Chapter 14 System of Particles

14.1 Introduction

system of particles

effective forces

mass center

variable system of particles

14.2 Application of Newton's Laws to the Motion of a System of Particles, Effective Forces

consider n particles in space

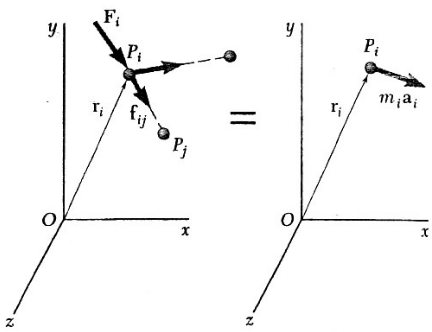
for the particle P_i, it has mass of m_i and acceleration $\boldsymbol{a}_i$

define $\boldsymbol{f}_{ij}$ the force exerted on P_i by another particles P_j of the system

$\boldsymbol{f}_{ij}$ is called internal force

thus the resultant of internal forces exerted on P_i is

$$R_i = \sum_{j=1}^{n} \boldsymbol{f}_{ij} \quad \text{with } \boldsymbol{f}_{ii} = 0$$



if $\boldsymbol{F}_i$ is the resultant of external force on P_i, by Newton's 2nd law for P_i, it is obtained

$$\boldsymbol{F}_i + \sum_{j=1}^{n} \boldsymbol{f}_{ij} = m_i \boldsymbol{a}_i \qquad (1)$$

if $\boldsymbol{r}_i$ is the position vector of P_i, then

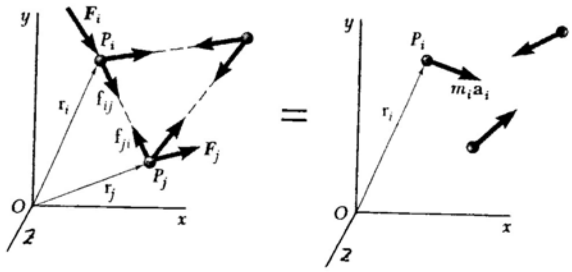
$$\boldsymbol{r}_i \times \boldsymbol{F}_i + \boldsymbol{r}_i \times \sum_{j=1}^{n} \boldsymbol{f}_{ij} = \boldsymbol{r}_i \times m_i \boldsymbol{a}_i \quad (2)$$

for $i = 1 \sim n$, n equations will be obtained for each particle

the vector $m_i \boldsymbol{a}_i$ are referred as the effective forces of the particles the internal forces $\boldsymbol{f}_{ij}$ always occur in pairs, $\boldsymbol{f}_{ij}$ and $\boldsymbol{f}_{ji}$

$\boldsymbol{f}_{ij}$: force exerted on P_i by P_j

$\boldsymbol{f}_{ji}$: force exerted on P_j by P_i



by Newton's 3rd law, we have

$$\boldsymbol{f}_{ij} + \boldsymbol{f}_{ji} = 0$$

and the sum of their moment about O is

$$\boldsymbol{r}_i \times \boldsymbol{f}_{ij} + \boldsymbol{r}_j \times \boldsymbol{f}_{ji} = \boldsymbol{r}_i \times (\boldsymbol{f}_{ij} + \boldsymbol{f}_{ji}) + (\boldsymbol{r}_j - \boldsymbol{r}_i) \times \boldsymbol{f}_{ji}$$

[$\boldsymbol{r}_j - \boldsymbol{r}_i$ and $\boldsymbol{f}_{ji}$ are collinear]

then $\boldsymbol{r}_i \times \boldsymbol{f}_{ij} + \boldsymbol{r}_j \times \boldsymbol{f}_{ji} = 0$

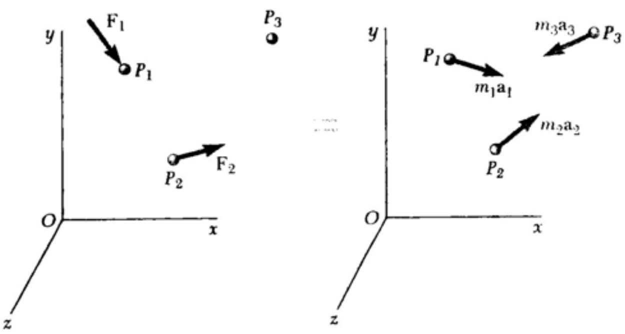
adding all internal forces of the system, and summing their moments about O, it is obtained

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{f}_{ij} = 0 \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} (\boldsymbol{r}_i \times \boldsymbol{f}_{ij}) = 0$$

summing the n equations of equations (1) and (2), it is obtained

$$\sum_{i=1}^{n} \boldsymbol{F}_i = \sum_{i=1}^{n} m_i \boldsymbol{a}_i \qquad (3)$$

$$\sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i = \sum_{i=1}^{n} (\boldsymbol{r}_i \times m_i \boldsymbol{a}_i) \qquad (4)$$



note that these two equations do not imply

$$\boldsymbol{F}_i = m_i \boldsymbol{a}_i \qquad \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{r}_i \times m_i \boldsymbol{a}_i$$

the system of the external forces acting on the particles and the system of effective forces of the particles are equipollent

$\Sigma\,\Sigma\,\boldsymbol{f}_{ij}$ express the fact that the system of the internal forces is equipollent to zero

