Chapter 13 Kinetics of Particles : Energy and Momentum Methods

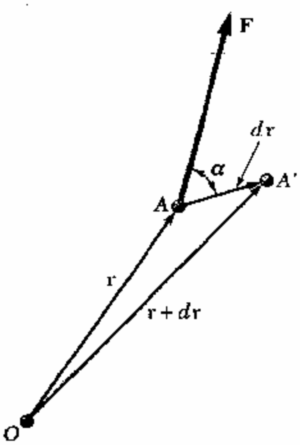
13.1 Introduction

Newton' s 2nd Law : $\Sigma \boldsymbol{F} = \dot{\boldsymbol{L}} = m\,\boldsymbol{a}$

two additional methods of analysis

1. method of work and energy
2. method of impulse and momentum

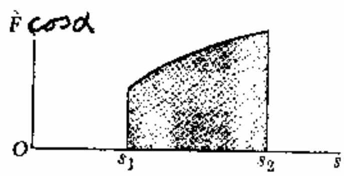
13.2 Work of a Force



the work of a force $\boldsymbol{F}$ due to a displacement $\boldsymbol{dr}$ is

$$dU = \boldsymbol{F} \bullet \boldsymbol{dr} = F\,ds\cos a$$

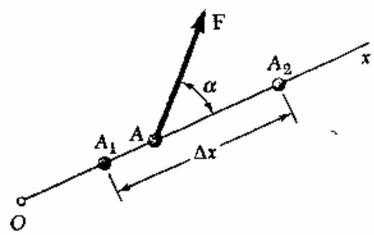
$$= F_x\,dx + F_y\,dy + F_z\,dz$$



the work of a force $\boldsymbol{F}$ due to a finite displacement from A_1 to A_2 is

$$U_{1\to2} = \int_{A_1}^{A_2} \boldsymbol{F} \bullet \boldsymbol{dr} = \int_{S_1}^{S_2} F\cos a\,ds$$

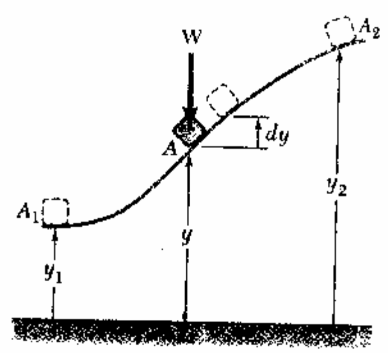
$$= \int_{A_1}^{A_2} (F_x\,dx + F_y\,dy + F_z\,dz)$$



work of a constant force in rectilinear motion

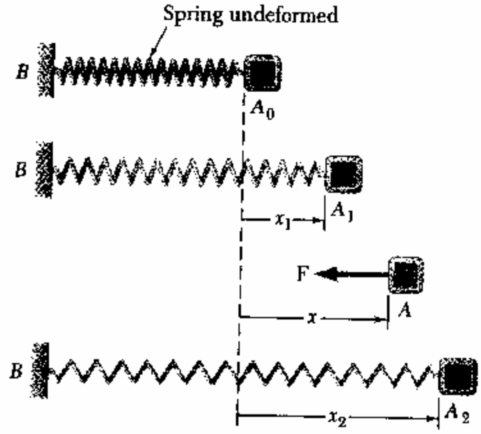
$$U_{1\to2} = (F\cos a)\,\Delta x$$

work of weight



$$dU = -W\,dy$$

$$U_{1\to2} = -W\,\Delta y = -W(y_2 - y_1)$$



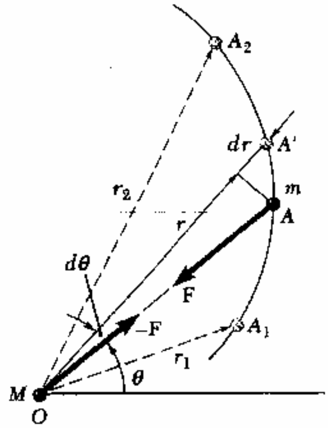
work of force exerted by a spring

$$F = k\,x$$

$$dU = -F\,dx$$

$$U_{1\to2} = -\int_{x_1}^{x_2} k\,x\,dx = \tfrac{1}{2}k\,(x_1^{\,2} - x_2^{\,2})$$

