $a_{ij} = a_{ji}$, then the three eigenvalues A_I, A_{II}, A_{III} are real

proof : A cubic equation must at least one real root, say A_I . Let the corresponding eigenvector be $\mathbf{n}^I$, rotate the coordinate system, so that $\mathbf{n}^I = \mathbf{e}_1'$

then

$\therefore$

$$\therefore \quad a'_{ij} = \begin{vmatrix} A_I & 0 & 0 \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{23} & a'_{33} \end{vmatrix}$$

$\therefore$

$$\begin{vmatrix} A_I - A & 0 & 0 \\ 0 & a'_{22} - A & a'_{23} \\ 0 & a'_{23} & a'_{33} - A \end{vmatrix} = 0$$

$\therefore$

examine the discriminate of quadratic

$\therefore$ the other two roots are also real for a symmetric second order tensor a_{ij}

Theorem : The principal direction (eigenvectors) $\mathbf{n}^I, \mathbf{n}^{II}, \mathbf{n}^{III}$ corresponding to A_I, A_{II}, A_{III} are mutually orthogonal if the eigenvalues are distinct.

recall the equation for eigenvalue

$$(1) \times \mathbf{n}_j^{II}$$

$$(2) \times \mathbf{n}_j^I$$

$\therefore i$ and j are dummy index for $a_{ij} n_i^{II} n_j^I$

$\therefore$

(b) becomes

(a) - (c)

$$\therefore A_I \neq A_{II} \quad \therefore$$

i.e. $\mathbf{n}^I \perp \mathbf{n}^{II}$ similarly for others

Theorem : $a_{ij} =$

proof : use (1) ~ (3) of the above theorem

$$a_{ij} (n_i^I n_k^I + n_i^{II} n_k^{II} + n_i^{III} n_k^{III})$$

$$=$$

let $\lambda_{1i} =$

similarly $\lambda_{2i} =$

$$\lambda_{3i} =$$

substitute into (a)

$$a_{ij} (\lambda_{1i} \lambda_{1k} + \lambda_{2i} \lambda_{2k} + \lambda_{3i} \lambda_{3k})$$

$$=$$

$$a_{ij} \lambda_{pi} \lambda_{pk} =$$

$$a_{ij} \delta_{ik} =$$

$$\therefore a_{kj} =$$

so if the eigenvectors are chosen as coordinate axes

i.e. $\mathbf{n}^I =$

$$\mathbf{n}^{II} =$$

$$\mathbf{n}^{III} =$$

then

$$a_{ij} = \begin{pmatrix} A_I & 0 & 0 \\ 0 & A_{II} & 0 \\ 0 & 0 & A_{III} \end{pmatrix}$$

that means if the coordinate rotate as $\mathbf{n}^I, \mathbf{n}^{II}, \mathbf{n}^{III}$, the matrix will be diagonal, and

$$\lambda_{ij} = \frac{\mathbf{n}^I}{\mathbf{n}^{II} \mathbf{n}^{III}} =$$

so that λ_{ij} is the transformation matrix used to diagonalize the matrix

$$\text{i.e. } a'_{\alpha\beta} =$$

Divergence Theorem (Gauss' Theorem) :

$$\int_B a_{ij \dots kl} n_r dA =$$

where $a_{ij \dots kl}$ be a tensor which are function of x_i , and continuously differentiable in D , and D is a regular region bounded by the closed surface B , and $\mathbf{n}$ is outer normal to B

Corollary

Divergence Theorem for plane

$$\int_B a_{ij \dots kl, r} dV =$$

where B is a regular region bounded by a close curve , $\mathbf{n}$ is the outer normal to

Strokes' Theorem :

B and are defined as previous

Green's Theorem :

$$\int_D (\nabla \phi \nabla \psi + \phi \nabla^2 \psi) dV =$$

$$\int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV =$$

$$\text{i.e. } \int_D (\phi_{,i} \psi_{,i} + \phi \psi_{,ii}) dV =$$

$$\int_D (\phi \psi_{,ii} - \psi \phi_{,ii}) dV =$$

for plane

$$\int_B \left(\frac{v_2}{x_1} - \frac{v_1}{x_2} \right) dA =$$

$$\text{i.e. } \int_B \varepsilon_{3jk} v_{k,j} dA =$$