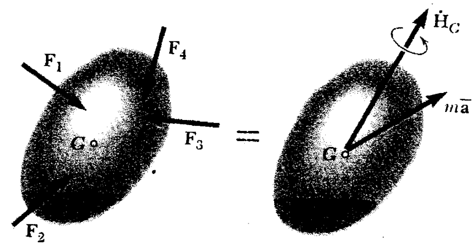
Chapter 18 Kinetics of Rigid Bodies in Three Dimensions

18.1 Introduction

two fundamental equations



$$\Sigma \boldsymbol{F} = m\,\underline{\boldsymbol{a}}$$

$$\Sigma \boldsymbol{M}_G = \dot{\boldsymbol{H}}_G$$

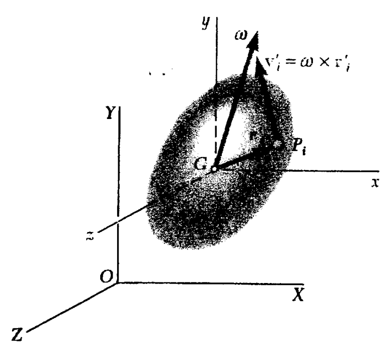
for plane motion $H_G = \underline{I}\,\omega$

for three-dimensional motion, $\boldsymbol{H}_G$ have to be discarded and replaced by the more general relation

18.2 Angular Momentum of a Rigid Body in Three Dimensions

consider a rigid body moves in space

the angular momentum of the body is



$$\boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times \boldsymbol{v}_i' \,\Delta m_i)$$

$$\boldsymbol{v}_i' = \omega \times \boldsymbol{r}_i'$$

$$\boldsymbol{H}_G = \sum_{i=1}^{n} [\boldsymbol{r}_i' \times (\omega \times \boldsymbol{r}_i')\,\Delta m_i]$$

let $\boldsymbol{H}_G = H_x\,\boldsymbol{i} + H_y\,\boldsymbol{j} + H_z\,\boldsymbol{k}$

$$\omega = \omega_x\,\boldsymbol{i} + \omega_y\,\boldsymbol{j} + \omega_z\,\boldsymbol{k}$$

$$\boldsymbol{r}_i' = x_i\,\boldsymbol{i} + y_i\,\boldsymbol{j} + z_i\,\boldsymbol{k}$$

then $H_x = \sum_{i=1}^{n} [y_i\,(\omega_x\,y_i - \omega_y\,x_i) - z_i\,(\omega_z\,x_i - \omega_x\,z_i)]\,\Delta m_i$

$$= \omega_x \sum_{i=1}^{n} (y_i^2 + z_i^2)\,\Delta m_i - \omega_y \sum_{i=1}^{n} x_i\,y_i\,\Delta m_i - \omega_z \sum_{i=1}^{n} x_i\,z_i\,\Delta m_i$$

for a rigid body, the numbers of particle $\rightarrow \infty$, then Σ can be replaced by $\int$, thus H_x can be written

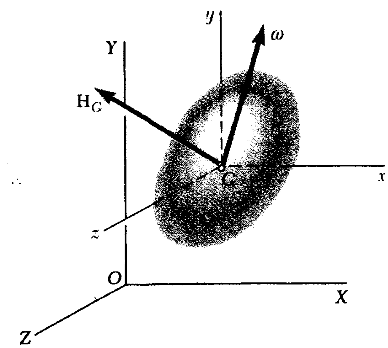
$$H_x = \omega_x \int (y^2 + z^2)\, dm - \omega_y \int x\, y\, dm - \omega_z \int x\, z\, dm$$

but $\quad I_x = \int (y^2 + z^2)\, dm$

$$I_{xy} = \int x\, y\, dm \qquad\qquad I_{xz} = \int x\, z\, dm$$

are the centroidal mass moments and products of inertia of the body, then

$$H_x = I_x\, \omega_x - I_{xy}\, \omega_y - I_{xz}\, \omega_z$$



similarly

$$H_y = - I_{xy}\, \omega_x + I_y\, \omega_y - I_{yz}\, \omega_z$$

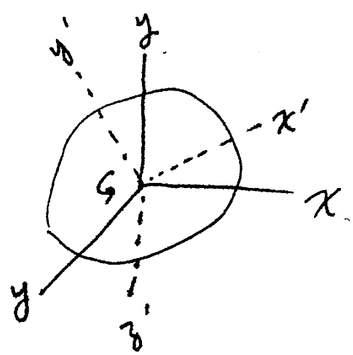
$$H_z = - I_{xz}\, \omega_x - I_{yz}\, \omega_y + I_z\, \omega_z$$

let $\quad \{H\} = \begin{vmatrix} H_x \\ H_y \\ H_y \end{vmatrix} \qquad \{\omega\} = \begin{vmatrix} \omega_x \\ \omega_y \\ \omega_y \end{vmatrix}$

and $\quad [I] = \begin{vmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{vmatrix}$

where $[I]$ defines the inertia tensor

then $\qquad \{H_G\} = [I]\, \{\omega\}$



if $Gx'y'z'$ are the principal axes of inertia

then $\qquad I_{x'y'} = I_{y'z'} = I_{x'z'} = 0$

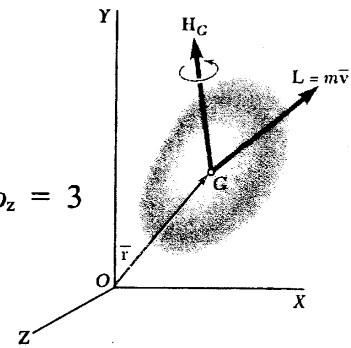
and $\qquad [I'] = \begin{vmatrix} I_x' & 0 & 0 \\ 0 & I_y' & 0 \\ 0 & 0 & I_z' \end{vmatrix}$