work of a gravitational force



$$F = G\,M\,m\,/\,r^2$$

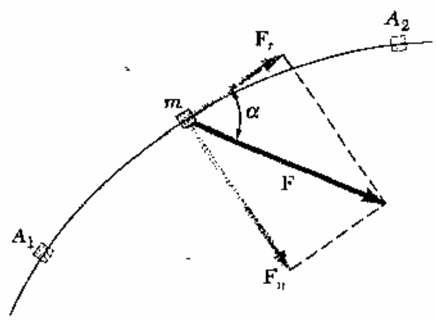
$$dU = -\boldsymbol{F}\bullet\boldsymbol{dr} \qquad (\boldsymbol{F} \text{ directed toward } O)$$

$$= -G\,\frac{M\,m}{r^2}\,dr$$

$$U_{1\to2} = \int_{r_1}^{r_2} dU = \frac{G\,M\,m}{r_2} - \frac{G\,M\,m}{r_1}$$

13.3 Kinetic Energy of a Particle, Principle of Work and Energy

consider a particle of mass m acted upon by a force $\boldsymbol{F}$ and moving along a path



Newton's 2nd Law

$$F_t = m\,a_t = m\,dv/dt$$

$$= m\,(dv/ds)(ds/dt) = m\,v\,(dv/ds)$$

$$F_t\,ds = m\,v\,dv$$

integrating from $A_1\ (s = s_1, v = v_1)$ to $A_2\ (s = s_2, v = v_2)$

$$U_{1\to2} = \int_{s_1}^{s_2} F_t\,ds = \int_{v_1}^{v_2} m\,v\,dv = \tfrac{1}{2}m\,v_2^{\,2} - \tfrac{1}{2}m\,v_1^{\,2}$$

define $\quad \boldsymbol{T \equiv \tfrac{1}{2}m\,v^2} \quad$ is the kinetic energy

$$\therefore \quad U_{1\to2} = T_2 - T_1$$

$$\text{or} \quad T_2 = T_1 + U_{1\to2}$$

the work of force $\boldsymbol{F}$ is equal to the change in kinetic energy

work and energy are scalar quantities

work may be positive or negative, but the kinetic energy is always positive (at least = 0)

if $\quad v_1 = 0 \qquad v_2 = v$

then $\quad T_1 = 0 \qquad T_2 = T$

$\therefore \quad U_{1\to2} = T \quad$ work done by the force equal to T

if $\quad v_1 = v \qquad v_2 = 0$

then $\quad T_1 = T \qquad T_2 = 0$

$\therefore \quad U_{1\to2} = -T$

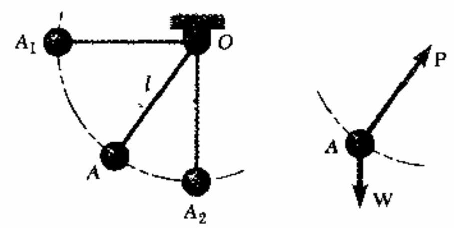
unit of $T \quad T = \frac{1}{2} m v^2$: kg-m²/s² = N-m = J

13.4 Application of the Principle of Work and Energy

consider a pendulum OA consisting of a bob A of weight W attached to a cord of length l

at A_1, no velocity

at point A, the forces acting on the bob are the normal force $\boldsymbol{P}$ and its weight $\boldsymbol{W}$