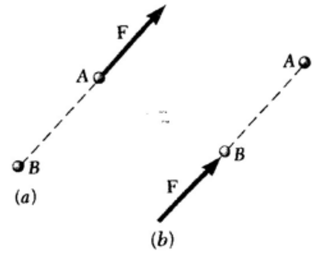
it does not mean $\boldsymbol{f}_{ij} = 0$ or $\Sigma\,\boldsymbol{f}_{ij} = 0$

it does not mean that the internal forces have no effect on the particles

equations (3) and (4) does not follow that two system of external forces which have same resultant and same moment resultant will have the same effect on a given system of particles

e.g. (a) A leave and B unaffected

(b) the force accelerates B and does not affect A



14.3 Linear and Angular Momentum of a System of Particles

defining $\quad \boldsymbol{L} \equiv \sum_{i=1}^{n} m_i \boldsymbol{v}_i$

$$\boldsymbol{H}_0 \equiv \sum_{i=1}^{n} (\boldsymbol{r}_i \times m_i \boldsymbol{v}_i)$$

where $\boldsymbol{L}$: linear momentum of the system of particles

$\boldsymbol{H}_0$: angular momentum of the system of particles

then $\quad \dot{\boldsymbol{L}} = \sum_{i=1}^{n} m_i \dot{\boldsymbol{v}}_i = \sum m_i \boldsymbol{a}_i$

$$\dot{\boldsymbol{H}}_0 = \sum_{i=1}^{n} (\dot{\boldsymbol{r}}_i \times m_i \boldsymbol{v}_i) + \sum (\boldsymbol{r}_i \times m_i \dot{\boldsymbol{v}}_i)$$

$$= \sum_{i=1}^{n} (\boldsymbol{v}_i \times m_i \boldsymbol{v}_i) + \sum_{i=1}^{n} (\boldsymbol{r}_i \times m_i \boldsymbol{a}_i) = \sum_{i=1}^{n} (\boldsymbol{r}_i \times \boldsymbol{F}_i)$$

thus it can be written

$$\sum \boldsymbol{F} = \dot{\boldsymbol{L}}$$

$$\sum \boldsymbol{M}_0 = \dot{\boldsymbol{H}}_0$$

the force resultant and the moment resultant about a fixed point O of the external forces are respectively equal to the rates of change of the linear momentum and angular momentum about O of the system of particles

14.4 Motion of the Mass Center of a System of Particles

if the mass center of the system is the point G is defined by the position vector $\underline{\boldsymbol{r}}(\underline{x}, \underline{y}, \underline{z})$, then

$$m\,\underline{\boldsymbol{r}} = \sum_{i=1}^{n} m_i\, \boldsymbol{r}_i$$

where $\quad m = \sum_{i=1}^{n} m_i \quad$ is the total mass of the system of particles

and $\quad m\,\underline{x} = \sum_{i=1}^{n} m_i\, x_i \qquad m\,\underline{y} = \sum_{i=1}^{n} m_i\, y_i \qquad m\,\underline{z} = \sum_{i=1}^{n} m_i\, z_i$

then $\quad m\,\dot{\underline{\boldsymbol{r}}} = \sum_{i=1}^{n} m_i\, \dot{\boldsymbol{r}}_i$

or $\quad m\,\underline{\boldsymbol{v}} = \sum_{i=1}^{n} m_i\, \boldsymbol{v}_i$

where $\underline{\boldsymbol{v}}$: velocity of the mass center G

hence $\quad \boldsymbol{L} = \sum_{i=1}^{n} m_i\, \boldsymbol{v}_i = m\,\underline{\boldsymbol{v}}$

and $\quad \dot{\boldsymbol{L}} = m\,\underline{\boldsymbol{a}}$

where $\underline{\boldsymbol{a}}$: acceleration of the mass center G

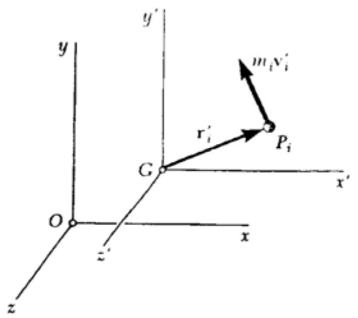
thus $\quad \Sigma\, \boldsymbol{F} = m\,\underline{\boldsymbol{a}}$

the mass center of a system of particles moves as if the entire mass of the system and all the external forces were concentrated at that point

14.5 Angular Momentum of a System of Particles About Its Mass Center

consider a moving frame $Gx'y'z'$ with its origin attached at the mass center of a system of particles G

denote $\boldsymbol{r}_i'$ and $\boldsymbol{v}_i'$ the position and velocity vectors of particle P_i relative to the moving frame, respectively



defined the angular momentum of the system about G by their relative velocities with respect to as

$$\boldsymbol{H}_G' = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times m_i \boldsymbol{v}_i')$$

$$\dot{\boldsymbol{H}}_G' = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times m_i \boldsymbol{a}_i') \qquad [\because \dot{\boldsymbol{r}}_i' \times m_i \boldsymbol{v}_i' = 0]$$

where $\boldsymbol{a}_i'$ is the acceleration of P_i relative to the moving frame

then $\qquad \boldsymbol{a}_i = \underline{\boldsymbol{a}} + \boldsymbol{a}_i'$

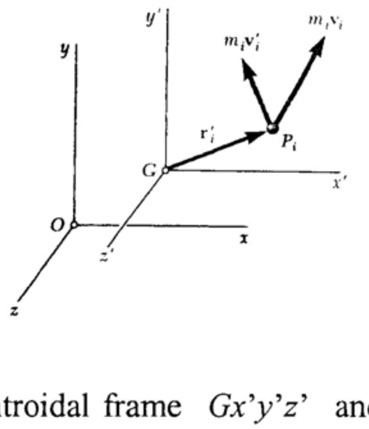
$$\therefore \quad \dot{\boldsymbol{H}}_G' = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times m_i \boldsymbol{a}_i) - (\sum_{i=1}^{n} m_i \boldsymbol{r}_i') \times \underline{\boldsymbol{a}}$$

