

Chapter 15 Kinematics of Rigid Bodies

15.1 Introduction

relationship between time, positions, velocities and accelerations of rigid bodies motion may be grouped as follows :

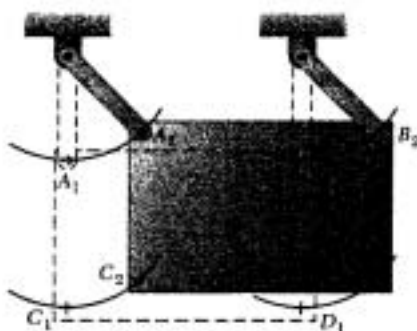
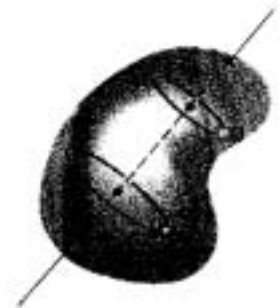
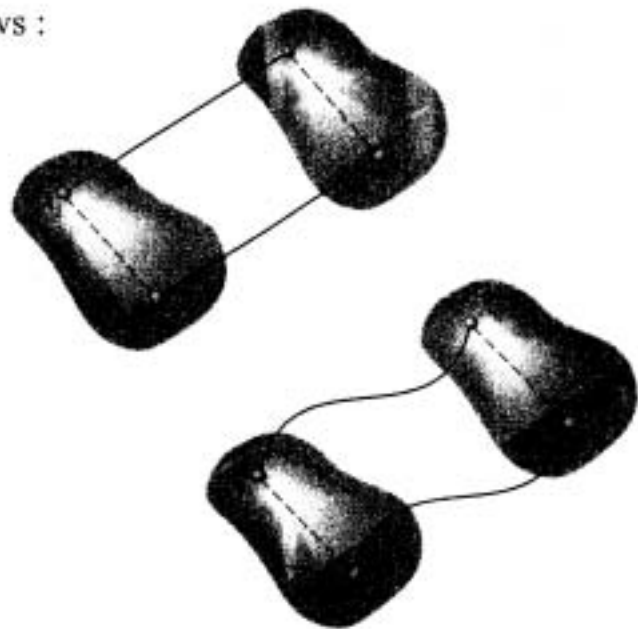
1. translation : if any straight line inside the body keeps the same direction during the motion

rectilinear translation : if the paths of motion of the particles are straight line

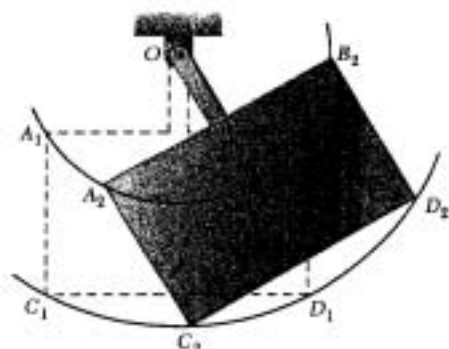
curvilinear translation : if the paths of motion of the particles are curved line

2. rotation about a fixed axis : AA' is called the axis of rotation, v and a are zero if the particle located on this axis

it is different from curvilinear translation



(a) Curvilinear translation



(b) Rotation

3. general plane motion :

plane motion : each particle moves in a given plane

any plane motion which is neither a rotation nor translation is referred to as a general motion

4. motion about a fixed point : the three-dimensional motion of a rigid body attached at a fixed point

5. general motion : any rigid motion does not fall in any of above



(a) Rolling wheel



(b) Sliding rod

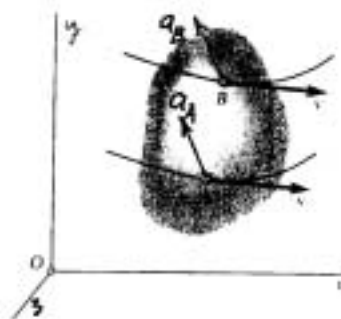
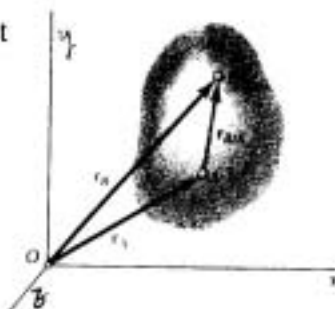


15.2 Translation

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

$\therefore \mathbf{r}_{B/A}$ must remain constant direction and magnitude

$\therefore \mathbf{r}_{B/A} = \text{constant}$



differentiate with respect to time, then

$$\dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A \quad \text{or} \quad \mathbf{v}_B = \mathbf{v}_A$$

also

$$\mathbf{a}_B = \mathbf{a}_A$$

that means when a rigid body is in translation, all the points of the body have the same velocity and the same acceleration at any given instant

15.3 Rotation About a Fixed Axis

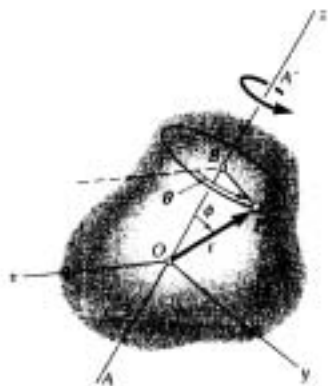
consider a rigid body rotates about a fixed axis AA' , and let P be a point of the body and r its position vector

let B be the projection of P on AA' , then

$$BP = r \sin \phi = \text{constant}$$

P will describe a circle of center at B

denote the angle θ the angular coordinate of the body, it is defined the angle between BP and xz plane



$$1 \text{ revolution} = 2\pi \text{ rad} = 360^\circ$$

$$v = \frac{dr}{dt} \quad v = \frac{ds}{dt}$$

$$\therefore \Delta s = BP (\Delta\theta) = r \sin \phi (\Delta\theta)$$

$$v = ds/dt = r \sin \phi \dot{\theta} \quad \dot{\theta} = d\theta/dt$$

θ depends on the position of P , but $\dot{\theta}$ is independent of P

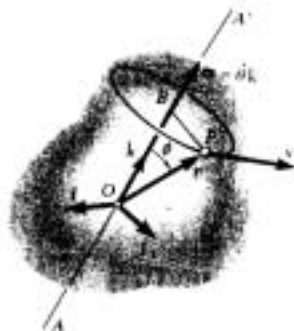
v is a vector $\perp$ the plane containing AA' and r

define $\omega = \dot{\theta} k$

the angular velocity of the rigid body

$$v = dr/dt = \omega \times r$$

$$\begin{aligned} a &= dv/dt = d(\omega \times r)/dt \\ &= (d\omega/dt) \times r + \omega \times (dr/dt) \\ &= \alpha \times r + \omega \times v \end{aligned}$$



where $\alpha = d\omega/dt$ is called angular acceleration

$$\alpha = \alpha k = \dot{\omega} k = \ddot{\theta} k$$

$\mathbf{a}$ is the sum of two vectors

$\boldsymbol{\alpha} \times \mathbf{r}$ is the tangent to the circle which is the tangential component of $\mathbf{a}$

$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is a vector triple product, $(\boldsymbol{\omega} \times \mathbf{r})$ is tangent to the circle, and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is directed to the center B , which is normal component of $\mathbf{a}$

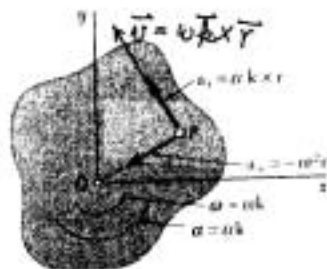
consider a plane containing the circle of point P , choose xy plane as the reference plane

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

and $v = r \omega$

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} - \omega^2 \mathbf{r} = \mathbf{a}_t + \mathbf{a}_n$$

where $\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}$ $a_t = a r$
 $\mathbf{a}_n = -\omega^2 \mathbf{r}$ $a_n = r \omega^2$



15.4 Equations Defining the Rotation of a Rigid Body About a Fixed Axis

the motion of rigid body rotating about a fixed axis is said to be known when $\theta(t)$ is known

$$\omega = d\theta / dt$$

$$\alpha = d\omega / dt = d^2\theta / dt^2$$

$$\alpha = d\omega / dt = (d\omega / d\theta) (d\theta / dt) = \omega (d\omega / d\theta)$$

two particular cases of rotation are encountered

1. uniform rotation

$$\alpha = 0 \quad \omega = \text{constant}$$

$$\text{then } \theta = \theta_0 + \omega t$$

2. uniformly accelerated rotation

$$\alpha = \text{constant}$$

$$\omega = \omega_0 + \alpha t \quad \omega^2 = \omega_0^2 + 2 \alpha (\theta - \theta_0)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