$\because$ $\boldsymbol{P}$ is normal to the moving path, $\therefore$ no work done, only force does work is its weight $\boldsymbol{W}$

$$\therefore \quad U_{1\to2} = W\,l$$

$$T_1 + U_{1\to2} = T_2$$

$$\because \quad v_1 = 0 \qquad \therefore \quad T_1 = 0$$

then $$T_2 = \frac{1}{2}\frac{W}{g}v_2^2 = W\,l$$

or $$v_2 = (2\,g\,l)^{1/2}$$

this example shows the advantages of the method of work and energy

1. no need to determine the acceleration

2. all quantities involved are scalars

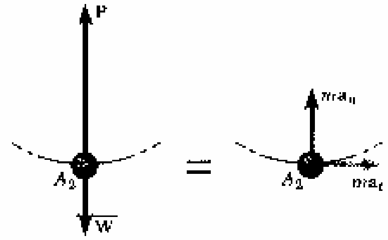
3. eliminated the force which do no work

but, this method cannot be used to directly determine an acceleration, in this problem, we must use Newton's 2nd law to calculate acceleration

at point A_2

$$F_t = 0 = m\,a_t \qquad a_t = 0$$

$$F_n = P - W = m\,a_n = \frac{W}{g}\,\frac{2gl}{l} = 2\,W$$



$$\therefore \qquad P = 3\,W$$

when a problem involves two or more particles, each particle may be considered separately and the principle of work and energy may be applied to each particle, adding T of the particles, and consider the work of all forces acting on them, we may write

$$T_1 + U_{1\rightarrow 2} = T_2$$

T : arithmetic sum of the kinetic energy of particles

$U_{1\rightarrow 2}$: work of all forces acting on the particles

13.5 Power and Efficiency

power : time rate at which work is done

$$\text{power} = \frac{dU}{dt} = \frac{\mathbf{F}\cdot d\mathbf{r}}{dt} = \mathbf{F}\cdot\mathbf{v}$$

unit of power : watt (W) = J/s = N-m/s

mechanical efficiency η is defined

$$\eta = \frac{\text{output work}}{\text{input work}} \qquad \text{for work done at a constant rate}$$

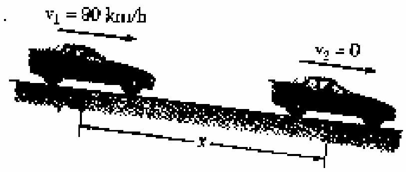
$$\eta = \frac{\text{power output}}{\text{power input}} < 1 \qquad \text{energy losses due to friction}$$

Sample Problem 13.1

$m = 2000$ kg $\quad v_1 = 90$ km/h

breaking force $= 7$ kN

determine the breaking distance

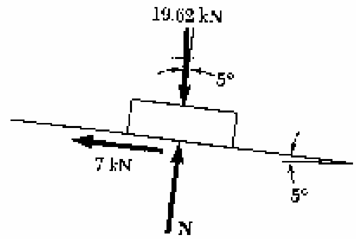


$W = m\,g = 2000 \times 9.81 = 19.62$ kN

$v_1 = 90$ km/h $= 25$ m/s

$T_1 = \frac{1}{2}\, m\, v_1^2 = \frac{1}{2}\, 2000 \times 25^2 = 625$ kJ

$v_2 = 0 \qquad T_2 = 0$



$U_{1\to2} = -\,7\,x + 19.62 \sin 5^{\circ}\, x = -\,5.29\, x$

$T_1 + U_{1\to2} = T_2$

$625 - 5.29\, x = 0 \qquad x = 118.15$ m

if the automobile is up grade

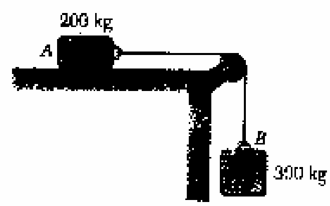
$U_{1\to2} = -\,7\,x - 19.62 \sin 5^{\circ}\, x = -\,8.71\, x$