$$\because \quad \sum_{i=1}^{n} m_i \boldsymbol{r}_i' = m \underline{\boldsymbol{r}}_i' \quad \text{but} \quad \underline{\boldsymbol{r}}_i' = 0 \qquad [G \text{ is the mass center}]$$

$$\therefore \quad \dot{\boldsymbol{H}}_G' = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times m_i \boldsymbol{a}_i) = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times \boldsymbol{F}_i) = \Sigma \boldsymbol{M}_G$$

the moment resultant about G of the external force is equal to the rate of change of the angular momentum about G of the system of particles

defined the angular momentum of the system about G by their absolute velocities as



$$\boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times m_i \boldsymbol{v}_i)$$

$$\because \quad \boldsymbol{v}_i = \underline{\boldsymbol{v}} + \boldsymbol{v}_i'$$

$$\therefore \quad \boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times m_i \boldsymbol{v}_i') + (\sum_{i=1}^{n} m_i \boldsymbol{r}_i') \times \underline{\boldsymbol{v}}$$

$$= \boldsymbol{H}_G'$$

note that this property is peculiar to the centroidal frame $Gx'y'z'$ and does not hole in general

so we can simplify to dropping the prime (')

$$\boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times m_i \boldsymbol{v}_i) = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times m_i \boldsymbol{v}_i')$$

i.e. $\Sigma \boldsymbol{M}_G = \dot{\boldsymbol{H}}_G = \dot{\boldsymbol{H}}_G'$

14.6 Conservation of Momentum for a System of Particles

if no external force acts on the particles of a system, then

$$\dot{\boldsymbol{L}} = 0 \qquad \dot{\boldsymbol{H}}_0 = 0$$

or $\boldsymbol{L}$ = constant $\quad \boldsymbol{H}_0$ = constant

if problem involving central forces, the moment about a fixed point O of each of the external force may be zero without any of forces being zero

$\boldsymbol{H}_0$ = constant still hole

if the sum of the external forces is zero, $\Sigma \boldsymbol{F} = 0$

$\boldsymbol{L}$ = constant = $m\,\boldsymbol{v}$

i.e. $\boldsymbol{v}$ = constant

that means the mass center G moves in a straight line and at constant speed

if the sum of moment at G by external forces is zero, $\Sigma \boldsymbol{M}_G = 0$

then $\boldsymbol{H}_G = \boldsymbol{H}_G'$ = constant

Sample Problem 14-1

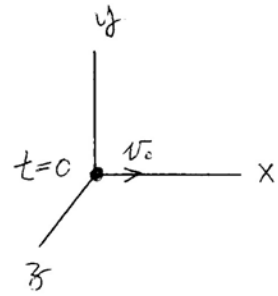
$m = 200$ kg $\qquad \boldsymbol{v}_0 = 150$ m/s $\boldsymbol{i}$ at $t = 0$

after explosion, it separates into 3 parts

$m_A = 100$ kg $\qquad m_B = 60$ kg $\qquad m_C = 40$ kg

at $t = 2.5$ s $\qquad \boldsymbol{r}_A = (555, -180, 240)$ m

$\boldsymbol{r}_B = (255, 0, -120)$ m



determine $\boldsymbol{r}_C$

∵ no external forces, the mass center G moves with a constant velocity

$\underline{\boldsymbol{v}} = \boldsymbol{v}_0 = 150 \text{ m/s } \boldsymbol{i}$

at $t = 2.5$ s $\quad \underline{\boldsymbol{r}} = \boldsymbol{v}_0\, t = 150 \times 2.5\, \boldsymbol{i} = 375 \text{ m } \boldsymbol{i}$

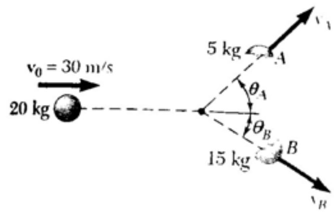
$\because \quad m\, \underline{\boldsymbol{r}} = \Sigma\, m_i\, \boldsymbol{r}_i$

$$200 \times 375\, \boldsymbol{i} = 100\,(555\, \boldsymbol{i} - 180\, \boldsymbol{j} + 240\, \boldsymbol{k}) + 60\,(255\, \boldsymbol{i} - 120\, \boldsymbol{k}) + 40\, \boldsymbol{r}_C$$

$$\boldsymbol{r}_C = (105\, \boldsymbol{i} - 450\, \boldsymbol{j} - 420\, \boldsymbol{k}) \text{ m}$$

Sample Problem 14-2

$m = 20$ kg $\quad \boldsymbol{v}_0 = 30 \text{ m/s } \boldsymbol{i}$ at $t = 0$



projectile explodes into two fragments A and B

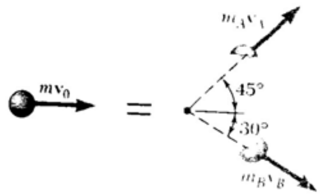
$m_A = 5$ kg $\quad m_B = 15$ kg

$\theta_A = 45^\circ \quad \theta_B = 30^\circ$

determine v_A and v_B

$\because$ no external forces, the linear momentum is constant

$$m_A\, \boldsymbol{v}_A + m_B\, \boldsymbol{v}_B = m\, \boldsymbol{v}_0$$



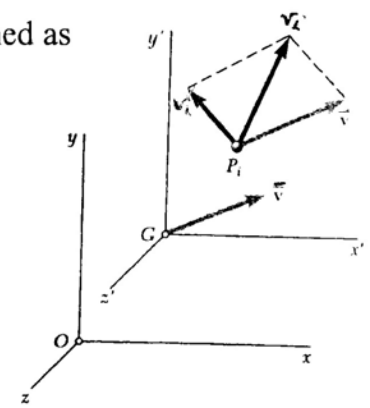
x-comp. $\quad 5\, v_A \cos 45^\circ + 15\, v_B \cos 30^\circ = 20 \times 30$