Sample Problem 15.1

$$v_C = 300 \text{ mm/s} \quad a_C = 225 \text{ mm/s}^2$$

determine

(a) numbers of revolutions at $t = 2 \text{ s}$ (b) v_B and Δy_B at $t = 2 \text{ s}$ (c) a_D at $t = 0$

$$(v_D)_0 = (v_C)_0 = 300 \text{ mm/s} \rightarrow$$

$$(a_D)_0 = a_C = 225 \text{ mm/s}^2 \rightarrow$$

$$(v_D)_0 = r \omega_0 \quad \omega_0 = 4 \text{ rad/s}$$

$$(a_D)_0 = r a \quad a = 3 \text{ rad/s}^2$$

at $t = 2 \text{ s}$

$$\omega = \omega_0 + a t = 4 + 3 \times 2 = 10 \text{ rad/s}$$

$$\theta = \omega_0 t + \frac{1}{2} a t^2 = 4 \times 2 + \frac{1}{2} 3 \times 2^2 = 14 \text{ rad}$$

$$\text{no. of rev.} = 14 / 2\pi = 2.23 \text{ rev.}$$

$$v_B = r \omega = 125 \times 10 = 1250 \text{ mm/s} \uparrow$$

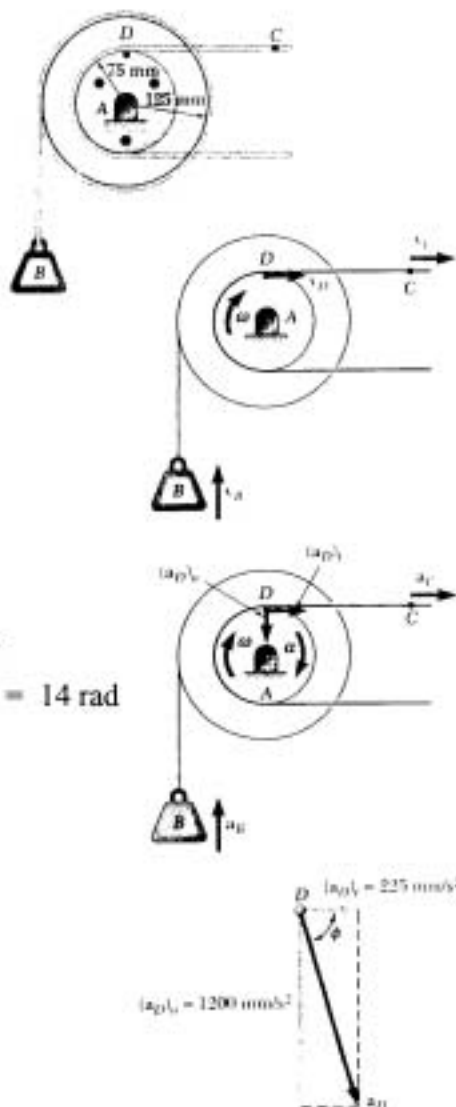
$$\Delta y_B = r \theta = 125 \times 14 = 1750 \text{ mm}$$

at $t = 0$

$$(a_D)_0 = 225 \text{ mm/s}^2$$

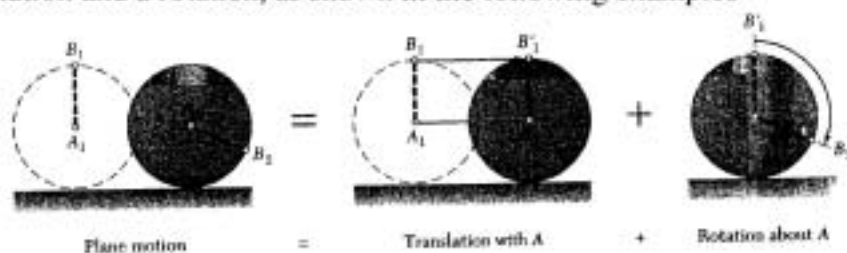
$$(a_D)_n = r_D \omega^2 = 1200 \text{ mm/s}^2$$

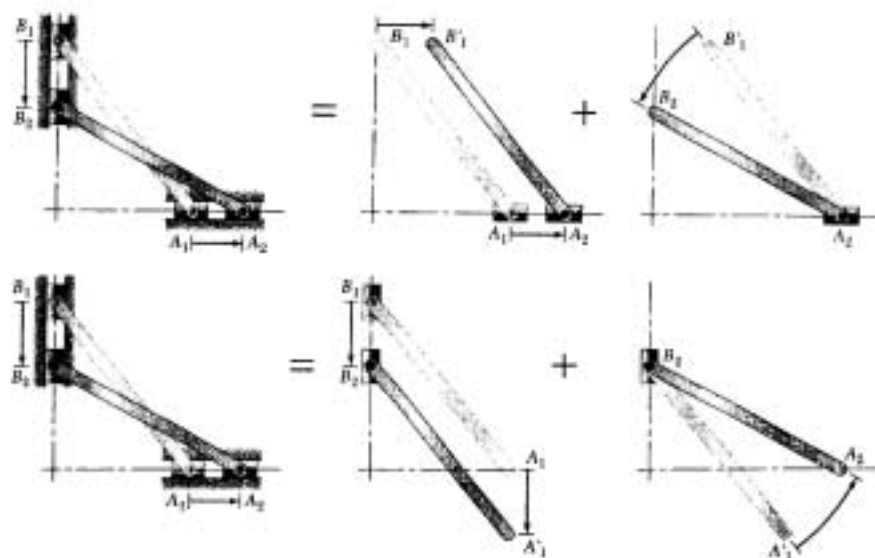
$$\text{i.e. } a_D = 1220 \text{ mm/s}^2 \searrow 79.4^\circ$$



15.5 General Plane Motion

a general plane motion can always be considered as the sum of a translation and a rotation, as shown in the following examples

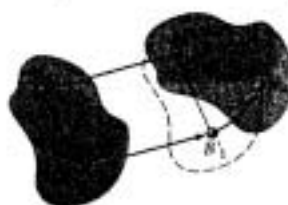




in the general case of motion

$$A_1B_1 \Rightarrow A_2B_1' \quad \text{by translation}$$

$$A_2B_1' \Rightarrow A_2B_2 \quad \text{by rotation}$$



15.6 Absolute and Relative Velocity in Plane Motion

the absolute velocity v_B of a particle B is obtained

$$v_B = v_A + v_{B/A}$$

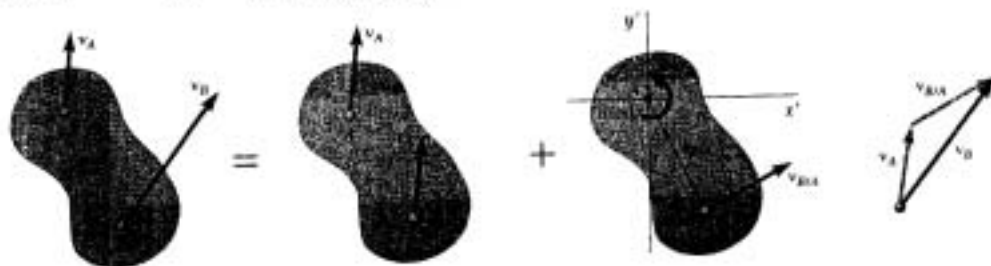
where

$$v_{B/A} = \omega \mathbf{k} \times \mathbf{r}_{B/A}$$

$$v_{B/A} = r \omega$$

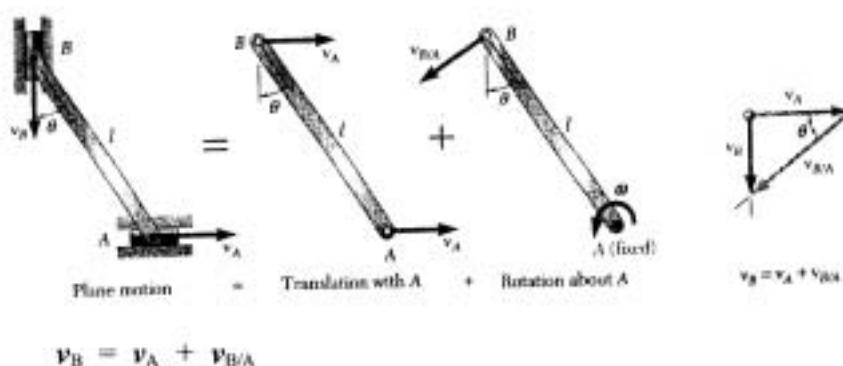
then

$$v_B = v_A + \omega \mathbf{k} \times \mathbf{r}_{B/A}$$



consider the rod AB , assuming that the velocity v_A is known, we want to find v_B and ω

take the reference point at A

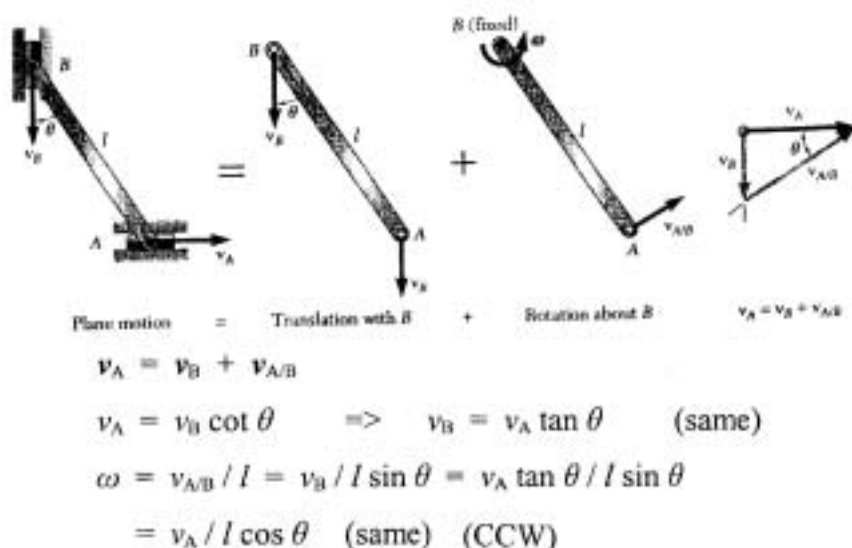


the directions of v_A , v_B and $v_{B/A}$ are all known

$$v_B = v_A \tan \theta$$

$$\omega = v_{B/A} / l = v_A / l \cos \theta \quad (\text{CCW})$$

take the reference point at B

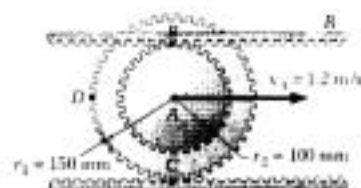


the angular velocity ω of a rigid body in plane motion is independent of the reference point