$T_1 + U_{1\to2} = T_2$

$625 - 8.71\, x = 0 \qquad x = 71.76$ m

the breaking distance for up grade is shorter

Sample Problem 13.2



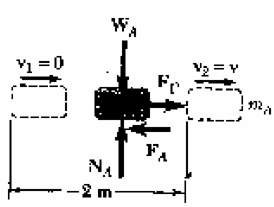
the system is release from rest

$m_A = 200$ kg $\qquad m_B = 300$ kg

$\mu_k = 0.25$

determine v_A after it has moved 2 m

for the system $\qquad v_A = v_B = v$



for block A

$N_A = W_A = 200 \times 9.81 = 1962$ N

$F_A = \mu_k N_A = 0.25 \times 1962 = 409.5\ N$

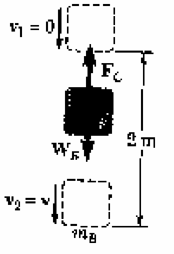
$v_1 = 0 \qquad T_1 = 0 \qquad T_2 = ½\, m_A\, v_2^2$

$U_{1\to2} = (F_C - F_A) \times 2 = (F_C - 409.5) \times 2$

$T_1 + U_{1\to2} = T_2$

$2\, F_C - 981 = ½\, m_A\, v_2^2 \qquad (1)$

for block B



$v_1 = 0 \qquad T_1 = 0 \qquad T_2 = ½\, m_B\, v_2^2$

$U_{1\to2} = (W_B - F_C) \times 2 = 5886 - 2\, F_C$

$T_1 + U_{1\to2} = T_2$

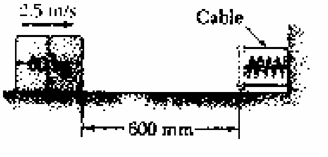
$5886 - 2\, F_C = ½\, m_B\, v_2^2 \qquad (2)$

$(1) + (2) \qquad ½\, (m_A + m_B)\, v_2^2 = 4905$

$½\, (200 + 300)\, v_2^2 = 4905$

$v_2^2 = 19.62 \qquad v_2 = 4.43$ m/s

Sample Problem 13.3



spring is initial compressed 120 mm, and the maximum addition deflection of the spring is 40 mm, for $v_1 = 2.5$ m/s and $k = 20$ kN/m, $m = 60$ kg, determine

a. μ_k

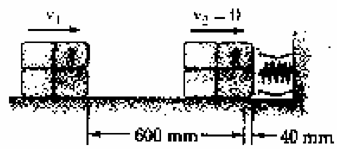
b. v for the block passes again through the original position

$T_1 = ½\, m\, v_1^2 = 187.5$ J

at [2] $\qquad v_2 = 0 \qquad T_2 = 0$

1. work done by friction force from [1] to [2]

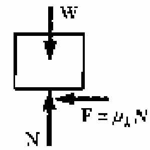
$(U_{1\to2})_f = - F\, x = - \mu_k\, m g\, x = - \mu_k\, 60 \times 9.81 \times 0.64 = - 377\, \mu_k$ (N-m) work done due to spring from [1] to [2]



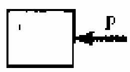
$$P_{min} = k\,x_0 = 20 \text{ x } 0.12 = 2400 \text{ N}$$

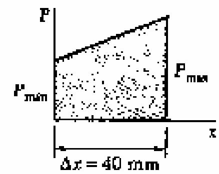
$$P_{max} = P_{min} + k\,\Delta x = 3200 \text{ N}$$

$$(U_{1\to2})_e = -\tfrac{1}{2}(P_{max} + P_{min})\,\Delta x = -112 \text{ J}$$



total work done from [1] to [2]