where I_x', I_y', I_z' represent the principal centroidal moments of inertia of the body, then

$$H_x' = I_x'\, \omega_x' \qquad H_y' = I_y'\, \omega_y' \qquad H_z' = I_z'\, \omega_z'$$

if $\underline{I}_x' = \underline{I}_y' = \underline{I}_z'$, then H_x', H_y', H_z' are proportional to the components $\omega_x', \omega_y', \omega_z'$

i.e. $\boldsymbol{H}_G' \;//\; \omega'$



in general, $\boldsymbol{H}_G$ and ω will have different direction

e.g. $I_x = 1,\ I_y = 2,\ I_z = 3$ and $\omega_x = 1,\ \omega_y = 2,\ \omega_z = 3$

then $H_x = 1 \quad H_y = 4 \quad H_z = 9$

thus $\boldsymbol{H}_G'$ not // ω'

except when two of the three components of ω are zero, i.e. ω directed along one of the principal axis

i.e. $\boldsymbol{H}_G$ and ω have the same direction if and only if ω directed along a principal axis of inertia

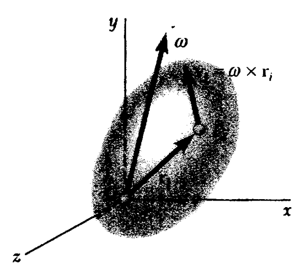
this condition is satisfied in the plane motion is of a rigid body symmetrical with respect to the reference plane

thus $\boldsymbol{H}_G = I\,\omega$ is correct for plane motion

this result cannot be extended to the case of plane motion of a nonsymmetrical body, or to the case of three dimensional motion

the angular momentum $\boldsymbol{H}_0$ may be expressed

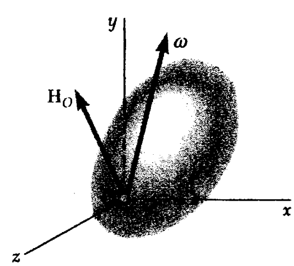
$$\boldsymbol{H}_0 = \boldsymbol{H}_G + \underline{r} \times m\,\underline{v}$$



(a)

for the body constrained to rotate about a fixed point O, $\boldsymbol{H}_0$ can be expressed

$$\boldsymbol{H}_0 = \sum_{i=1}^{n} (\boldsymbol{r}_i \times \boldsymbol{v}_i\,\Delta m_i)$$



(b)

where $\boldsymbol{v}_i = \omega \times \boldsymbol{r}_i$

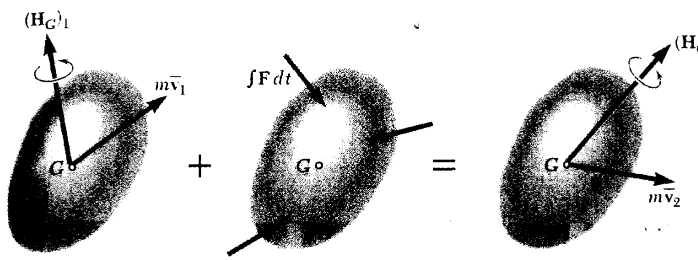
similar as previous, it is obtained

$$H_x = I_x\,\omega_x - I_{xy}\,\omega_y - I_{xz}\,\omega_z$$
$$H_y = -I_{xy}\,\omega_x + I_y\,\omega_y - I_{yz}\,\omega_z$$
$$H_z = -I_{xz}\,\omega_x - I_{yz}\,\omega_y + I_z\,\omega_z$$

where I_x, I_y, I_z, I_{xy}, I_{xz}, and I_{yz} are the moments and products of inertia with respect to the frame $Oxyz$

18.3 Application of the Principle of Impulse and Momentum to the Three Dimensional Motion of a Rigid Body

principle of impulse and momentum



syst momenta$_1$ + syst ext. Imp$_{1\rightarrow 2}$ = syst momenta$_2$

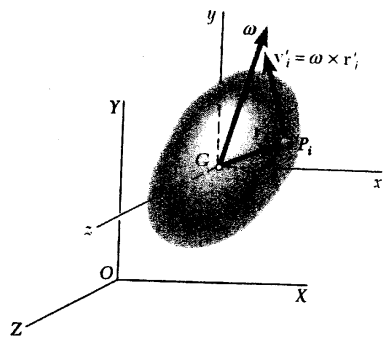
this principle have 6 equations, 3 linear equations and 3 angular equations

in solving problem dealing with the motion of the body about a fixed point O, it will be convenient to eliminate the impulse of the reaction at O by writing an equation involving the moments of momenta and impulses about O