y-comp. $\quad 5\, v_A \sin 45^\circ - 15\, v_B \sin 30^\circ = 0$

$$v_A = 62.1 \text{ m/s} \qquad v_B = 29.3 \text{ m/s}$$

14.7 Kinetic Energy of a System of Particles

the kinetic energy of the system of particles is defined as



$$T = \tfrac{1}{2} \sum_{i=1}^{n} m_i\, v_i^2$$

$$\because \quad \boldsymbol{v}_i = \underline{\boldsymbol{v}} + \boldsymbol{v}_i'$$

$$T = \tfrac{1}{2} \sum_{i=1}^{n} m_i\, (\underline{\boldsymbol{v}} + \boldsymbol{v}_i') \bullet (\underline{\boldsymbol{v}} + \boldsymbol{v}_i')$$

$$= \tfrac{1}{2} \sum_{i=1}^{n} m_i\, \underline{\boldsymbol{v}}^2 + \underline{\boldsymbol{v}} \bullet \sum_{i=1}^{n} m_i\, \boldsymbol{v}_i'^2 + \tfrac{1}{2} \sum_{i=1}^{n} m_i\, \boldsymbol{v}_i'^2$$

$$T = = \tfrac{1}{2} \sum_{i=1}^{n} m_i \bar{v}^2 + \tfrac{1}{2} \sum_{i=1}^{n} m_i \boldsymbol{v}_i'^2$$

the kinetic energy T may be obtained by adding kinetic energy of the mass center G and the kinetic energy of the system in its motion relative to $Gx'y'z'$

14.8 Work-Energy Principle, Conservation of Energy for a System of Particles

$$T_1 + U_{1\to 2} = T_2 \qquad \text{for each particle}$$

where $U_{1\to 2}$ represents the work done by $\boldsymbol{f}_{ij}$ and $\boldsymbol{F}_i$ on P_i

for the entire system

$$T_1 + \Sigma\, U_{1\to 2} = T_2$$

$\Sigma\, U_{1\to 2}$ includes the work done by external and internal forces, note that the work done by internal forces $\boldsymbol{f}_{ij}$ and $\boldsymbol{f}_{ji}$ will not cancel out in general, $\because$ the displacements of P_i and P_j are not the same

if the forces acting on the particles of the system are conservative, then

$$T_1 + V_1 = T_2 + V_2$$

where V represents the potential energy associated with the internal and external forces acting on the particles of the system (conservation of energy)

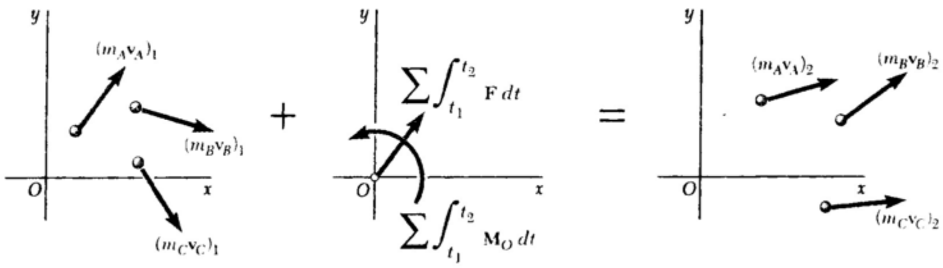
14.9 Principle of Impulse and Momentum for a System of Particles

$$\Sigma\, \boldsymbol{F} = \dot{\boldsymbol{L}} \qquad \Sigma\, \boldsymbol{M}_0 = \dot{\boldsymbol{H}}_0$$

this two equations can be expressed

$$\Sigma \int_{t_1}^{t_2} \boldsymbol{F}\, dt = \boldsymbol{L}_2 - \boldsymbol{L}_1$$

$$\Sigma \int_{t_1}^{t_2} \boldsymbol{M}_0 \, dt = (\boldsymbol{H}_0)_2 - (\boldsymbol{H}_0)_1$$



where $\Sigma \int \boldsymbol{L} \, dt$ is called linear impulse

$\Sigma \int \boldsymbol{M}_0 \, dt$ is called angular impulse

these two equation may be written as

$$\boldsymbol{L}_1 + \Sigma \int_{t_1}^{t_2} \boldsymbol{F} \, dt = \boldsymbol{L}_2$$

$$(\boldsymbol{H}_0)_1 + \Sigma \int_{t_1}^{t_2} \boldsymbol{M}_0 \, dt = (\boldsymbol{H}_0)_2$$

if no external forces applied

$$\boldsymbol{L}_1 = \boldsymbol{L}_2 \qquad (\boldsymbol{H}_0)_1 = (\boldsymbol{H}_0)_2$$

conservation of linear and angular momenta

Sample Problem 14-3

same condition as in sample problem 14-1

$m_A = 100$ kg $\quad m_B = 60$ kg $\quad m_C = 40$ kg

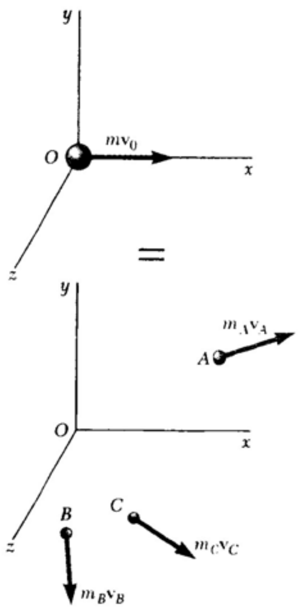
at $t = 2.5$ s $\quad \boldsymbol{r}_A = (555, -180, 240)$ m