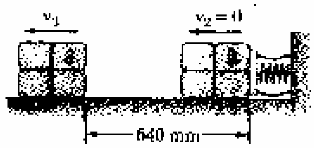
$$U_{1\to2} = -377\,\mu_k - 112.0$$

$$T_1 + U_{1\to2} = T_2$$

$$187.5 - 377\,\mu_k - 112.0 = 0$$

$$\mu_k = 0.2$$

b. work and energy for positions [2] and [3]



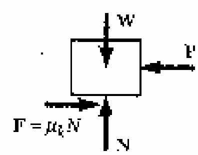
$$T_2 = 0 \qquad T_3 = \tfrac{1}{2} m\,v_3^2$$

$$U_{2\to3} = -377\,\mu_k + 112.0 = 36.5 \text{ J}$$

$$T_2 + U_{2\to3} = T_3$$

$$0 + 36.5 = \tfrac{1}{2}\,60\,v_3^2$$

$$v_3 = 1.103 \text{ m/s}$$

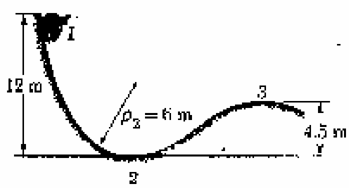


Sample Problem 13.4

$$m = 1000 \text{ kg} \qquad \rho_2 = 6 \text{ m}$$

(a) determine force exerted by the track at [2]

(b) determine the safe value for ρ_3



(a) because the car is rest at position [1]

$$T_1 = 0 \qquad T_2 = \tfrac{1}{2} m\,v_2^2$$

$$U_{1\to2} = m\,g\,h = 12\,mg$$

$$T_1 + U_{1\to2} = T_2$$

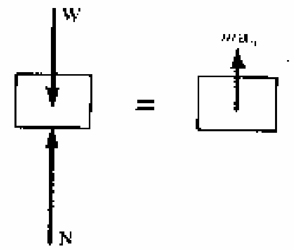
$$0 + 12\,mg = \tfrac{1}{2} m\,v_2^2$$

$v_2^2 = 24\,g = 24 \times 9.81 \qquad v_2 = 15.34$ m/s

$a_n = v_2^2 / \rho_2 = 24\,g / 6 = 4\,g$

$\Sigma F_n = m\,a_n \qquad N - W = 4\,mg$

$N = 5$ mg $= 49.1$ kN

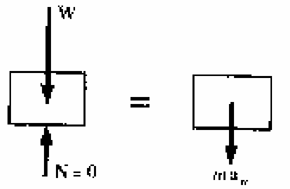


(b) work and energy from [1] to [3]

$T_1 + U_{1\to3} = T_3$

$\frac{1}{2}\,m\,v_3^2 = mg\,(12 - 4.5)$

$v_3^2 = 15\,g \qquad v_3 = 12.13$ m/s



minimum ρ occurs when $N = 0$

$\Sigma F_n = m\,a_n \qquad m\,g = m\,a_n = m\,v_3^2 / \rho_3$

$(\rho_3)_{min} = 15\,g / g = 15$ m

Sample Problem 13.5

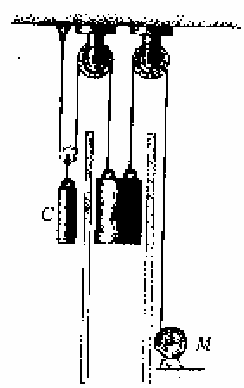
$m_C = 200$ kg $\qquad W_C = 1962$ N

$m_D = 150$ kg $\qquad W_D = 1472$ N

determine the power of the motor when

(a) $v_D = 2.4$ m/s $=$ constant $\uparrow$

(b) $v_D = 2.4$ m/s $\uparrow$ and $a_D = 0.75$ m/s² $\uparrow$



for the constant length of the cable

$2\,y_C + y_D =$ constant

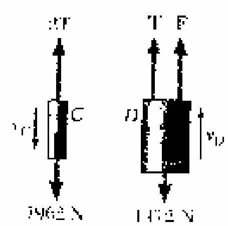
$2\,v_C + v_D = 0 \qquad 2\,a_C + a_D = 0$