Sample Problem 15.2

$$v_A = 1.2 \text{ m/s} \rightarrow$$

determine ω , v_B , and v_D



after one revolution of the gear

$$x_A = 2\pi r_1 \quad \theta = 2\pi$$

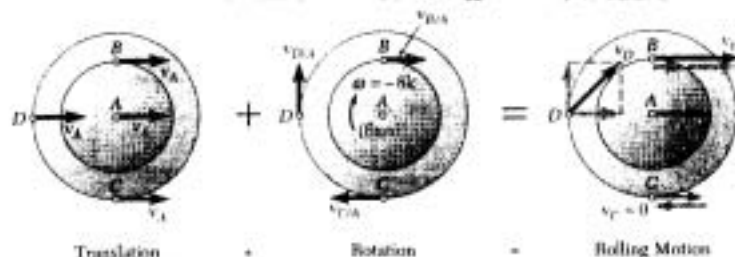
then $x_A / 2\pi r_1 = -\theta / 2\pi$

or $x_A = -r_1 \theta$

and $v_A = -r_1 \dot{\theta} = -r_1 \omega$

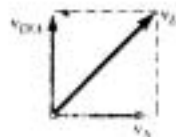
$$1.2 = -0.15 \omega$$

$$\omega = -8 \text{ rad/s} \quad \text{or} \quad \omega = -8 \mathbf{k} \text{ rad/s}$$



$$\begin{aligned} \mathbf{v}_B &= \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \omega \mathbf{k} \times \mathbf{r}_{B/A} \\ &= 1.2 \mathbf{i} + (-8 \mathbf{k}) \times (0.1 \mathbf{j}) = 2 \mathbf{i} = 2 \text{ m/s} \rightarrow \end{aligned}$$

$$\begin{aligned} \mathbf{v}_D &= \mathbf{v}_A + \mathbf{v}_{D/A} \\ &= 1.2 \mathbf{i} + (-8 \mathbf{k}) \times (-1.5 \mathbf{i}) = 1.2 \mathbf{i} + 1.2 \mathbf{j} \\ &= 1.697 \text{ m/s} \angle 45^\circ \end{aligned}$$



Sample Problem 15.3

$\omega_{AB} = 200 \text{ rpm clockwise}$

determine ω_{BD} and $\mathbf{v}_P$



$$\omega_{AB} = 200 \times 2\pi / 60 = 209 \text{ rad/s (CW)}$$

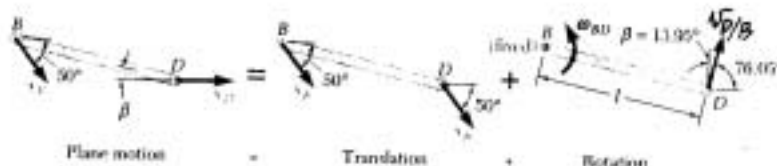
$$\mathbf{v}_B = (AB) \omega_{AB} = 75 \times 209 = 15.71 \text{ m/s} \angle 50^\circ$$

compute the angle β

$$75 \sin 40^\circ = 200 \sin \beta \quad \beta = 13.95^\circ$$

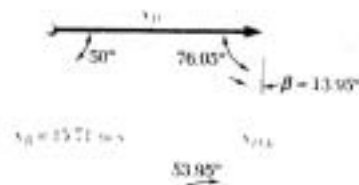
$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$





by the law of sine

$$\frac{v_D}{\sin 53.9^\circ} = \frac{v_B}{\sin 76.1^\circ} = \frac{v_{D/B}}{\sin 50^\circ}$$



with $v_B = 15.71$ m/s known

$$v_D = 13.09 \text{ m/s} \quad v_{D/B} = 12.40 \text{ m/s}$$

$$v_P = v_D = 13.09 \text{ m/s} \rightarrow v_{D/B} = 12.40 \text{ m/s} \angle 76.05^\circ$$

$$v_{D/B} = (BD) \omega_{BD}$$

$$\omega_{BD} = v_{D/B} / (BD) = 12.40 / 0.2 = 62.0 \text{ rad/s} \quad (\text{CCW})$$

15.7 Instantaneous Center of Rotation in Plane Motion

consider the general plane motion in slab, for a given time, the velocities of the various particles of the slab are the same if the slab were rotating about a certain axis perpendicular to the plane of slab is called **instantaneous axis of rotation** which intersect the slab at a point C , is called **instantaneous center of rotation**

$$\therefore v_B = v_A + \omega \mathbf{k} \times \mathbf{r}_{B/A}$$

a general plane motion may be described by a translation and rotation, it may be described by v_A and ω which rotates about C



if v_A and ω are given, these velocities could be obtained by letting the slab rotate with the angular velocity ω about a point C located on the perpendicular to v_A at distance $r = v_A / \omega$

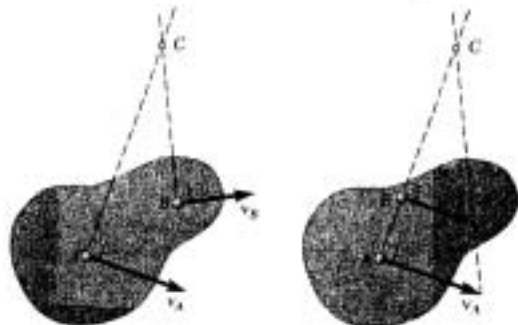
$$v_A \perp AC \quad r = AC = v_A / \omega$$

the slab seems to rotate about the instantaneous center C

the position of the instantaneous center can be defined in two other ways

1. intersection of two lines perpendicular to v_A and v_B

2. if $v_A \parallel v_B$, and $AB \perp v_A$, C can be found by intersecting the line AB with the line joining the extremities of the vectors v_A and v_B



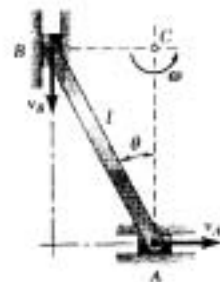
the instantaneous center C may be located either on or outside the slab

e.g. consider the rod AB

C can easily be located (C is not inside AB)

$$\omega = v_A / AC = v_B / BC$$

$$v_B = v_A BC / AC = v_A \tan \theta$$



if C located on slab (say C'), then $v_{C'} = 0$ at this instant t (only at this given instant)

since the instantaneous center will move with time

$$\therefore v_{C'} = 0 \text{ at } t \quad \text{but} \quad v_{C'} \neq 0 \text{ at } t + \Delta t$$

thus $a_{C'} \neq 0$

but the acceleration at this point cannot be determined directly

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_t + \mathbf{a}_n = r \alpha \mathbf{i}_t - v^2 / \rho \mathbf{i}_n \\ &= 0 \mathbf{i}_t - 0 / 0 \mathbf{i}_n \quad (\text{no meaning}) \end{aligned}$$

space centroid : instantaneous center describes one curve in space

body centroid : the locus of the position of instantaneous center on the body

two curves are tangent at point C at instant
body centroid curve rolls on the space
centroid curve during motion of the body



Sample Problem 15.4

$$v_A = 1.2 \text{ m/s} \rightarrow$$

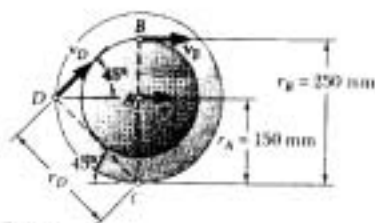
determine ω , v_R , and v_D

point C is the instantaneous center

$$\omega = v_A / AC = 1.2 / 0.15 = 8 \text{ rad/s (CW)}$$

$$v_R = v_B = r_B \omega = 0.25 \times 8 = 2 \text{ m/s} \rightarrow$$

$$v_D = r_D \omega = 0.212 \times 8 = 1.697 \text{ m/s} \angle 45^\circ$$



Sample Problem 15.5

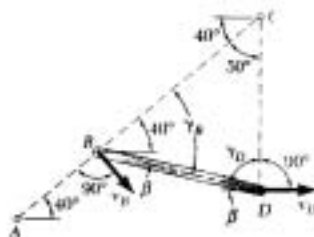
$$\omega_{AB} = 200 \text{ rpm clockwise}$$

determine ω_{BD} and v_P

$$\omega_{AB} = 209 \text{ rad/s}$$

$$v_B = 15.71 \text{ m/s} \searrow 50^\circ$$

$$\beta = 13.95^\circ$$



from the directions of v_B and v_D , C is located

$$\text{in } \triangle BCD, \gamma_B = 40^\circ + \beta = 53.9^\circ \quad \gamma_D = 90^\circ - \beta = 76.1^\circ$$

with $BD = 200 \text{ mm}$, BC and CD can be found

$$\frac{BC}{\sin 76.1^\circ} = \frac{CD}{\sin 53.9^\circ} = \frac{200}{\sin 90^\circ}$$

$$BC = 253.4 \text{ mm} \quad CD = 211.1 \text{ mm}$$

$$v_B = \omega_{BD} BC \quad \omega_{BD} = 62.0 \text{ rad/s clockwise}$$

$$v_D = \omega_{BD} CD = 62.0 \times 0.2111 = 13.09 \text{ m/s} \rightarrow$$

15.8 Absolute and Relative Acceleration in Plane Motion

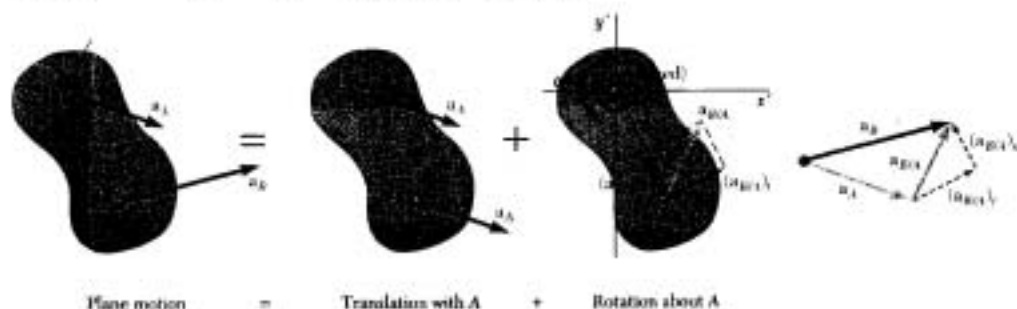
$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

$$\mathbf{a}_{B/A} = (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

$$(\mathbf{a}_{B/A})_t = \alpha \mathbf{k} \times \mathbf{r}_{B/A} \quad (\mathbf{a}_{B/A})_t = \alpha r$$