$\boldsymbol{r}_B = (255, 0, -120)$ m

$\boldsymbol{r}_C = (105, 450, -420)$ m

$\boldsymbol{v}_A = (270, -120, 160)$ m/s $\quad \boldsymbol{v}_B = [(v_B)_x, 0, (v_B)_z]$

determine $\boldsymbol{v}_B$ and $\boldsymbol{v}_C$



since no external forces, thus

$$\boldsymbol{L}_1 = \boldsymbol{L}_2 \qquad (\boldsymbol{H}_0)_1 = (\boldsymbol{H}_0)_2$$

i.e. $m\,\boldsymbol{v}_0 = m_A\,\boldsymbol{v}_A + m_B\,\boldsymbol{v}_B + m_C\,\boldsymbol{v}_C$

$0 = (\boldsymbol{r}_A \times m_A\,\boldsymbol{v}_A) + (\boldsymbol{r}_B \times m_B\,\boldsymbol{v}_B) + (\boldsymbol{r}_C \times m_C\,\boldsymbol{v}_C)$

$200 \times 150\,\boldsymbol{i} = 100\,(270, -120, 160) + 60\,[(v_B)_x, 0, (v_B)_z]$

$+ 40\,[(v_C)_x, (v_C)_y, (v_C)_z]$

$$0 = 100\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 555 & -180 & 240 \\ 270 & -120 & 160 \end{vmatrix} + 60\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 255 & 0 & -120 \\ (v_B)_x & 0 & (v_B)_z \end{vmatrix} + 40\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 105 & 450 & -420 \\ (v_C)_x & (v_C)_y & (v_C)_z \end{vmatrix}$$

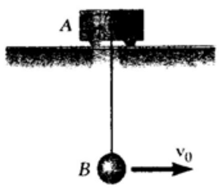
now we have 3 equations for linear momentum and another 3 equations for angular momentum, it can be solved

$\boldsymbol{v}_B = 70\,\boldsymbol{i} - 80\,\boldsymbol{k}$ (m/s)

$\boldsymbol{v}_C = -30\,\boldsymbol{i} + 300\,\boldsymbol{j} - 280\,\boldsymbol{k}$ (m/s)

actually, $(v_B)_y = 0$ is not necessary to known, 6 unknowns can be solved by 6 equations, $(v_B)_y = 0$ can be solved

Sample Problem 14-4

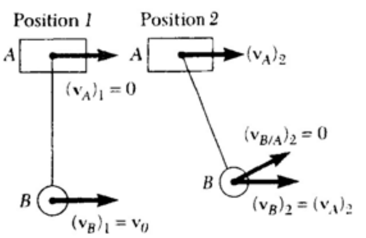


at $t = 0$ $\quad v_B = v_0$, $\quad$ determine

(a) v_B when B reaches maximum elevation

(b) maximum elevation h of B

$(v_A)_1 = 0 \quad (v_B)_1 = v_0$

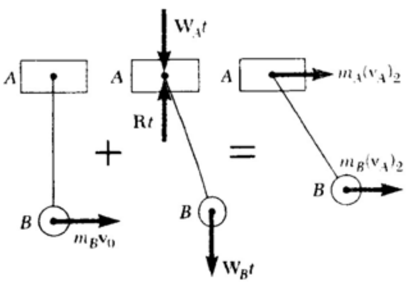


when B reaches the maximum elevation

$(v_{B/A})_2 = 0$

i.e. $(v_B)_2 = (v_A)_2$

impulse-momentum principle



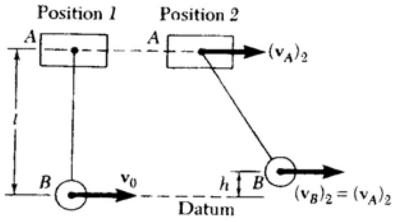
$\Sigma\,(m\,v)_1 + \Sigma\,Imp_{1\to2} = \Sigma\,(m\,v)_2$

in x-direction, no external impulse

$$m_B v_0 = (m_A + m_B)(v_A)_2$$

$$(v_A)_2 = (v_B)_2 = \frac{m_B}{m_A + m_B} v_0 \rightarrow$$

conservation of energy



$$T_1 = \tfrac{1}{2} m_B v_0^2 \qquad V_1 = m_A g l$$

$$T_2 = \tfrac{1}{2}(m_A + m_B)(v_A)_2^2$$

$$V_2 = m_A g l + m_B g h$$

$$T_1 + V_1 = T_2 + V_2$$

$$\tfrac{1}{2} m_B v_0^2 + m_A g l = \tfrac{1}{2}(m_A + m_B)(v_A)_2^2 + m_A g l + m_B g h$$

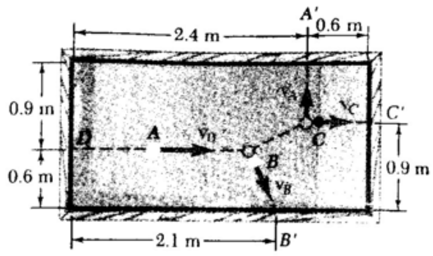
$$h = \frac{v_0^2}{2g} - \frac{m_A + m_B}{m_B}\frac{(v_A)_2^2}{2g} = \frac{m_A}{m_A + m_B}\frac{v_0^2}{2g}$$

for $m_A >> m_B$, $\quad h = v_0^2/2g$, $\quad$ oscillates as a simple pendulum

for $m_B >> m_A$, $\quad (v_A)_2 = (v_B)_2 = v_0$ and $h = 0$

A and B move with the same constant velocity

Sample Problem 14-5

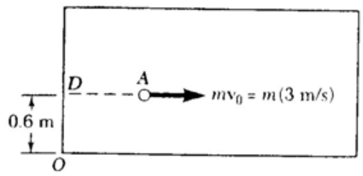


$m_A = m_B = m_C = m \qquad v_0 = 3$ m/s $\boldsymbol{i}$

frictionless and perfectly elastic impact

determine $\boldsymbol{v}_A$, $\boldsymbol{v}_B$ and $\boldsymbol{v}_C$ after impact