18.4 Kinetic Energy of a Rigid Body in Three Dimensions

for system of particles, the kinetic energy is expressed

$$T = \tfrac{1}{2}\, m\, \underline{v}^2 + \tfrac{1}{2}\sum_{i=1}^{n} \Delta m_i\, v_i'^2$$



$$v_i'^2 = |v_i'|^2 = |\omega \times \boldsymbol{r}_i'|^2$$

define $$T' = \tfrac{1}{2}\sum_{i=1}^{n} \Delta m_i\, |\omega \times \boldsymbol{r}_i'|^2$$

$$\omega = \omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k}$$

$$\boldsymbol{r}_i' = x_i \boldsymbol{i} + y_i \boldsymbol{j} + z_i \boldsymbol{k}$$

expressing the square $|\omega \times \boldsymbol{r}_i'|^2$ in terms of rectangular components of the vector product, and replace Σ by $\int$ for the rigid body, it can be obtained

$$2T' = \int [(\omega_x y - \omega_y x)^2 + (\omega_y z - \omega_z y)^2 + (\omega_z x - \omega_x z)^2]\, dm$$

$$= \omega_x^2 \int (y^2 + z^2)\, dm + \omega_y^2 \int (x^2 + z^2)\, dm + \omega_z^2 \int (x^2 + y^2)\, dm$$

$$- 2\omega_x \omega_y \int x y\, dm - 2\omega_y \omega_z \int y z\, dm - 2\omega_x \omega_z \int x z\, dm$$

$$T' = \tfrac{1}{2}(I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 - 2 I_{xy} \omega_x \omega_y - 2 I_{yz} \omega_y \omega_z$$

$$- 2 I_{xz} \omega_x \omega_z)$$

and the kinetic energy can be expressed

$$T = \tfrac{1}{2} m \underline{v}^2 + \tfrac{1}{2}(I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 - 2 I_{xy} \omega_x \omega_y$$

$$- 2 I_{yz} \omega_y \omega_z - 2 I_{xz} \omega_x \omega_z)$$

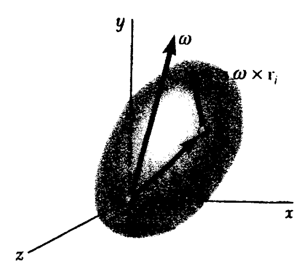
if we choose x', y', z' as the principal axes, then

$$I_{x'y'} = I_{y'z'} = I_{x'z'} = 0$$

and $$T = \tfrac{1}{2} m \underline{v}^2 + \tfrac{1}{2}(I_{x'} \omega_{x'}^2 + I_{y'} \omega_{y'}^2 + I_{z'} \omega_{z'}^2)$$

for plane motion, $\omega_{x'} = \omega_{y'} = 0$, $\omega_{z'} = \omega$

$$T = \tfrac{1}{2} m \underline{v}^2 + \tfrac{1}{2} I \omega^2$$



for the body rotates about a fixed point

$$T = \tfrac{1}{2}\sum_{i=1}^{n} \Delta m_i\, v_i^2$$

$$v_i^2 = |\boldsymbol{v}_i|^2 = |\omega \times \boldsymbol{r}_i|^2$$

as $n \rightarrow \infty$, $\Sigma \rightarrow \int$, then

$$T = \tfrac{1}{2}\,(I_x\,\omega_x^2 + I_y\,\omega_y^2 + I_z\,\omega_z^2 - 2\,I_{xy}\,\omega_x\,\omega_y - 2\,I_{yz}\,\omega_y\,\omega_z - 2\,I_{xz}\,\omega_x\,\omega_z)$$

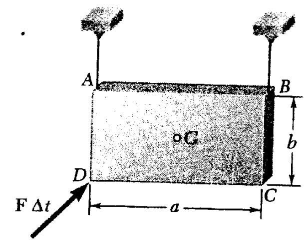
if the principal axes x', y', z' are chosen as the coordinate axes, then

$$T = \tfrac{1}{2}\,(I_{x'}\,\omega_{x'}^2 + I_{y'}\,\omega_{y'}^2 + I_{z'}\,\omega_{z'}^2)$$

Sample Problem 18.1

a rectangular plate of mass m

determine $\underline{v}$ and ω of the plate immediately after impact



assume the wires remain taut

$$\underline{v} = v_x\,\boldsymbol{i} + v_z\,\boldsymbol{k}$$

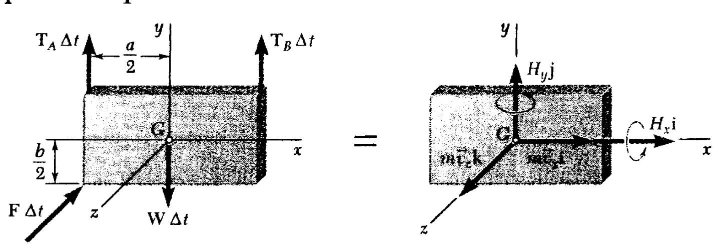
$$\boldsymbol{\omega} = \omega_x\,\boldsymbol{i} + \omega_y\,\boldsymbol{j}$$

since x, y, z are principal axes of inertia

then $\quad \boldsymbol{H}_G = \underline{I}_x\,\omega_x\,\boldsymbol{i} + \underline{I}_y\,\omega_y\,\boldsymbol{j}$

$$= (1/12)\,m\,b^2\,\omega_x\,\boldsymbol{i} + (1/12)\,m\,a^2\,\omega_y\,\boldsymbol{j}$$

principle of impulse and momentum

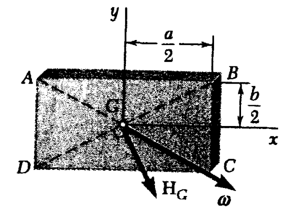


linear momentum

x-comp. $\quad 0 = m\,\underline{v}_x \qquad \underline{v}_x = 0$

z-comp. $\quad -F\,\Delta t = m\,\underline{v}_z \qquad \underline{v}_z = -F\,\Delta t\,/\,m$