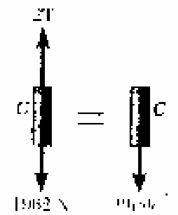
(a) $\quad v_D = 2.4$ m/s $\uparrow$ $\quad$ then $\quad v_C = 1.2$ m/s $\downarrow$

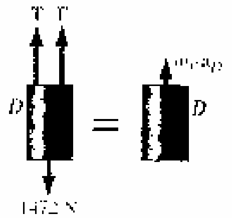
$a_C = a_D = 0$

for body $C \qquad \Sigma F_y = 0 \qquad 2\,T = 1962$ N

$T = 981$ N







for body D $\quad \Sigma F_y = 0 \quad F + T = 1472$ N

$F = 491$ N

Power $= F v_D = 491 \times 2.4 = 1178$ J/s $= 1178$ W

(b) $\quad a_D = 0.75$ m/s$^2 \uparrow$ then $\quad a_C = 0.375$ m/s$^2 \downarrow$

for body C $\quad \Sigma F_y = m_C a_C$

$1962 - 2T = 200 \times 0.375$

$T = 944$ N

for body D $\quad \Sigma F_y = m_D a_D$

$F + T - 1472$ N $= \underline{150 \times 0.75}$

$F = 640$ N

Power $= F v_D = 640 \times 2.4 = 1536$ W

Remark : without block C

$\Sigma F_y = m_D a_D \qquad F - 150\,g = 150 \times 0.75$

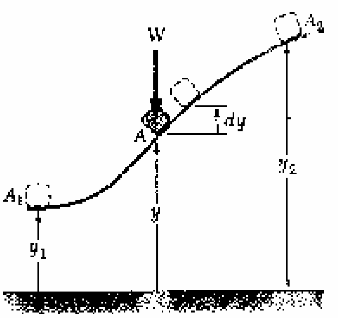
$F = 1584$ N

Power $= F v_D = 1584 \times 2.4 = 3801.6$ W $\qquad$ [2.475 times]

13.6 Potential Energy

consider a weight W moves along a curved path

$$U_{1\to2} = W y_1 - W y_2$$



define the potential energy of the body with respect to force of gravity W as

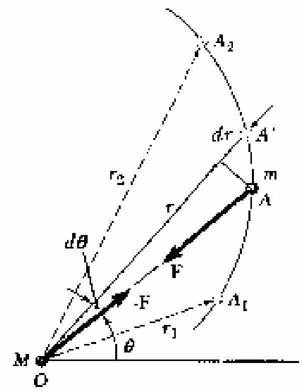
$$V_g = W y \quad \text{then} \quad U_{1\to2} = (V_g)_1 - (V_g)_2$$

in this example $\quad (V_g)_2 > (V_g)_1$

if the potential energy increases during displacement, then the work $U_{1\to2}$ is negative

$W = mg$, g may change with location, W is assumed to remain constant when the displacement are very small compared with the radius of earth

for the space vehicle



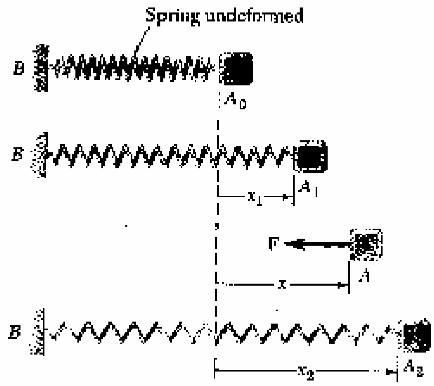
$$U_{1\to 2} = \frac{G\,M\,m}{r_2} - \frac{G\,M\,m}{r_1}$$

define the potential energy V_g when the variation of the of gravity cannot be neglected is