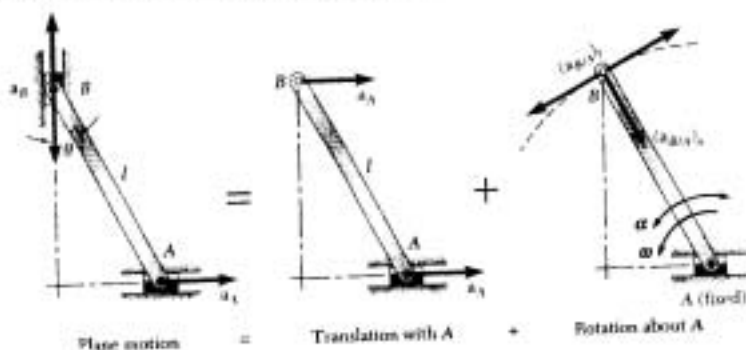
$$(\mathbf{a}_{B/A})_n = -\omega^2 \mathbf{r}_{B/A} \quad (\mathbf{a}_{B/A})_n = r \omega^2$$

thus
$$\mathbf{a}_B = \mathbf{a}_A + \alpha \mathbf{k} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$



e.g. consider the rod AB , with l , θ , $\mathbf{v}_A$ and $\mathbf{a}_A$ given

how to determine $\mathbf{a}_{AB}$ and $\mathbf{a}_B$



l , θ , $\mathbf{v}_A \rightarrow \mathbf{v}_B$ and ω_{AB}
(method of I. C.)

ω_{AB} and $\mathbf{r}_{B/A} \rightarrow (\mathbf{a}_{B/A})_n$

$\mathbf{a}_A$, $(\mathbf{a}_{B/A})_n$ and the directions of $\mathbf{a}_B$, $(\mathbf{a}_{B/A})_t$
 $\rightarrow$ magnitudes of $\mathbf{a}_B$, $(\mathbf{a}_{B/A})_t$ by vector polygon

$(\mathbf{a}_{B/A})_t$ and $l \rightarrow \alpha_{AB}$

in the vector polygon of accelerations

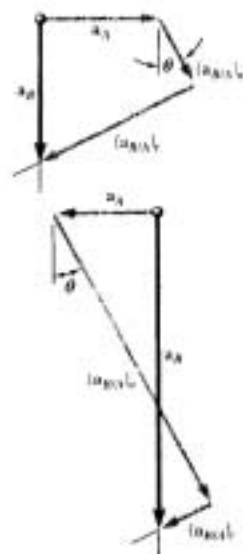
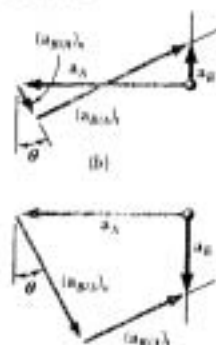
$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

$$x\text{-component} \quad 0 = a_A + l \omega^2 \sin \theta - l a \cos \theta$$

$$y\text{-component} \quad -a_B = -l \omega^2 \cos \theta - l a \sin \theta$$

two unknowns a_B and a can be solved

other three cases of vector polygons can be solved in the similarly way



15.9 Analysis of Plane Motion in Terms of a Parameter

consider the same example in section 15.8

$$x_A = l \sin \theta \quad y_B = l \cos \theta$$

$$v_A = \dot{x}_A = -l \cos \theta \dot{\theta}$$

$$a_A = \dot{v}_A = -l \ddot{\theta} \sin \theta + l \dot{\theta}^2 \cos \theta$$

$$v_B = \dot{y}_B = -l \sin \theta \dot{\theta}$$

$$a_B = \dot{v}_B = -l \ddot{\theta} \cos \theta - l \dot{\theta}^2 \sin \theta$$

$$\therefore \quad \dot{\theta} = \omega \quad \ddot{\theta} = \alpha$$

$$\text{then} \quad v_A = -l \cos \theta \omega \quad v_B = -l \sin \theta \omega$$

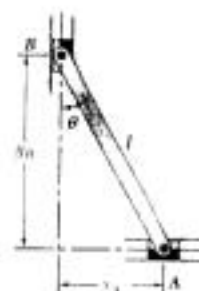
$$\text{and} \quad \omega = v_A / l \cos \theta \quad \text{and} \quad v_B = v_A \tan \theta \quad \text{can be solved}$$

another two equations

$$a_A = -l \omega^2 \sin \theta + l a \cos \theta$$

$$a_B = -l \omega^2 \cos \theta - l a \sin \theta$$

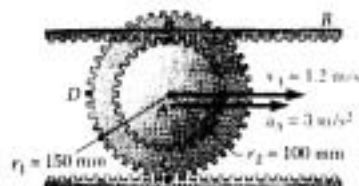
two unknowns a_B and a can also be solved



Sample Problem 15.6

$$v_A = 1.2 \text{ m/s} \rightarrow a_A = 3 \text{ m/s}^2 \rightarrow$$

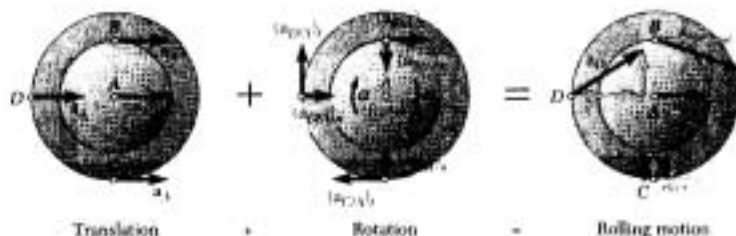
determine a , a_B , a_C and a_D



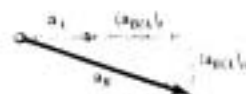
$$x_A = -r_1 \theta$$

$$v_A = -r_1 \omega \quad \omega = -8 \text{ rad/s}$$

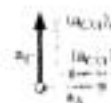
$$a_A = -r_1 \alpha \quad \alpha = -20 \text{ rad/s}^2$$



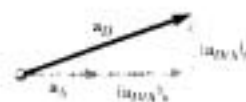
$$\begin{aligned} \mathbf{a}_B &= \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n \\ &= 3 \mathbf{i} - 20 \mathbf{k} \times 0.1 \mathbf{j} - 8^2 \times 0.1 \mathbf{j} \\ &= 5 \mathbf{i} - 6.4 \mathbf{j} \text{ (m/s}^2\text{)} \end{aligned}$$



$$\begin{aligned} \mathbf{a}_C &= \mathbf{a}_A + (\mathbf{a}_{C/A})_t + (\mathbf{a}_{C/A})_n \\ &= 3 \mathbf{i} - 20 \mathbf{k} \times (-0.15 \mathbf{j}) - 8^2 \times 0.15 \mathbf{j} \\ &= -9.6 \mathbf{j} \text{ (m/s}^2\text{)} \end{aligned}$$



$$\begin{aligned} \mathbf{a}_D &= \mathbf{a}_A + (\mathbf{a}_{D/A})_t + (\mathbf{a}_{D/A})_n \\ &= 3 \mathbf{i} - 20 \mathbf{k} \times (-0.15 \mathbf{i}) - 8^2 \times (-0.15 \mathbf{j}) \\ &= 12.6 \mathbf{i} - 3 \mathbf{j} \text{ (m/s}^2\text{)} \end{aligned}$$



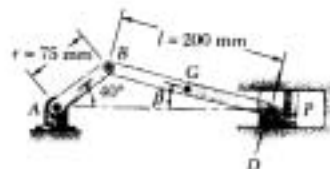
Sample Problem 15.7

$$\omega_{AB} = 2000 \text{ rpm} = 209.4 \text{ rad/s clockwise}$$

$$\omega_{BD} = 62.0 \text{ rad/s counterclockwise}$$

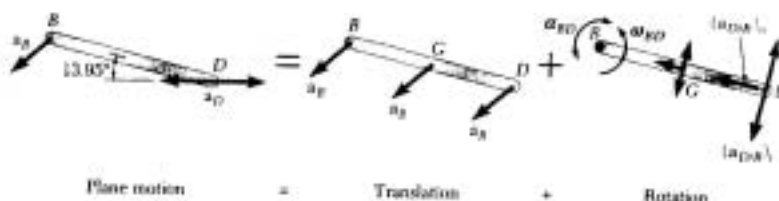
$$\beta = 13.95^\circ$$

determine a_{BD} and a_D



$$\mathbf{a}_B = \omega^2 AB = 209.4^2 \times 0.075 = 3289 \text{ m/s}^2 \angle 40^\circ$$