then $\quad \underline{v} = -(F\Delta t/m)\,\boldsymbol{k}$

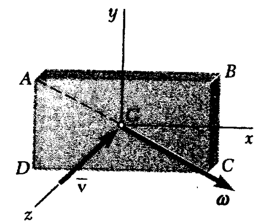


angular momentum

x-comp. $\quad F\Delta t\,(b/2) \;=\; H_x \;=\; (1/12)\, m\, b^2\, \omega_x$

y-comp. $\quad -\,F\Delta t\,(a/2) \;=\; H_y \;=\; (1/12)\, m\, a^2\, \omega_y$

$$\omega_x \;=\; \frac{6\,F\Delta t}{m\,b} \qquad\qquad \omega_y \;=\; -\,\frac{6\,F\Delta t}{m\,a}$$



$$\boldsymbol{\omega} \;=\; \omega_x\,\boldsymbol{i} \;+\; \omega_y\,\boldsymbol{j}$$

$$=\; (6F\Delta t/mab)(a\,\boldsymbol{i} \;-\; b\,\boldsymbol{j})$$

and $\quad \boldsymbol{H}_G \;=\; H_x\,\boldsymbol{i} \;+\; H_y\,\boldsymbol{j} \;=\; \tfrac{1}{2}\,F\Delta t\,(b\,\boldsymbol{i} \;-\; a\,\boldsymbol{j})$

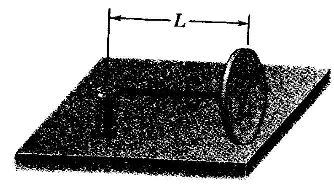
note that $\boldsymbol{H}_G$ and ω are in different direction

Sample Problem 18.2

OG has negligible mass

mass of disk $=\; m$

determine (a) ω of disk (b) $\boldsymbol{H}_0$ (c) T (d) $\boldsymbol{H}_G$ and $m\underline{\boldsymbol{v}}$



$$\omega \;=\; \omega_1\,\boldsymbol{i} \;-\; \omega_2\,\boldsymbol{j}$$

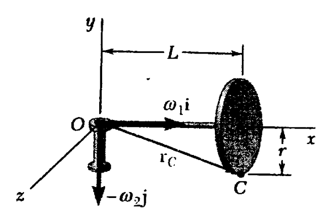
since $\boldsymbol{v}_C \;=\; 0$, $\quad$ i.e. $\quad \omega \times \boldsymbol{r}_C \;=\; 0$

$$(\omega_1\,\boldsymbol{i} \;+\; \omega_2\,\boldsymbol{j}) \times (L\,\boldsymbol{i} \;-\; r\,\boldsymbol{j}) \;=\; 0$$

$$(L\,\omega_2 \;-\; r\,\omega)\,\boldsymbol{k} \;=\; 0$$

$\therefore \qquad \omega_2 \;=\; r\,\omega_1\,/\,L$

i.e. $\quad \omega \;=\; \omega_1\,\boldsymbol{i} \;-\; (r\,\omega_1\,/\,L)\,\boldsymbol{j}$



since x, y, z are principal axes of inertia of the disk

$$H_x \;=\; I_x\,\omega_x \;=\; \tfrac{1}{2}\,m\,r^2\,\omega_1$$

$$H_y \;=\; I_y\,\omega_y \;=\; (mL^2 \;+\; \tfrac{1}{4}mr^2)\,(-\,r\omega_1/L) \;=\; -\,m\,(L^2 + \tfrac{1}{4}\,r^2)\,(r\omega_1/L)$$

$$H_z \;=\; I_z\,\omega_z \;=\; 0$$

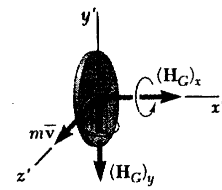
$$\boldsymbol{H}_0 \;=\; H_x\,\boldsymbol{i} \;+\; H_y\,\boldsymbol{j} \;+\; H_z\,\boldsymbol{k}$$

$$=\; \tfrac{1}{2}\,m\,r^2\,\omega_1\,\boldsymbol{i} \;-\; m\,(L^2 + \tfrac{1}{4}\,r^2)\,(-\,r\omega_1/L)\,\boldsymbol{j}$$

$$T = \tfrac{1}{2}(I_x\,\omega_x^2 + I_x\,\omega_x^2 + I_x\,\omega_x^2)$$
$$= \tfrac{1}{2}[\tfrac{1}{2}\,m\,r^2\,\omega_1^2 + m\,(L^2 + \tfrac{1}{4}\,r^2)(-\,r\omega_1/L)^2]$$
$$= \tfrac{1}{8}\,m\,r^2\,(6 + r^2/L^2)\,\omega_1^2$$

$$\underline{v} = \omega \times \underline{r} = [\omega_1\,\boldsymbol{i} - (r\omega_1/L)\,\boldsymbol{j}] \times L\,\boldsymbol{i} = r\,\omega_1\,\boldsymbol{k}$$
$$m\,\underline{v} = m\,r\,\omega_1\,\boldsymbol{k}$$

$$I_{y'} = I_{z'} = m\,r^2/4 \qquad I_{x'} = m\,r^2/2$$
$$\boldsymbol{H}_G = I_{x'}\,\omega_x\,\boldsymbol{i} + I_{y'}\,\omega_y\,\boldsymbol{j} + I_{z'}\,\omega_z\,\boldsymbol{k}$$
$$= \tfrac{1}{2}\,m\,r^2\,\omega_1\,(\boldsymbol{i} - r/2L\,\boldsymbol{j})$$