$$V_g = -\frac{G\,M\,m}{r} = -\frac{W\,R^2}{r}$$

then $U_{1\to 2} = (V_g)_1 - (V_g)_2$

for the spring



$$U_{1\to 2} = \tfrac{1}{2}\,k\,x_1^2 - \tfrac{1}{2}\,k\,x_2^2$$

define the potential energy V_e for spring is

$$V_g = \tfrac{1}{2}\,k\,x^2$$

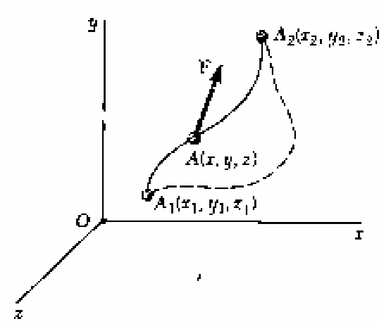
then $U_{1\to 2} = (V_e)_1 - (V_e)_2$

in all the above 3 cases, it can be written

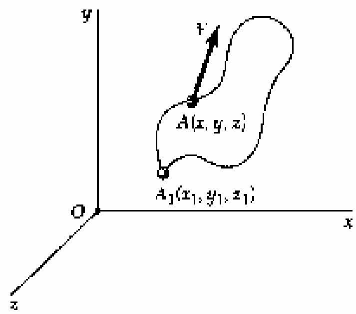
$$U_{1\to 2} = V_1 - V_2$$

13.7 Conservative Force

force $\boldsymbol{F}$ acting on a particle A is said to be conservative if its work $U_{1\to 2}$ is independent of path followed by the particle A as it moves from A_1 to A_2, we may write



$$U_{1\to 2} = V(x_1, y_1, z_1) - V(x_2, y_2, z_2)$$



or $\quad U_{1\to2} = V_1 - V_2$

where $V(x, y, z)$ is called potential function of $\boldsymbol{F}$

for a closed loop path $\quad A_2 = A_1 \quad$ then

$$U_{1\to2} = V_1 - V_1 = 0$$

i.e. $\quad \oint \boldsymbol{F}\cdot d\boldsymbol{r} = 0 \quad$ if $\boldsymbol{F}$ is conservative

for two neighboring points $A(x, y, z)$ and $A'(x+dx, y+dy, z+dz)$

$$dU = V(x, y, z) - V(x+dx, y+dy, z+dz)$$

or $\quad dU = -dV(x, y, z) \qquad$ [exact differential, independent of path]

$$F_x\,dx + F_y\,dy + F_z\,dz = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right)$$

then $\quad F_x = -\dfrac{\partial V}{\partial x} \qquad F_y = -\dfrac{\partial V}{\partial y} \qquad F_z = -\dfrac{\partial V}{\partial z}$

thus $\quad \boldsymbol{F} = -\left(\dfrac{\partial V}{\partial x}\boldsymbol{i} + \dfrac{\partial V}{\partial y}\boldsymbol{j} + \dfrac{\partial V}{\partial z}\boldsymbol{k}\right)$

$\qquad\qquad = -\nabla V = -\text{grad } V \qquad$ [gradient of a scalar function]

$\boldsymbol{F}$ is a conservative force if and only if

$$\boldsymbol{F} = -\nabla V$$

e.g. $\quad V_g = Wy \qquad \nabla V_g = W\boldsymbol{j} = -\boldsymbol{W}$

$\qquad V_c = \frac{1}{2}kx^2 \qquad \nabla V_e = kx\,\boldsymbol{i} = -\boldsymbol{F}$

$\qquad V_g = -\dfrac{GMm}{r} \qquad \nabla V_g = \dfrac{GMm}{r^2}\boldsymbol{e}_r = -\boldsymbol{F}$