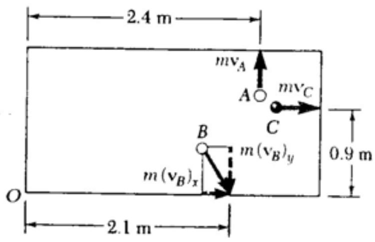
∵ no external force, conservation of momentum



x-component $\qquad m \times 3 = m(v_B)_x + m v_C$

y-component $\qquad 0 = m v_A - m(v_B)_y$

angular momentum about O



$$-0.6 \times m \times 3 = 2.4 \times m \times v_A - 2.1 \times m \times (v_B)_y - 0.9 \times m \times v_C$$

3 equations for 4 unknowns, it can not be solved completely

solving the 3 equations, it can be obtained

$$v_A = (v_B)_y = 3\,v_C - 6 \qquad (v_B)_x = 3 - v_C$$

∵ perfectly elastic impact, conservation of energy

$$\tfrac{1}{2}\,m\,v_0^2 = \tfrac{1}{2}\,m\,v_A^2 + \tfrac{1}{2}\,m\,[(v_B)_x^2 + (v_B)_y^2] + \tfrac{1}{2}\,m\,v_C^2$$

$$v_A^2 + (v_B)_x^2 + (v_B)_y^2 + v_C^2 = v_0^2 = 9$$

solving for v_C, and find $\qquad v_C = 1.5$ or 2.4 m/s

for $v_C = 1.5$ m/s => $v_A = (v_B)_y = -$ (N.G.)

for $v_C = 2.4$ m/s => $v_A = (v_B)_y = 1.2$ m/s $\quad (v_B)_x = 0.6$ m/s

then the velocities after impact are

$$\mathbf{v}_A = 1.2 \text{ m/s} \uparrow \qquad \mathbf{v}_B = 1.34 \text{ m/s} \searrow 63.4^\circ$$

$$\mathbf{v}_C = 2.4 \text{ m/s} \rightarrow$$

14.10 Variable System of Particles

system are continuously gaining or losing particles, or doing both at the same time

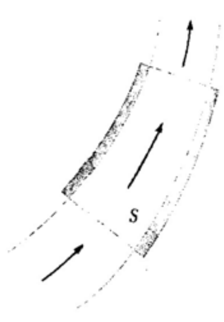
e.g. hydraulic turbine, rockets

14.11 Steady Stream of Particles

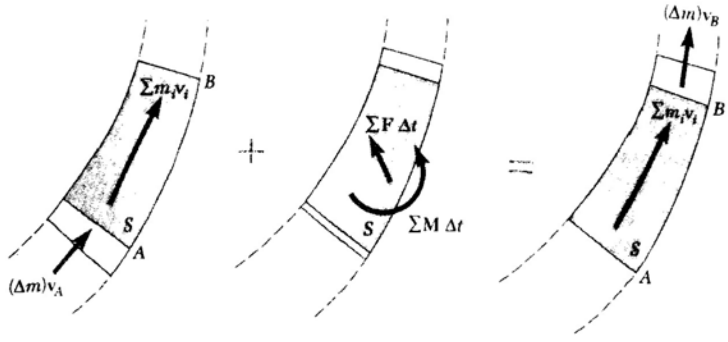
consider a steady stream of particles S

S is a variable system of particles, gains particles flowing in and loses an equal number of particles flowing out

thus the kinetics principles that have been established before cannot be directly applied to S



consider at time t the system S and the particles $\triangle m$ which will enter S, at time $t + \triangle t$, the system S and the particles $\triangle m$ which have left S



in time interval t to $t + \triangle t$

$(\triangle m)\, \boldsymbol{v}_A$ inside S $\qquad$ $(\triangle m)\, \boldsymbol{v}_B$ outside S

principle of impulse and momentum

$$(\triangle m)\, \boldsymbol{v}_A + \Sigma\, m_i\, \boldsymbol{v}_i + \Sigma\, \boldsymbol{F}\, \triangle t = (\triangle m)\, \boldsymbol{v}_B + \Sigma\, m_i\, \boldsymbol{v}_i$$

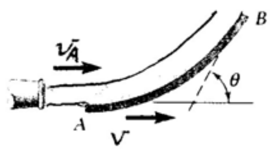
divided by $\triangle t$ in both sides

$$\Sigma\, \boldsymbol{F} = \frac{dm}{dt}(\boldsymbol{v}_B - \boldsymbol{v}_A)$$

this principle may be used to analyze a large number of engineering applications

(a) fluid stream diverted by a vane

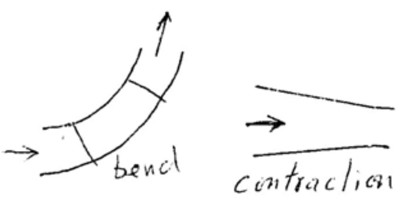
may be applied to find the force $\boldsymbol{F}$ exerted by the vane on the stream, for a fixed vane, $\boldsymbol{F}$ is the only force exerted by the stream