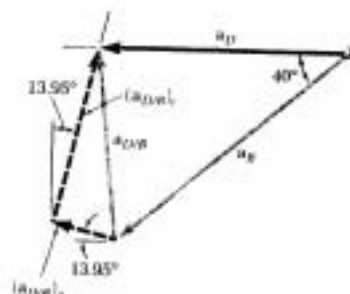
$$\mathbf{a}_D = \mathbf{a}_B + (\mathbf{a}_{D/B})_t + (\mathbf{a}_{D/B})_n$$

$$\mathbf{a}_D = \mathbf{a}_D \longleftrightarrow$$

$$(\mathbf{a}_{D/B})_t = 0.2 a_{BD} \nearrow \text{or} \nwarrow 76.1^\circ$$

$$(\mathbf{a}_{D/B})_n = 62^2 \times 0.2 = 769 \searrow 13.95^\circ$$

$$\mathbf{a}_D \longleftrightarrow = 3289 \swarrow 40^\circ + 0.2 a_{BD} \nearrow 76.1^\circ + 769 \searrow 13.95^\circ$$



x-comp

$$-a_D = -3289 \cos 40^\circ - 769 \cos 13.95^\circ + 0.2 a_{BD} \sin 76.1^\circ$$

y-comp

$$0 = -3289 \sin 40^\circ + 769 \sin 13.95^\circ + 0.2 a_{BD} \cos 76.1^\circ$$

solving the equations and obtained

$$a_{BD} = 9940 \text{ rad/s}^2 \text{ counterclockwise}$$

$$a_D = 2787 \text{ m/s}^2 \leftarrow$$

Sample Problem 15.8

linkage ABDE

$$\omega_1 = 20 \text{ k rad/s} = \text{constant (CCW)}$$

determine ω_{BD} , ω_{DE} , a_{BD} and a_{DE}

$$\omega_{BD} = \omega_{BD} \mathbf{k} \quad \omega_{DE} = \omega_{DE} \mathbf{k}$$

$$\mathbf{r}_{B/A} = 200 \mathbf{i} + 350 \mathbf{j} \text{ (mm)}$$

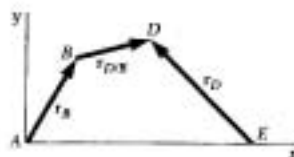
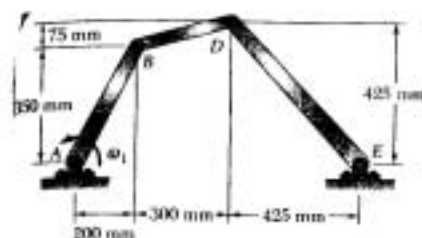
$$\mathbf{r}_{D/B} = 300 \mathbf{i} + 75 \mathbf{j} \text{ (mm)}$$

$$\mathbf{r}_{D/E} = -425 \mathbf{i} + 425 \mathbf{j} \text{ (mm)}$$

$$\alpha_{AB} = 0$$

$$\alpha_{BD} = a_{BD} \mathbf{k}$$

$$\alpha_{DE} = a_{DE} \mathbf{k}$$



$$\mathbf{v}_D = \mathbf{v}_E + \mathbf{v}_{D/E} = \mathbf{v}_B + \mathbf{v}_{D/B}$$

$$\omega_{DE} \mathbf{k} \times \mathbf{r}_{D/E} = \omega_{AB} \mathbf{k} \times \mathbf{r}_{B/A} + \omega_{DB} \mathbf{k} \times \mathbf{r}_{D/B}$$

with $\omega_{AB} = 20 \text{ rad/s}$, two unknowns ω_{BD} and ω_{DE} can be solved by the above vector equation (two scalar equations)

$$\omega_{BD} = -29.33 \text{ rad/s} \quad (\text{clockwise})$$

$$\omega_{DE} = 11.29 \text{ rad/s} \quad (\text{counterclockwise})$$

$$\mathbf{a}_D = \mathbf{a}_E + \mathbf{a}_{D/E} = \mathbf{a}_B + \mathbf{a}_{D/B}$$

$$\omega_{DE} \mathbf{k} \times \mathbf{r}_{D/E} - \omega_{DE}^2 \mathbf{r}_{D/E} = -\omega_{AB}^2 \mathbf{r}_{B/A} + \omega_{BD} \mathbf{k} \times \mathbf{r}_{D/B} - \omega_{BD}^2 \mathbf{r}_{D/B}$$

two unknowns a_{BD} and a_{DE} can be solved by the above vector equation (two scalar equations)

$$a_{BD} = -645 \text{ rad/s}^2 \quad (\text{clockwise})$$

$$a_{DE} = 809 \text{ rad/s}^2 \quad (\text{counterclockwise})$$

15.10 Rate of Change of a Vector with Respect to a Rotating Frame

$OXYZ$: frame of reference

$Oxyz$: rotating frame with rotating axis OA

consider a function $\mathbf{Q} = \mathbf{Q}(t)$

define $(\dot{\mathbf{Q}})_{XYZ}$: rate of change of $\mathbf{Q}$ w.r.t. $OXYZ$

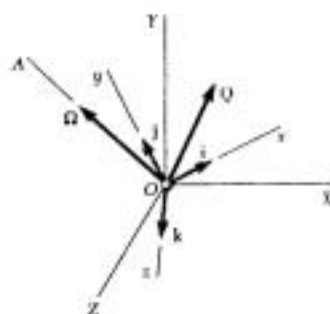
$(\dot{\mathbf{Q}})_{xyz}$: rate of change of $\mathbf{Q}$ w.r.t. $Oxyz$

$$\mathbf{Q} = Q_x \mathbf{i} + Q_y \mathbf{j} + Q_z \mathbf{k}$$

$$(\dot{\mathbf{Q}})_{xyz} = \dot{Q}_x \mathbf{i} + \dot{Q}_y \mathbf{j} + \dot{Q}_z \mathbf{k}$$

$$\text{and } (\dot{\mathbf{Q}})_{XYZ} = \dot{Q}_x \mathbf{i} + \dot{Q}_y \mathbf{j} + \dot{Q}_z \mathbf{k} + Q_x \frac{d\mathbf{i}}{dt} + Q_y \frac{d\mathbf{j}}{dt} + Q_z \frac{d\mathbf{k}}{dt}$$

$$(\dot{\mathbf{Q}})_{xyz} = 0 \text{ if } \mathbf{Q} \text{ is fixed on } Oxyz$$



$(\dot{\mathbf{Q}})_{xyz}$: the velocity of particle located at the tip of $\mathbf{Q}$ and belonging to a body rigidly attached to the frame $Oxyz$

if $\mathbf{Q}$ is fixed on $Oxyz$

$$(\dot{\mathbf{Q}})_{xyz} = Q_x \frac{d\mathbf{i}}{dt} + Q_y \frac{d\mathbf{j}}{dt} + Q_z \frac{d\mathbf{k}}{dt} = \mathbf{v} = \boldsymbol{\Omega} \times \mathbf{Q}$$

then in general

$$(\dot{\mathbf{Q}})_{xyz} = (\dot{\mathbf{Q}})_{xyz} + \boldsymbol{\Omega} \times \mathbf{Q}$$

the last term $\boldsymbol{\Omega} \times \mathbf{Q}$ is induced by rotation of $Oxyz$

15.11 Plane Motion of a Particle Relative to a Rotating Frame, Coriolis

Acceleration

consider two frames of reference on a plane

OXY : fixed frame

Oxy : rotating frame

let $\mathbf{v}_P$ = absolute velocity of P

then

$$\begin{aligned} \mathbf{v}_P &= (\dot{\mathbf{r}})_{OXY} = (\dot{\mathbf{r}})_{Oxy} + \boldsymbol{\Omega} \times \mathbf{r} \\ &= \mathbf{v}_{P/F} + \mathbf{v}_{P'} \end{aligned}$$

$\mathbf{v}_{P'}$: velocity of point P' on the moving frame which coincide with P at the instant

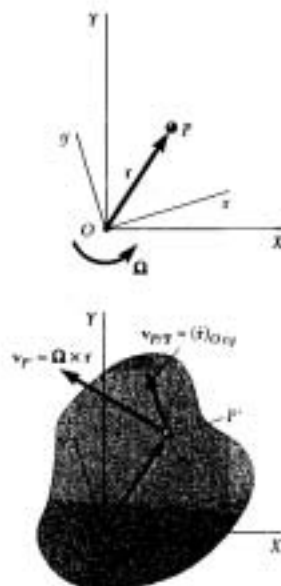
$\mathbf{v}_{P/F}$: velocity of P relative to Oxy

let $\mathbf{a}_P$ be the absolute acceleration

$$\mathbf{a}_P = \dot{\mathbf{v}}_P = \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times \dot{\mathbf{r}} + \frac{d}{dt} [(\dot{\mathbf{r}})_{Oxy}]$$

$$\frac{d}{dt} [(\dot{\mathbf{r}})_{Oxy}] = [(\dot{\mathbf{r}})_{Oxy}]_{OXY} = (\dot{\mathbf{r}})_{Oxy} + \boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxy}$$

$$\dot{\mathbf{r}} = (\dot{\mathbf{r}})_{OXY} = (\dot{\mathbf{r}})_{Oxy} + \boldsymbol{\Omega} \times \mathbf{r}$$



$$\begin{aligned}
 \text{then } \mathbf{a}_P &= \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2 \boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{OXY} + (\ddot{\mathbf{r}})_{OXY} \\
 &= (\mathbf{a}_{P'})_i + (\mathbf{a}_{P'})_n + \mathbf{a}_C + \mathbf{a}_{P/F} \\
 \mathbf{a}_P &= \mathbf{a}_{P'} + \mathbf{a}_{P/F} + \mathbf{a}_C
 \end{aligned}$$

$\mathbf{a}_{P'}$: acceleration of point P' of the moving frame coincide with P at instant

$\mathbf{a}_{P/F}$: acceleration of P relative to moving frame

$\mathbf{a}_C$: $\mathbf{a}_C = 2 \boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{OXY} = 2 \boldsymbol{\Omega} \times (\mathbf{v}_{P/F})$ is the complementary or Coriolis acceleration