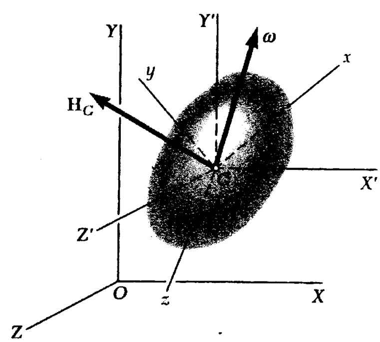
note that $\boldsymbol{H}_G$ and ω are not in the same direction

18.5 Motion of a Rigid Body in Three Dimensions

OXYZ : fixed frame

GX'Y'Z' : translating frame

Gxyz : rotating frame attached to the body



fundamental equations

$$\Sigma\,\boldsymbol{F} = m\,\underline{\boldsymbol{a}}$$

$$\Sigma\,\boldsymbol{M}_G = \dot{\boldsymbol{H}}_G$$

$$\boldsymbol{H}_G = H_x\,\boldsymbol{i} + H_y\,\boldsymbol{j} + H_z\,\boldsymbol{k}$$
$$= H_{X'}\,\boldsymbol{I'} + H_{Y'}\,\boldsymbol{J'} + H_{Z'}\,\boldsymbol{K'}$$

since $\quad \boldsymbol{H}_G = [I]\,\{\omega\}$

then $\quad \dot{\boldsymbol{H}}_G = \dfrac{d\,[I]}{d\,t}\,\{\omega\} + [I]\,\{\dfrac{d\,\omega}{d\,t}\}$

but $dI_{X'}/\mathrm{dt}$, $dI_{X'Y'}/dt$ etc. are not east to find, thus it is difficult to calculate $d\boldsymbol{H}_G/dt$ by this way define $(\dot{\boldsymbol{H}}_G)_{Gxyz}$ is the rate of change of $\boldsymbol{H}_G$ with respect to rotating frame *Gxyz*, then

$$(\dot{\boldsymbol{H}}_G)_{Gxyz} = \frac{d\,[\underline{I}]}{d\,t}\,\{\omega\} + [\underline{I}]\,\{\frac{d\,\omega}{d\,t}\}$$

since the moments and products of inertia will maintain the same values during the motion when observed from the body fixed coordinates *Gxyz*

then $\quad (\dot{\boldsymbol{H}}_G)_{Gxyz} = [\underline{I}]\,\{\alpha\}$

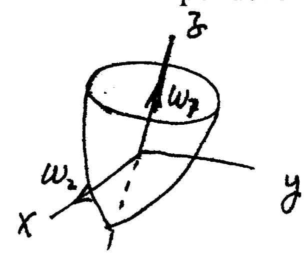
$\because \quad \dot{\boldsymbol{H}}_G = (\dot{\boldsymbol{H}}_G)_{Gxyz} + \Omega \times \boldsymbol{H}_G$

Ω : angular velocity of the rotating frame *Gxyz*

thus $\quad \Sigma \boldsymbol{M}_G = \dot{\boldsymbol{H}}_G = (\dot{\boldsymbol{H}}_G)_{Gxyz} + \Omega \times \boldsymbol{H}_G$

in this case, Ω is identically equal to the angular velocity ω of the body, if *Gxyz* is not a body fixed coordinate, Ω and ω are independent

for an axisymmetrical body



$\omega = \omega_1 + \omega_2$

choose $\Omega = \omega_2$, then $[\underline{I}]$ remain constants in *Gxyz* when body in motion

it is clearly that $\Omega \neq \omega$

18.6 Euler's Equations of Motion, Extension of d'Alembert's Principle to the Motion of a Rigid Body in Three Dimensions

if x, y, z axes are chosen as principal axes of inertia of the body, then

$$\boldsymbol{H}_G = \underline{I}_x\,\omega_x\,\boldsymbol{i} + \underline{I}_y\,\omega_y\,\boldsymbol{j} + \underline{I}_z\,\omega_z\,\boldsymbol{k}$$

setting $\Omega = \omega$

$$\Sigma \boldsymbol{M}_G = \dot{\boldsymbol{H}}_G = (\dot{\boldsymbol{H}}_G)_{Gxyz} + \Omega \times \boldsymbol{H}_G$$

$$\Omega \times \boldsymbol{H}_G = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_x & \omega_y & \omega_z \\ I_x \omega_x & I_y \omega_y & I_z \omega_z \end{vmatrix}$$

thus the three scalar equations can be expressed

$$\Sigma M_x = I_x \dot{\omega}_x - (I_y - I_z)\, \omega_y\, \omega_z$$

$$\Sigma M_y = I_y \dot{\omega}_y - (I_z - I_x)\, \omega_z\, \omega_x$$

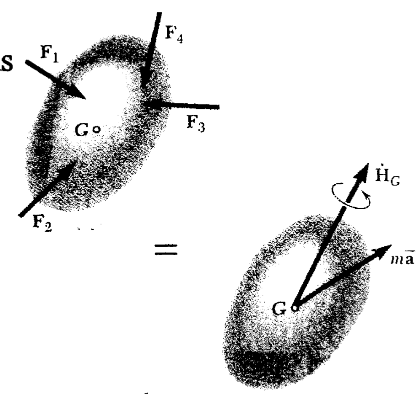
$$\Sigma M_z = I_z \dot{\omega}_z - (I_x - I_y)\, \omega_x\, \omega_y$$

this equations are called Euler's equations of motion

with $\quad \Sigma F_x = m\, a_x \qquad \Sigma F_y = m\, a_y \qquad \Sigma F_z = m\, a_z$

we have 6 equations to solve the dynamic problems

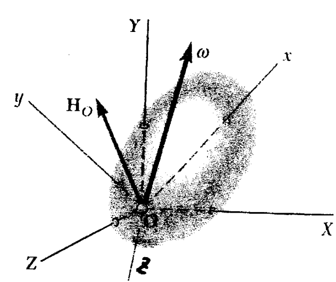
the system of the external forces acting on a rigid body in three dimensional motion is equivalent to the system consisting of the vector $m\,\boldsymbol{a}$ attached at the mass center G of the body and the couple of moment $\dot{\boldsymbol{H}}_G$