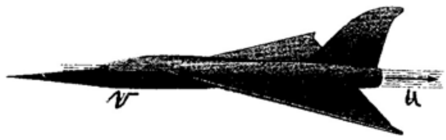
if the vane moves with a constant velocity, the stream is not steady, but it will appear steady to an observer moving with the vane

(b) fluid flowing through a pipe

the force exerted by the fluid on a pipe transition such as a bend or a contraction can be determined



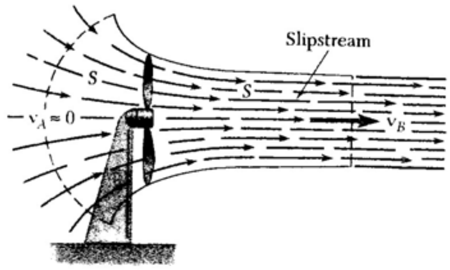
(c) jet engine



air enters with no velocity and leaves the engine with high velocity, the energy is obtained by burning fuel, the thrust of the engine can be determined

the air particles entering velocity v is equal the velocity of the plane, and leaves with a velocity u relative to the plane

(d) fan

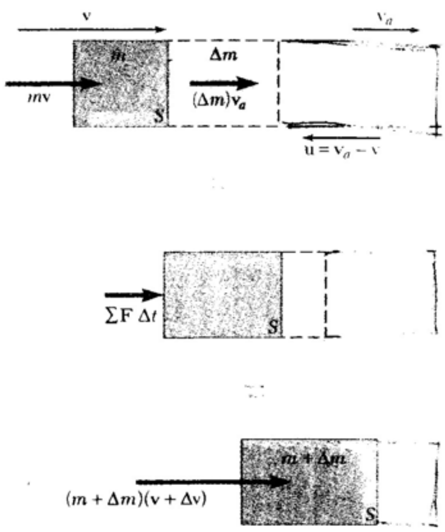


the velocity $\boldsymbol{v}_A$ of the particles entering the system is assumed zero, $\boldsymbol{v}_B$ is the velocity of the particles leaving the system, the only external force acting on S is the thrust of the fan

(e) helicopter

the velocity $\boldsymbol{v}_A$ of the air particles approach the blades is assumed zero, $\boldsymbol{v}_B$ is the velocity of the particles leaving the system, the only external force acting on S is the thrust of the fan

14.12 Systems Gaining or Losing Mass



consider a system S with mass m at t, which increases by Δm in time interval Δt

principle of impulse and momentum

$$m\,\boldsymbol{v} + \Delta m\,\boldsymbol{v}_A + \Sigma\,\boldsymbol{F}\Delta t = (m + \Delta m)(\boldsymbol{v} + \Delta\boldsymbol{v})$$

or $\quad \Sigma\,\boldsymbol{F}\Delta t = m\Delta\boldsymbol{v} + \Delta m(\boldsymbol{v} - \boldsymbol{v}_A) + \Delta m\Delta\boldsymbol{v}$

let $\quad \boldsymbol{u} = \boldsymbol{v}_A - \boldsymbol{v} \qquad$ relative velocity clearly that $v_A < u$, neglecting the last term (high order term), we have

$$\Sigma \boldsymbol{F} \triangle t = m \triangle \boldsymbol{v} - \triangle m\, \boldsymbol{u}$$

dividing by $\triangle t$ and let $\triangle t \to 0$

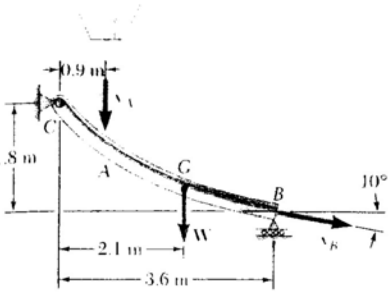
$$\Sigma \boldsymbol{F} = m \frac{d\boldsymbol{v}}{dt} - \frac{dm}{dt} \boldsymbol{u}$$

or $$\Sigma \boldsymbol{F} + \frac{dm}{dt} \boldsymbol{u} = m \frac{d\boldsymbol{v}}{dt}$$

since $\boldsymbol{u}$ is directed to left, which tends slow down the motion of S

similarly for losing mass system

Sample Problem 14-5

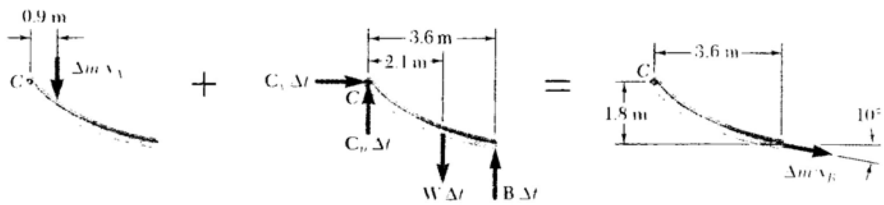


$\boldsymbol{v}_A = 6$ m/s ↓ $\boldsymbol{v}_B = 4.5$ m/s ↘ 10°

$\boldsymbol{W} = 3$ kN $\triangle m / \triangle t = 120$ kg/s

determine the reactions at B and C

for the time interval $\triangle t$, principle of impulse and momentum



x-components $C_x \triangle t = (\triangle m) v_B \cos 10^{\circ}$

y-components $-(\triangle m) v_A + C_y \triangle t - W \triangle t + B \triangle t$

$= -(\triangle m) v_B \sin 10^{\circ}$

moment about C $-0.9\,(\triangle m) v_A - 2.1\, W \triangle t + 3.6\, B \triangle t$

$= 1.8\,(\triangle m) v_B \cos 10^{\circ} - 3.6\,(\triangle m) v_B \sin 10^{\circ}$