$\mathbf{a}_C \perp \boldsymbol{\Omega}$ and $\mathbf{v}_{P/F}$, $\mathbf{a}_C = 0$ when $\boldsymbol{\Omega}$ or $\mathbf{v}_{P/F} = 0$

consider a collar P which is made to slide along a rotating rod OB

time from t to $t + \Delta t$

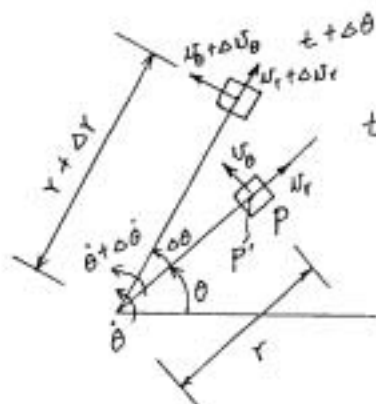
$$r \rightarrow r + \Delta r$$

$$\theta \rightarrow \theta + \Delta \theta$$

$$\dot{\theta} \rightarrow \dot{\theta} + \Delta \dot{\theta}$$

$$v_r \rightarrow v_r + \Delta v_r$$

$$v_\theta \rightarrow v_\theta + \Delta v_\theta$$



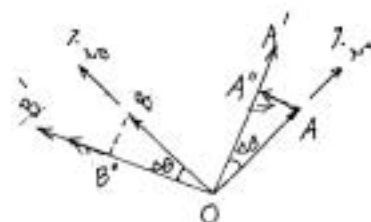
for the velocity vector polygon

$$OA = v_r = \dot{r}$$

$$OB = v_\theta = r \dot{\theta}$$

$$OA' = v_r + \Delta v_r$$

$$\begin{aligned}
 OB' &= v_\theta + \Delta v_\theta = (r + \Delta r)(\dot{\theta} + \Delta \dot{\theta}) \\
 &= r \dot{\theta} + \Delta r \dot{\theta} + r \Delta \dot{\theta} + \Delta r \Delta \dot{\theta}
 \end{aligned}$$



$$\text{at time } t \quad \mathbf{v} = \mathbf{OA} + \mathbf{OB}$$

$$\text{at time } t + \Delta t \quad \mathbf{v} + \Delta \mathbf{v} = \mathbf{OA'} + \mathbf{OB'}$$

$$\begin{aligned}
 \text{then } \Delta \mathbf{v} &= (\mathbf{OA'} + \mathbf{OA}) + (\mathbf{OB'} + \mathbf{OB}) \\
 &= \mathbf{AA''} + \mathbf{A''A'} + \mathbf{BB''} + \mathbf{B''B'}
 \end{aligned}$$

$$AA'' = OA \Delta\theta \mathbf{i}_\theta = \dot{r} \Delta\theta \mathbf{i}_\theta$$

$$A''A' = \Delta v_t \mathbf{i}_t$$

$$BB'' = -OB \Delta\theta \mathbf{i}_\theta = -r \dot{\theta} \Delta\theta \mathbf{i}_t$$

$$B''B' = (\Delta r \dot{\theta} + r \Delta\dot{\theta}) \mathbf{i}_\theta$$

$$\Delta \mathbf{v} = (\Delta v_t - r \dot{\theta} \Delta\theta) \mathbf{i}_t + (\dot{r} \Delta\theta + \Delta r \dot{\theta} + r \Delta\dot{\theta}) \mathbf{i}_\theta$$

take $\Delta t \rightarrow 0$

$$\mathbf{a}_P = \lim \frac{\Delta \mathbf{v}}{\Delta t} = \left(\frac{dv_t}{dt} - r \dot{\theta} \frac{d\theta}{dt} \right) \mathbf{i}_t + \left(\dot{r} \frac{d\theta}{dt} + \frac{dr}{dt} \dot{\theta} + r \frac{d\dot{\theta}}{dt} \right) \mathbf{i}_\theta$$

$$= (\dot{r} - r \dot{\theta}^2) \mathbf{i}_t + (\dot{r} \dot{\theta} + 2 \dot{r} \dot{\theta}) \mathbf{i}_\theta$$

$$= (-r \dot{\theta}^2 \mathbf{i}_t + \dot{r} \dot{\theta} \mathbf{i}_\theta) + \dot{r} \dot{\theta} \mathbf{i}_t + 2 \dot{r} \dot{\theta} \mathbf{i}_\theta$$

$$= \mathbf{a}_{P'} + \mathbf{a}_{P''} + \mathbf{a}_C$$

$\mathbf{a}_C = 2 \dot{r} \dot{\theta} \mathbf{i}_\theta$ include two components :

1. the direction of relative velocity changed (AA'')
2. the distance from O is changed, thus the tangential velocity component changed ($B''B'$)

Sample Problem 15.9

$\omega_D = 10 \text{ rad/s}$ counterclockwise

$\theta = 135^\circ$ $R = 50 \text{ mm}$

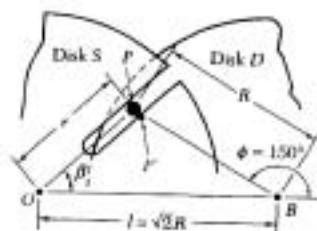
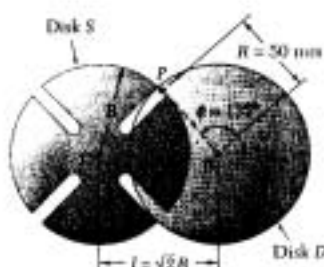
when $\phi = 150^\circ$, determine ω_S and $v_{P/S}$

$$r^2 = R^2 + l^2 - 2Rl \cos 30^\circ = 0.551 R^2$$

$$r = 37.1 \text{ mm}$$

$$\frac{50}{\sin \beta} = \frac{37.1}{\sin 30^\circ} \Rightarrow \beta = 42.4^\circ$$

$\therefore P$ is attached on D



$$\therefore v_P = R \omega_D = 50 \times 10 = 500 \text{ mm/s} \angle 60^\circ$$

from disk S

$$v_P = v_{P'} + v_{P/S}$$

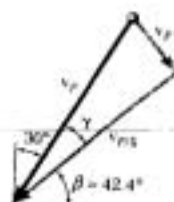
$$\gamma = 90^\circ - 30^\circ - 42.4^\circ = 17.6^\circ$$

$$v_{P'} \perp v_{P/S} \quad v_{P'} = v_P \sin \gamma = 151.2 \text{ mm/s} \angle 42.4^\circ$$

$$v_{P/S} = v_P \cos \gamma = 477 \text{ mm/s} \angle 42.4^\circ$$

$$\text{and} \quad v_P = r \omega_S$$

$$\omega_S = 4.08 \text{ rad/s clockwise}$$



Sample Problem 15.10

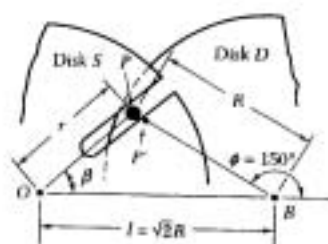
same condition as state in 15.9

determine a_S

from sample problem 15.9

$$\omega_S = 4.08 \text{ rad/s clockwise}$$

$$\beta = 42.4^\circ \quad v_{P/S} = 477 \text{ mm/s} \angle 42.4^\circ$$



$$\therefore \quad a_P = a_{P'} + a_{P/S} + a_C$$

$$a_P = (a_P)_n = -R \omega_D^2 i_n = 5000 \text{ mm/s}^2 \angle 30^\circ$$

[$\because \omega_D = \text{constant}, (a_P)_t = 0$]

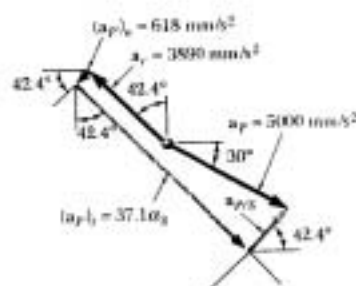
$$a_{P'} = (a_{P'})_t + (a_{P'})_n = r a_S i_t - r \omega_S^2 i_n$$

$$= 37.1 a_S i_t - 618 i_n$$

$$a_{P/S} = a_{P/S} \angle \text{ or } \angle 42.4^\circ$$

$$a_C = 2 \omega_S \times v_{P/S} = 2 \times 4.08 \times 447$$

$$= 3890 \text{ mm/s}^2 \angle 42.4^\circ$$



from the vector polygon diagram

$$(a_P)_t = a_C + a_P \cos 17.6^\circ$$

$$37.1 a_S = 3890 + 5000 \cos 17.6^\circ$$

$$a_S = 233 \text{ rad/s}^2 \text{ clockwise}$$

$$a_{P/S} = (a_P)_n + a_P \sin 17.6^\circ = 768 \text{ mm/s}^2 \angle 42.4^\circ$$

15.12 Motion about a Fixed Point

at instant, motion about a fixed point may be considered as a rotation of the body about an axis (called instantaneous axis of rotation) through O

consider the body rotates about O

$$(A_1B_1) \rightarrow (A_2B_2)$$

$$\text{but } A_1B_1 = A_2B_2 \quad (\text{rigid body})$$

let A_2 at B_1 , and

let C be the point of intersection of these two bisecting arcs, then

$$A_1C = A_2C = B_2C$$

$$\therefore A_1B_1 = A_2B_2$$

$$\therefore \triangle A_1CB_1 = \triangle A_2CB_2$$

denote $\theta = \angle A_1CB_1$

the motion of the sphere during Δt may be considered as a rotation through $\Delta\theta$ about OC

let $\Delta t \rightarrow 0$, OC is called the instantaneous axis of rotation

at this instant

$$\mathbf{v} = d\mathbf{r} / dt = \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a} = \mathbf{a} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where $\mathbf{a} = d\boldsymbol{\omega} / dt$



(a)



$\therefore \omega$ is change in direction and magnitude with time

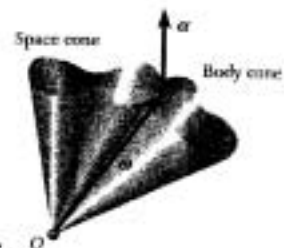
thus a is not directed along the instantaneous axis of rotation

a is tangent to the curve described in the space by the tip of vector ω

space cone : instantaneous axis of rotation describes on cone in space

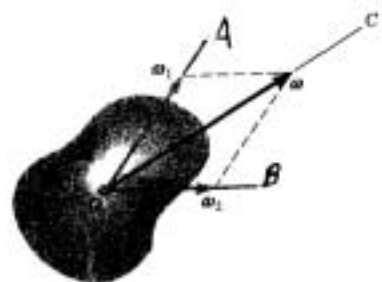
body cone : the locus of the position of instantaneous axis of rotation in the body

at any instant, two cones are tangent along the instantaneous axis of rotation, the body cone appears to roll on the space cone



the vector ω moves within the body as well as in space

consider a rigid body fixed at O , rotates simultaneously about OA and OB , may be considered equivalent a single rotation of $\vec{\omega}$