18.7 Motion of a Rigid Body About a Fixed Point

OXYZ : fixed frame

Oxyz : rotating frame



$$\Sigma \boldsymbol{M}_0 = \dot{\boldsymbol{H}}_0 = (\dot{\boldsymbol{H}}_0)_{oxyz} + \Omega \times \boldsymbol{H}_0$$

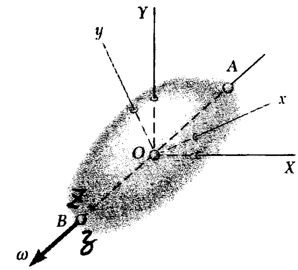
$(\dot{\boldsymbol{H}}_0)_{oxyz}$: rate of change of $\boldsymbol{H}_0$ with respect to the rotating frame *Oxyz*

Ω : angular velocity of *Oxyz*

if *Oxyz* is attached to the body, then $\qquad \Omega = \omega$

if $Oxyz$ is not a body fixed axes, then $\quad \Omega \neq \omega$

18.8 Rotation of a Rigid Body About a Fixed Axis



AB : fixed axis (not a principal axis)

chosen Z and z axes coincide with AB

then $\quad \boldsymbol{\omega} = \omega \boldsymbol{k} \quad \omega_x = \omega_y = 0$

thus $\quad H_x = -I_{xz}\,\omega \quad H_y = -I_{yz}\,\omega \quad H_z = I_z\,\omega$

since $Oxyz$ is attached to the body, then $\Omega = \omega = \omega \boldsymbol{k}$

thus
$$\Sigma \boldsymbol{M}_0 = (\dot{\boldsymbol{H}}_0)_{oxyz} + \Omega \times \boldsymbol{H}_0$$
$$= (-I_{xz}\boldsymbol{i} - I_{yz}\boldsymbol{j} + I_z\boldsymbol{k})\,\dot{\omega}$$
$$+\ \omega \boldsymbol{k} \times (-I_{xz}\boldsymbol{i} - I_{yz}\boldsymbol{j} + I_z\boldsymbol{k})$$
$$= (-I_{xz}\boldsymbol{i} - I_{yz}\boldsymbol{j} + I_z\boldsymbol{k})\,a + (-I_{xz}\boldsymbol{j} - I_{yz}\boldsymbol{i})\,\omega^2$$

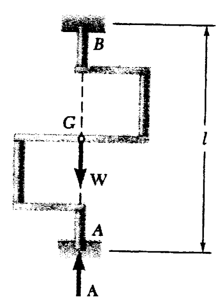
i.e.
$$\Sigma M_x = -I_{xz}\,a + I_{yz}\,\omega^2$$
$$\Sigma M_y = -I_{yz}\,a - I_{xz}\,\omega^2$$
$$\Sigma M_z = I_z\,a$$

if the rotating body is symmetrical with respect to xy plane, then $I_{xz} = I_{yz} = 0$ [that means z is a principal axis], then

$$\Sigma M_x = \Sigma M_y = 0$$
$$\Sigma M_z = I_z\,a \qquad \text{[plane motion problem]}$$

if I_{xz}, $I_{yz} \neq 0$, but $a = 0$ (constant ω), then

$$\Sigma M_x = I_{yz}\,\omega^2$$
$$\Sigma M_y = -I_{xz}\,\omega^2$$
$$\Sigma M_z = 0$$



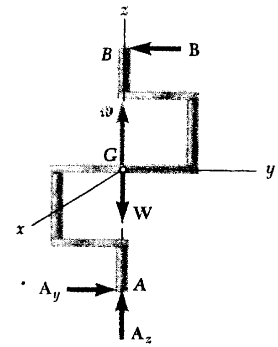
consider the balancing of a rotating shaft

for static balance $\qquad A_z = W$

$$A_y = B = 0$$

if the shaft is rotating with constant ω

chosen $Gxyz$ as the body fixed axis, then



$I_{xy} = I_{xz} = 0 \qquad [\because \text{ all } x = 0]$

$I_{yz} = + \neq 0$

$\therefore \quad \Sigma M_x = I_{yz}\,\omega^2 \neq 0$

$\Sigma M_y = \Sigma M_z = 0$

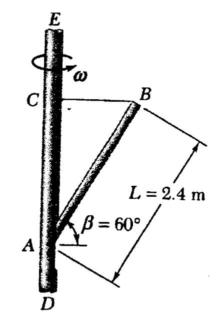
but $\quad \Sigma M_x = A_y\, l = B\, l$

$\therefore \quad A_y = B = I_{yz}\,\omega^2 / l$

A_y and B are proportional to ω^2, A_y and B are called dynamic reactions, which are contained in y-z plane, it cause the structure support to vibrate,

dynamic balancing : if we let I_{yz} becomes to zero by adding corrective mass, the dynamic reaction may reduce to zero