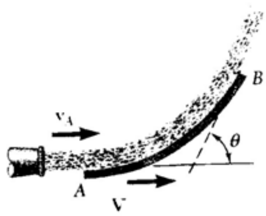
$W = 3000$ N $v_A = 6$ m/s $v_B = 4.5$ m/s

$\triangle m / \triangle t = 120$ kg/s

3 unknowns can be solved by the above 3 equations

$$C_x = 532 \text{ N} \rightarrow \qquad C_y = 1524 \text{ N} \uparrow \qquad B = 2102 \text{ N} \uparrow$$

Sample Problem 14-6



v_A : velocity of water $\quad$ V : velocity of the blade

V = constant

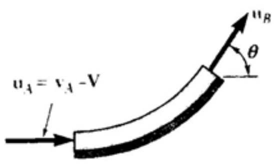
determine (a) F exerted by the blade on the stream

(b) V for which maximum power is developed

choose the coordinate system with the blade (constant velocity, no acceleration)

let $\quad \boldsymbol{u}_A = \boldsymbol{v}_A - \boldsymbol{V}$

$\boldsymbol{u}_B$ be the relative velocity of the particles leave the blade



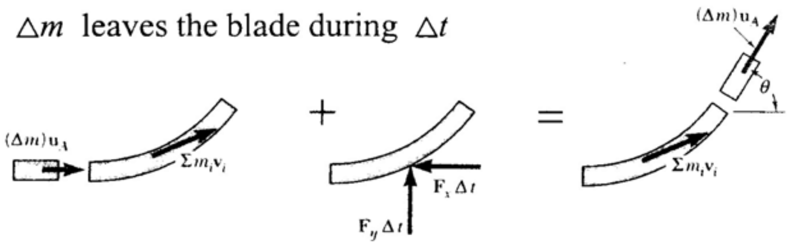
then $\qquad u_A = u_B \qquad$ in magnitude

and let $\qquad \rho$: density of water $\qquad A$: cross section

during time interval Δt, the mass striking the blade is

$$\Delta m = A\,\rho\,(v_A - V)\,\Delta t$$

same Δm leaves the blade during Δt



principle of impulse and momentum

x-components $\qquad (\Delta m)\,u - F_x\,\Delta t = (\Delta m)\,u \cos\theta \qquad (1)$

y-components $\qquad F_y\,\Delta t = (\Delta m)\,u \sin\theta \qquad (2)$

(1) => $\qquad F_x = (\Delta m/\Delta t)\,u\,(1 - \cos\theta)$

$$= A\,\rho\,(v_A - V)^2\,(1 - \cos\theta) \leftarrow$$

(2) => $\quad F_y = A\,\rho\,(v_A - V)^2 \sin\theta \;\uparrow$

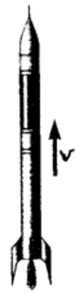
$$\text{Power} = F_x V = A\,\rho\,(v_A - V)^2 (1 - \cos\theta)\,V$$

let $\quad dP/dV = 0 = A\,\rho\,(v_A^2 - 4\,v_A V + 3\,V^2)(1 - \cos\theta)$

$V = v_A \qquad P = 0$

$V = \frac{1}{3}\,v_A \qquad P = P_{max}$

Sample Problem 14-7

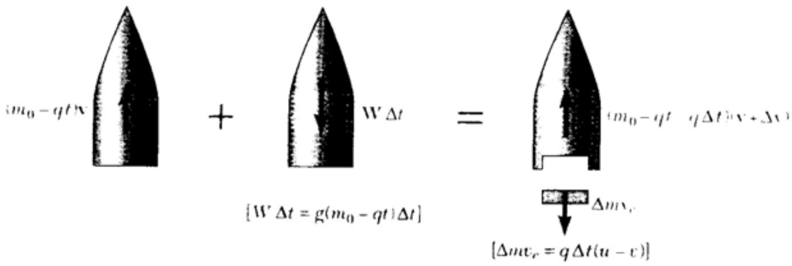


m_0 : initial mass at $t = 0$

the fuel is consumed at constant rate $q = dm/dt$

and expelled at a constant speed u relative to the rocket

determine $v(t)$



principle of impulse and momentum between time t and $t + \Delta t$

$$(m_0 - qt)v - (m_0 - qt)g\Delta t = (m_0 - qt - q\Delta t)(v + \Delta v) - q\Delta t(u - v)$$
$$= (m_0 - qt)v + (m_0 - qt)\Delta v - qv\Delta t - q\Delta t\Delta v - q\Delta t(u - v)$$

divided by Δt and let $\Delta t \to 0$

$$-(m_0 - qt)g = (m_0 - qt)\frac{dv}{dt} - q\,u - q\Delta v$$

$$dv = \left(\frac{q\,u}{m_0 - qt} - g\right) dt$$

$$\int_0^v dv = \int_0^t \left(\frac{q\,u}{m_0 - qt} - g\right) dt$$

$$v = [- u\, ln(m_0 - qt) - gt]\,\Big|_0^t$$

$$= u\, ln \frac{m_0}{m_0 - qt} - gt$$

let the final time for the fuel has been expended be t_f, then the mass of the rocket is $m_s = m_0 - q\, t_f$

$$v_{max} = u\, ln \frac{m_0}{m_s} - g\, t_f$$

assumed that t_f is short, then $g\, t_f$ may neglected

i.e. $v_{max} \simeq u\, ln\, (m_0/m_s)$

the escape velocity for gravitational field is about 11.18 km/s, assuming that $u = 2200$ m/s, then

$$m_0/m_s = 161$$

thus, to project each kg of the rocket shell into space, it is necessary to consume more than 161 kg of fuel