$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

let point P inside the body of position vector r

denote v_1 : velocity of P when the body rotates about OA

v_2 : velocity of P when the body rotates about OB

then $v_1 = \vec{\omega}_1 \times r$ $v_2 = \vec{\omega}_2 \times r$

and $v = v_1 + v_2 = (\vec{\omega}_1 + \vec{\omega}_2) \times r = \vec{\omega} \times r$

thus $\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$

15.13 General Motion

general motion = translation + rotation about a fixed point

$$r_B = r_A + r_{B/A}$$

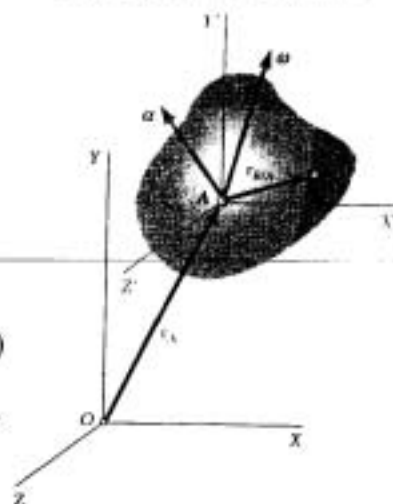
$$v_B = v_A + v_{B/A}$$

$$= \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

where $\boldsymbol{\omega}$ is the angular velocity of the body at instant

$$\begin{aligned}\mathbf{a}_B &= \mathbf{a}_A + \mathbf{a}_{B/A} \\ &= \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})\end{aligned}$$

where $\boldsymbol{\alpha}$ is the angular acceleration of the body at instant

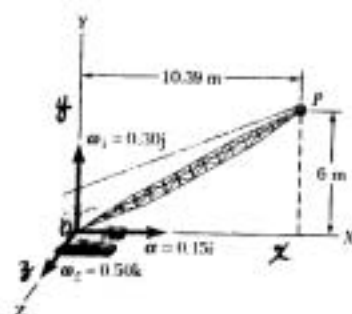


Sample Problem 15.11

$$\omega_1 = 0.3 \text{ rad/s}$$

$$\omega_2 = 0.5 \text{ rad/s relative to the cab } (OXYZ)$$

determine ω_{OP} , a_{OP} , $\mathbf{v}_P$ and $\mathbf{a}_P$



$$\omega_{OP} = \omega_1 + \omega_2 = 0.3 \mathbf{j} + 0.5 \mathbf{k} \text{ (rad/s)}$$

$$\begin{aligned}\mathbf{a}_{OP} &= [\dot{\omega}_{OP}]_{OXYZ} = [\dot{\omega}_1]_{OXYZ} + [\dot{\omega}_2]_{OXYZ} \\ &= 0 + (\dot{\omega}_2)_{OXYZ} + \omega_1 \times \omega_2 \\ &= 0.3 \mathbf{j} \times 0.5 \mathbf{k} = 0.15 \mathbf{i} \text{ (rad/s}^2\text{)}\end{aligned}$$

$$\begin{aligned}\mathbf{v}_P &= \omega_{OP} \times \mathbf{r}_P = (0.3 \mathbf{j} + 0.5 \mathbf{k}) \times (10.39 \mathbf{i} + 6 \mathbf{j}) \\ &= -3 \mathbf{i} + 5.2 \mathbf{j} - 3.12 \mathbf{k} \text{ (m/s)}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_P &= \mathbf{a}_{OP} \times \mathbf{r}_P + \omega_{OP} \times (\omega_{OP} \times \mathbf{r}_P) \\ &= -3.54 \mathbf{i} - 1.5 \mathbf{j} + 1.8 \mathbf{k} \text{ (m/s}^2\text{)}\end{aligned}$$

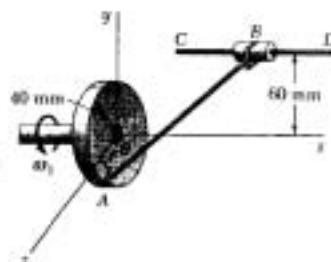
Sample Problem 15.12

$$\omega_1 = 12 \mathbf{i} \text{ rad/s} \quad CD \text{ on } xy \text{ plane}$$

determine $\mathbf{v}_B$ and ω_{AB} for $\theta = 0$

$$\mathbf{v}_A = \omega_1 \times \mathbf{r}_A = 12 \mathbf{i} \times 40 \mathbf{k} = -480 \mathbf{j} \text{ (mm/s)}$$

$$\mathbf{v}_B = v_B \mathbf{i} \quad \omega_{AB} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$$



$$v_B = v_A + \omega \times r_{B/A}$$

$$v_B i = -480 j + \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ 120 & 60 & -40 \end{vmatrix}$$

$$v_B = -40\omega_y - 60\omega_z \quad (1)$$

$$-480 = 40\omega_x + 120\omega_z \quad (2)$$

$$0 = 60\omega_x - 120\omega_y \quad (3)$$

3 equations for 4 unknowns

$$(1) \times 6 + (2) \times 3 + (3) \times (-2)$$

$$6v_B = -1440 \quad v_B = -240 i$$

$\therefore AB$ rotates about CD , also rotates about an axis $\perp$ plane containing AB and CD

$$\omega_1 \perp BE \text{ and } AB \quad \text{plane } ABE$$

$$\omega_2 \perp BE \text{ and } BF \quad \text{plane } BEF$$

$$\text{i.e. } \omega_1 \parallel BC \times BA$$

$$\omega_2 \parallel BC$$

thus ω in on the plane containing vectors $(BC \times BA)$ and BC

$$\text{i.e. } \omega \perp BC \times (BC \times BA)$$

$$\therefore BE \parallel BC \times (BC \times BA)$$

$$\text{thus } \omega \perp BE$$

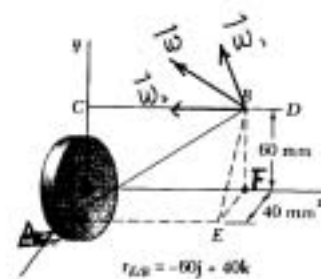
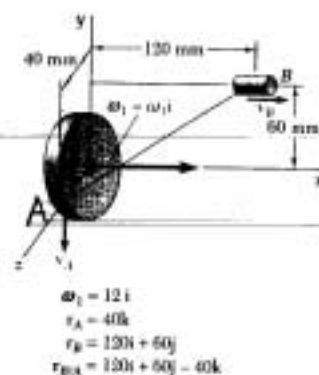
$$\text{or } \omega \cdot BE = 0$$

$$(\omega_x i + \omega_y j + \omega_z k) \cdot (-60 j + 40 k) = 0$$

$$-60\omega_y + 40\omega_z = 0 \quad (4)$$

from equations (1) to (4), it is obtained

$$\omega = 3.69 i + 1.846 j + 2.77 k \text{ (rad/s)}$$



15.14 Three-Dimensional Motion of a Particle Relative to a Rotating Frame, Coriolis Acceleration

$OXYZ$: fixed coordinate

$Oxyz$: rotating coordinate

OA : instantaneous axis of rotation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q}$$

$$\mathbf{v}_P = (\dot{\mathbf{r}})_{OXYZ} = (\dot{\mathbf{r}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{r}$$

$$= \mathbf{v}_{P/F} + \mathbf{v}_{P'}$$

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/F} + \mathbf{a}_C$$

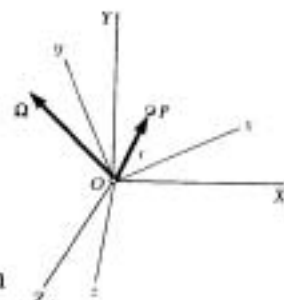
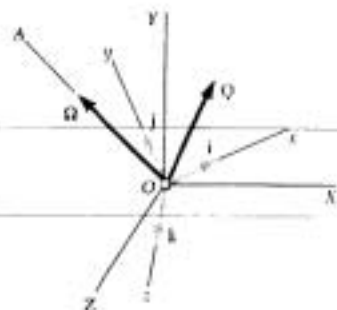
$$\mathbf{a}_{P'} = \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

$$\mathbf{a}_{P/F} = (\ddot{\mathbf{r}})_{Oxyz}$$

$$\mathbf{a}_C = 2 \boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxyz} \quad \text{Coriolis acceleration}$$

$$\mathbf{a}_C \perp \boldsymbol{\Omega} \text{ and } \mathbf{v}_{P/F} \text{ but } \boldsymbol{\Omega} \text{ not } \perp \mathbf{v}_{P/F} \text{ in general}$$

$$\text{thus } \mathbf{a}_C \neq 2 \boldsymbol{\Omega} \times \mathbf{v}_{P/F} \text{ in general}$$