Sample Problem 18.3



$L = 2.4$ m $\qquad m = 20$ kg

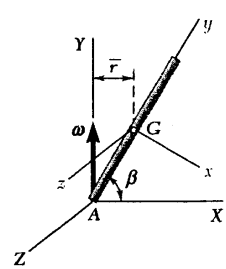
$\omega = 15$ rad/s $\qquad \beta = 60°$

determine T_{BC} and $\boldsymbol{R}_A$

$\underline{r} = (L/2)\cos\beta$

$\underline{\boldsymbol{a}} = \underline{\boldsymbol{a}}_n = -\underline{r}\,\omega^2 \boldsymbol{I} = -135\,\boldsymbol{I}$ (m/s^2)

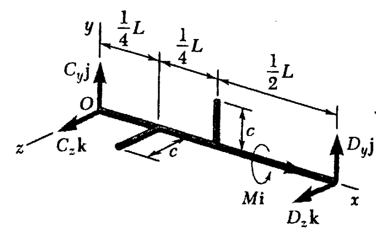
$m\,\underline{\boldsymbol{a}} = -2700\,\boldsymbol{I}$ (N)



$\boldsymbol{H}_G = H_x\,\boldsymbol{i} + H_y\,\boldsymbol{j} + H_z\,\boldsymbol{k}$

since $Gxyz$ is the principal axes of rotation

$\underline{I}_x = \underline{I}_z = m\,L^2 / 12 \qquad \underline{I}_y = 0$



let $Oxyz$ be the body fixed axes, then $\Omega = \omega$

equations of motion

$\Sigma \boldsymbol{M}_0$:

$$\Sigma \boldsymbol{M}_0 = \dot{\boldsymbol{H}}_0 = (\dot{\boldsymbol{H}}_0)_{Oxyz} + \Omega \times \boldsymbol{H}_0$$

$$= (I_x \boldsymbol{i} - I_{xy} \boldsymbol{j} - I_{xz} \boldsymbol{k})\, a + \omega\, \boldsymbol{i} \times (I_x \boldsymbol{i} - I_{xy} \boldsymbol{j} - I_{xz} \boldsymbol{k})\, \omega$$

$$= I_x\, a\, \boldsymbol{i} - (I_{xy}\, a - I_{xz}\, \omega^2)\, \boldsymbol{j} - (I_{xz}\, a + I_{xy}\, \omega^2)\, \boldsymbol{k}$$

but

$$\Sigma \boldsymbol{M}_0 = L\, \boldsymbol{i} \times (D_y \boldsymbol{j} + D_z \boldsymbol{k}) + M \boldsymbol{i}$$

$$= M \boldsymbol{i} - D_z L \boldsymbol{j} + D_y L \boldsymbol{k}$$

note that the masses of rods A and B are neglected

thus

$$M = I_x\, a = 2\,(\tfrac{1}{3}\, m\, c^2)\, a$$

$$a = 3\, M / 2\, m\, c^2$$

and

$$D_y = -\,(I_{xz}\, a + I_{xy}\, \omega^2) / L$$

$$D_z = (I_{xy}\, a - I_{xz}\, \omega^2) / L$$

$$I_{xy} = \Sigma\, m\, x\, y = m\,(\tfrac{1}{2} L)\,(\tfrac{1}{2} c) = \tfrac{1}{4}\, m\, L\, c$$

$$I_{xz} = \Sigma\, m\, x\, z = m\,(\tfrac{1}{4} L)\,(\tfrac{1}{2} c) = \tfrac{1}{8}\, m\, L\, c$$

then

$$D_y = -\frac{3}{16}\frac{M}{c} - \frac{1}{4}\, m\, c\, \omega^2$$

$$D_z = \frac{3}{8}\frac{M}{c} - \frac{1}{8}\, m\, c\, \omega^2$$

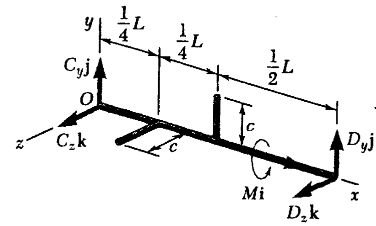
with $\omega = 125.7$ rad/s $\quad c = 0.1$ m $\quad M = 6$ N-m $\quad m = 0.3$ kg

$$D_y = -\,129.8 \text{ N} \qquad D_z = -\,36.8 \text{ N}$$

similarly, take $\Sigma \boldsymbol{M}_D$, it is obtained

$$C_y = -\,152.2 \text{ N} \qquad C_z = -\,155.2 \text{ N}$$

note that $\Sigma F_y \neq 0$ and $\Sigma F_z \neq 0$, since G is not located on CD, thus $\boldsymbol{a} \neq 0$, $\Sigma \boldsymbol{F} = m\, \boldsymbol{a} \neq 0$



let $Oxyz$ be the body fixed axes, then $\Omega = \omega$

equations of motion

$\Sigma \boldsymbol{M}_0$: $\quad \Sigma \boldsymbol{M}_0 = \dot{\boldsymbol{H}}_0 = (\dot{\boldsymbol{H}}_0)_{Oxyz} + \Omega \times \boldsymbol{H}_0$

$$= (I_x \boldsymbol{i} - I_{xy} \boldsymbol{j} - I_{xz} \boldsymbol{k})\, a + \omega \boldsymbol{i} \times (I_x \boldsymbol{i} - I_{xy} \boldsymbol{j} - I_{xz} \boldsymbol{k})\, \omega$$

$$= I_x\, a\, \boldsymbol{i} - (I_{xy}\, a - I_{xz}\, \omega^2) \boldsymbol{j} - (I_{xz}\, a + I_{xy}\, \omega^2) \boldsymbol{k}$$

but $\quad \Sigma \boldsymbol{M}_0 = L \boldsymbol{i} \times (D_y \boldsymbol{j} + D_z \boldsymbol{k}) + M \boldsymbol{i}$

$$= M \boldsymbol{i} - D_z L \boldsymbol{j} + D_y L \boldsymbol{k}$$

note that the masses of rods A and B are neglected

thus $\quad M = I_x\, a = 2\,(\tfrac{1}{3}\, m\, c^2)\, a$

$$a = 3\, M / 2\, m\, c^2$$

and $\quad D_y = -(I_{xz}\, a + I_{xy}\, \omega^2) / L$

$$D_z = (I_{xy}\, a - I_{xz}\, \omega^2) / L$$

$$I_{xy} = \Sigma\, m\, \underline{x}\, \underline{y} = m\, (\tfrac{1}{2} L)\, (\tfrac{1}{2} c) = \tfrac{1}{4}\, m\, L\, c$$

$$I_{xz} = \Sigma\, m\, \underline{x}\, \underline{z} = m\, (\tfrac{1}{4} L)\, (\tfrac{1}{2} c) = \tfrac{1}{8}\, m\, L\, c$$

then $\quad D_y = -\frac{3}{16}\frac{M}{c} - \frac{1}{4}\, m\, c\, \omega^2$

$$D_z = \frac{3}{8}\frac{M}{c} - \frac{1}{8}\, m\, c\, \omega^2$$

with $\omega = 125.7$ rad/s $\quad c = 0.1$ m $\quad M = 6$ N-m $\quad m = 0.3$ kg

$D_y = -129.8$ N $\qquad D_z = -36.8$ N

similarly, take $\Sigma \boldsymbol{M}_D$, it is obtained

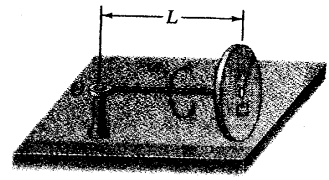
$C_y = -152.2$ N $\qquad C_z = -155.2$ N

note that $\Sigma F_y \neq 0$ and $\Sigma F_z \neq 0$, since G is not located on CD, thus $\boldsymbol{a} \neq 0$, $\Sigma \boldsymbol{F} = m\, \boldsymbol{a} \neq 0$

Sample Problem 18.5

the disk rolls on the floor with ω_1

determine the reactions at O and on the floor



from sample problem 18.2

$$\omega_2 = r\,\omega_1 / L$$

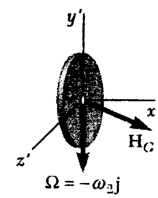
$$\boldsymbol{a} = -L\,\omega_2^2\,\boldsymbol{i}$$

thus $\qquad m\,\boldsymbol{a} = -m\,L\,\omega_2^2\,\boldsymbol{i} = -m\,r^2\,\omega_2^2 / L\,\boldsymbol{i}$.

choose $Gx'y'z'$ as the rotating frame (not the body fixed axes), then

$$\Omega = -\omega_2\,\boldsymbol{j}$$

note that $\Omega \neq \omega$, but $[I]$ = constant on the frame during motion



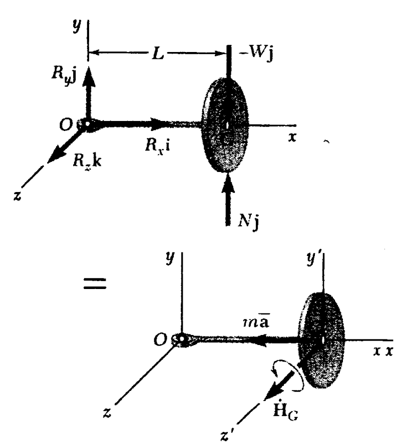
$$\boldsymbol{H}_G = \frac{1}{2} m\,r^2\,\omega_1\,(\boldsymbol{i} - \frac{r}{2L}\boldsymbol{j})$$

$$\dot{\boldsymbol{H}}_G = (\dot{\boldsymbol{H}}_G)_{Gx'y'z'} + \Omega \times \boldsymbol{H}_G$$

$$= 0 - (\frac{r\,\omega_1}{L}\boldsymbol{j}) \times [\frac{1}{2} m\,r^2\,\omega_1\,(\boldsymbol{i} - \frac{r}{2L}\boldsymbol{j})]$$

$$= \frac{1}{2} m\,r^2\,(\frac{r}{L})\,\omega_1^2\,\boldsymbol{k}$$

equations of motion



$\Sigma\,\boldsymbol{M}_0$: $\qquad L\,\boldsymbol{i} \times (N\boldsymbol{j} - W\boldsymbol{j}) = \dot{\boldsymbol{H}}_G$

$$(N - W)\,L\,\boldsymbol{k} = \frac{1}{2} m\,r^2\,(\frac{r}{L})\,\omega_1^2\,\boldsymbol{k}$$

$$N = W + \frac{1}{2} m\,r^2\,(\frac{r}{L})\,\omega_1^2$$

with ω_1 increase, N is increased

$\Sigma\,\boldsymbol{F}$: $\qquad \boldsymbol{R} + N\boldsymbol{j} - W\boldsymbol{j} = m\,\boldsymbol{a}$

$$R = -\frac{m\, r^2\, \omega_1^{\,2}}{L}\left(\boldsymbol{i} + \frac{r}{2L}\boldsymbol{j}\right)$$

also, $\boldsymbol{R}$ increased as ω_1 increased

18.9 Motion of a Gyroscope, Eulerian Angles

18.10 Steady Precession of a Gyroscope

18.11 Motion of an Axisymmetrical Body Under No Force