15.15 Frame of Reference in General Motion

$OXYZ$: fixed coordinate

$AX'Y'Z'$: translation coordinate

$Axyz$: rotating coordinate

$\boldsymbol{\Omega}$: angular velocity of $Axyz$ at instant

$$\mathbf{r}_P = \mathbf{r}_A + \mathbf{r}_{P/A}$$

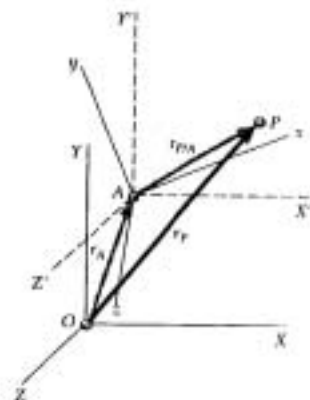
$$\mathbf{v}_P = \mathbf{v}_A + [\dot{\mathbf{r}}_{P/A}]_{AX'Y'Z'}$$

$$= \mathbf{v}_A + (\dot{\mathbf{r}}_{P/A})_{Axyz} + \boldsymbol{\Omega} \times \mathbf{r}_{P/A}$$

$$\mathbf{a}_P = [\dot{\mathbf{v}}_P]_{OXYZ} = \dot{\mathbf{v}}_A + [\dot{\mathbf{v}}_{P/A}]_{AX'Y'Z'}$$

$$= \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{P/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{P/A}) + (\ddot{\mathbf{r}}_{P/A})_{Axyz} + 2 \boldsymbol{\Omega} \times (\dot{\mathbf{r}}_{P/A})_{Axyz}$$

$$= \mathbf{a}_A + \mathbf{a}_{P'} + \mathbf{a}_{P/A} + \mathbf{a}_C$$



Sample Problem 15.13

$$\Omega = -20 \mathbf{j} \text{ rad/s} \quad \dot{\Omega} = -200 \mathbf{j} \text{ rad/s}^2$$

$$\mathbf{v}_{D/F} = 1.25 \text{ m/s} \nearrow 60^\circ \quad \mathbf{a}_{D/F} = 1.25 \text{ m/s}^2 \nearrow 60^\circ$$

$$OD = 200 \text{ mm}$$

determine $\mathbf{v}_D$ and $\mathbf{a}_D$

choose OAB as the rotating frame

$$\mathbf{r}_D = 0.2 (\sin 30^\circ \mathbf{i} + \cos 30^\circ \mathbf{j})$$

$$= 0.1 \mathbf{i} + 0.1732 \mathbf{j} \text{ (m)}$$

$$\mathbf{v}_{D/F} = 0.625 \mathbf{i} + 1.083 \mathbf{j} \text{ (m/s)}$$

$$\mathbf{a}_{D/F} = 7.5 \mathbf{i} + 12.99 \mathbf{j} \text{ (m/s}^2\text{)}$$

$$\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/F}$$

$$= -20 \mathbf{j} \times (0.1 \mathbf{i} + 0.1732 \mathbf{j}) + (0.625 \mathbf{i} + 1.083 \mathbf{j})$$

$$= 0.625 \mathbf{i} + 1.083 \mathbf{j} + 2 \mathbf{k}$$

$$\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/F} + \mathbf{a}_C$$

$$\mathbf{a}_{D'} = \dot{\Omega} \times \mathbf{r} + \Omega \times (\Omega \times \mathbf{r})$$

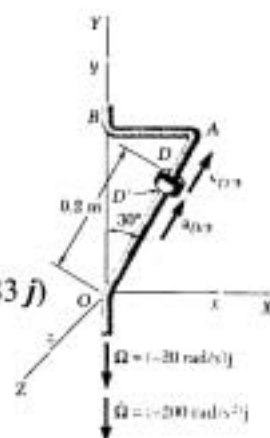
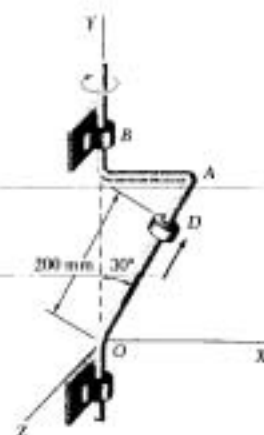
$$= -200 \mathbf{j} \times (0.1 \mathbf{i} + 0.1732 \mathbf{j}) - 20 \mathbf{j} \times 2 \mathbf{k}$$

$$= 40 \mathbf{i} + 20 \mathbf{k}$$

$$\mathbf{a}_C = 2 \Omega \times \mathbf{v}_{D/F} = 2 (-20 \mathbf{j}) \times (0.625 \mathbf{i} + 1.083 \mathbf{j})$$

$$= 25 \mathbf{k}$$

$$\mathbf{a}_D = -32.5 \mathbf{i} + 12.99 \mathbf{j} + 45 \mathbf{k} \text{ (m/s}^2\text{)}$$



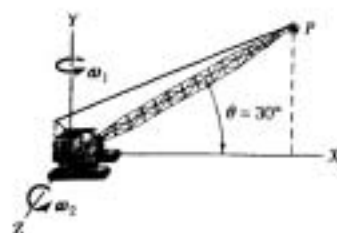
Sample Problem 15.14

$$\omega_1 = 0.3 \mathbf{j} = \Omega$$

$$\omega_2 = 0.5 \mathbf{k} = \omega_{B/F}$$

$$OP = 12 \text{ m}$$

determine $\mathbf{v}_P$ and $\mathbf{a}_P$



choose the cab as the rotating coordinate

then $\mathbf{r} = 10.39 \mathbf{i} + 6 \mathbf{j}$

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_1 = 0.3 \mathbf{j}$$

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P'/F}$$

$$= \boldsymbol{\Omega} \times \mathbf{r} + \boldsymbol{\omega}_{B/F} \times \mathbf{r}$$

$$= 0.3 \mathbf{j} \times (10.39 \mathbf{i} + 6 \mathbf{j}) + 0.5 \mathbf{k} \times (10.39 \mathbf{i} + 6 \mathbf{j})$$

$$= -3 \mathbf{i} + 5.2 \mathbf{j} - 3.12 \mathbf{k} \text{ (m/s)}$$

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P'/F} + \mathbf{a}_C$$

$$\mathbf{a}_{P'} = \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

$$= 0 + 0.3 \mathbf{j} \times 3.12 \mathbf{k} = -0.94 \mathbf{i}$$

$$\mathbf{a}_{P'/F} = \boldsymbol{\omega}_{B/F} \times (\boldsymbol{\omega}_{B/F} \times \mathbf{r})$$

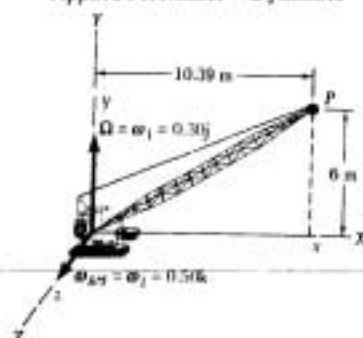
$$= 0.5 \mathbf{k} \times (-3 \mathbf{i} + 5.2 \mathbf{j}) = -2.6 \mathbf{i} - 1.5 \mathbf{j}$$

$$\mathbf{a}_C = 2 \boldsymbol{\Omega} \times \mathbf{v}_{P'/F}$$

$$= 2(0.3 \mathbf{j}) \times (-3 \mathbf{i} + 5.2 \mathbf{j}) = 1.8 \mathbf{k}$$

$$\mathbf{a}_P = -3.54 \mathbf{i} - 1.5 \mathbf{j} + 1.8 \mathbf{k} \text{ (m/s}^2\text{)}$$

the results are the same as in sample problem 15.11



Sample Problem 15.15

ω_1 and ω_2 are given as constants

determine $\mathbf{v}_P$, $\mathbf{a}_P$, and $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ of the disk

choose $Axyz$ as the rotating coordinate

then $\boldsymbol{\Omega} = \omega_1 \mathbf{j}$ and $\boldsymbol{\omega}_{D/F} = \omega_2 \mathbf{k}$

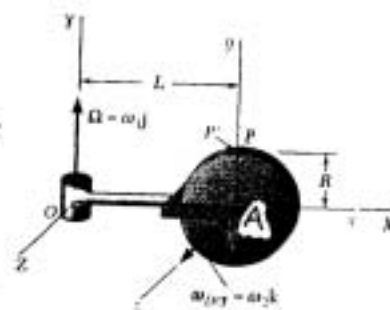
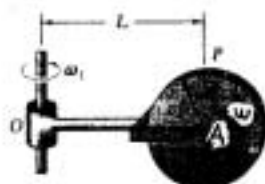
$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{P/A} = \mathbf{v}_A + (\mathbf{v}_{P'/A} + \mathbf{v}_{P'/F})_{Axyz}$$

$$\mathbf{v}_A = \boldsymbol{\omega}_1 \times \mathbf{OA} = \omega_1 \mathbf{j} \times L \mathbf{i} = -\omega_1 L \mathbf{k}$$

$$\mathbf{v}_{P'/A} = \boldsymbol{\Omega} \times \mathbf{AP}' = \omega_1 \mathbf{j} \times R \mathbf{j} = 0$$

$$\mathbf{v}_{P'/F} = \omega_2 \mathbf{k} \times R \mathbf{j} = -\omega_2 R \mathbf{i}$$

$$\mathbf{v}_P = -\omega_2 R \mathbf{i} - \omega_1 L \mathbf{k}$$



$$\mathbf{a}_P = [\mathbf{a}_P]_{OXYZ} + \mathbf{a}_{P/F} + \mathbf{a}_C = (\mathbf{a}_A + \mathbf{a}_{P'/F}) + \mathbf{a}_{P/F} + \mathbf{a}_C$$

$$\mathbf{a}_A + \mathbf{a}_{P'/F} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times L \mathbf{i}) + 0$$

$$= \omega_1 \mathbf{j} \times (-\omega_1 L \mathbf{k}) = -\omega_1^2 L \mathbf{i} = \mathbf{a}_A$$

$$\mathbf{a}_{P/F} = \omega_2 \times (\omega_2 \times AP)$$

$$= \omega_2 \mathbf{k} \times (-\omega_2 R \mathbf{i}) = -\omega_2^2 R \mathbf{j}$$

$$\mathbf{a}_C = 2 \boldsymbol{\Omega} \times \mathbf{v}_{P/F}$$

$$= 2 \omega_1 \mathbf{j} \times (-\omega_2 R \mathbf{i}) = 2 \omega_1 \omega_2 R \mathbf{k}$$

$$\mathbf{a}_P = -\omega_1^2 L \mathbf{i} - \omega_2^2 R \mathbf{j} + 2 \omega_1 \omega_2 R \mathbf{k}$$

$$\boldsymbol{\omega}_D = \boldsymbol{\Omega} + \boldsymbol{\omega}_{D/F} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k}$$

$$\dot{\boldsymbol{\omega}}_D = (\dot{\boldsymbol{\omega}}_D)_{OXYZ} = (\dot{\boldsymbol{\omega}}_D)_{Axyz} + \boldsymbol{\Omega} \times \boldsymbol{\omega}_D$$

$$= 0 + \omega_1 \mathbf{j} \times (\omega_1 \mathbf{j} + \omega_2 \mathbf{k})$$

$$= \omega_1 \omega_2 \mathbf{i}$$