13.8 Conservation of Energy

if a particle moves under conservative force

$$U_{1\to2} = V_1 - V_2$$

but $\quad U_{1\to2} = T_2 - V_1$

then $\qquad V_1 - V_2 = T_2 - V_1$

or $\qquad T_1 + V_1 = T_2 + V_2$

this is $\qquad E = T_1 + V_1 =$ total mechanical energy $=$ constant

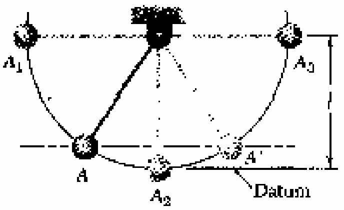
the sum of kinetic energy and potential energy remains constant

e.g. at point 1 $\quad T_1 = 0 \qquad V_1 = W\,l$

at point 2 $\quad T_2 = \frac{1}{2} m v_2^2 \qquad V_2 = 0$

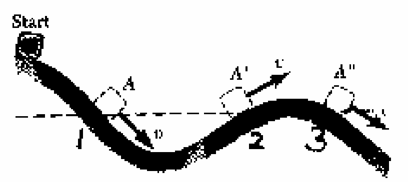
$T + V =$ constant

$$0 + W\,l = \tfrac{1}{2} m v_2^2 + 0$$



i.e. $\qquad v_2 = (2\,g\,l)^{1/2}$

on a frictionless surface



$$V_1 = V_2 = V_3 \qquad \Rightarrow \qquad T_1 = T_2 = T_3$$

or $\qquad v_1 = v_2 = v_3$

but if friction force is considered, then $v_1 > v_2 > v_3$

friction forces are nonconservative forces, since the work done due to friction force depends on path, it cannot be expressed as a change in potential energy

because the friction force is always in the opposite direction of motion, thus the work done of friction force is always negative

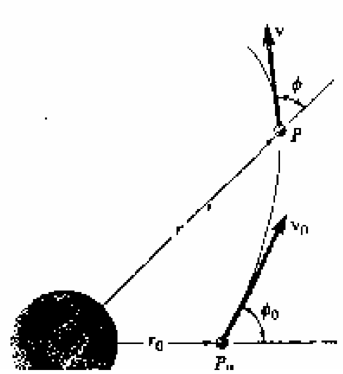
the work done due to friction force transformed into heat, thus the sum of mechanical energy and thermal energy of the system remains constant

13.9 Motion Under a Conservative Central Force, Application to Space Mechanics

if the force $\boldsymbol{F}$ is central force, then

$$\boldsymbol{H}_0 = \text{constant}$$

$$r_0\, m\, v_0 \sin\phi_0 = r\, m\, v \sin\phi \qquad (1)$$



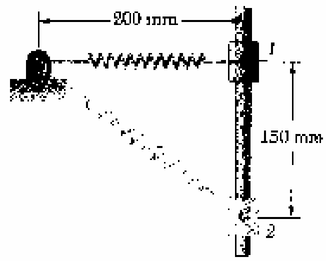
if $\boldsymbol{F}$ is also conservative force, then

$$E = T + V = \text{constant}$$

$$\tfrac{1}{2}\, m\, v_0^2 - G\, M\, m \,/\, r_0 = \tfrac{1}{2}\, m\, v^2 - G\, M\, m \,/\, r \qquad (2)$$

when r is known, v and ϕ can be solved by equations (1) and (2), it is more convenient than the method indicated in chapter 12

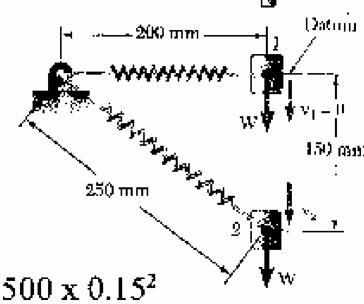
Sample Problem 13-6



$m = 10$ kg $\qquad k = 500$ N/m

undeformed length of spring = 100 mm

$v_1 = 0 \qquad v_2 = ?$



at point 1 $\quad v_1 = 0 \qquad T_1 = 0$

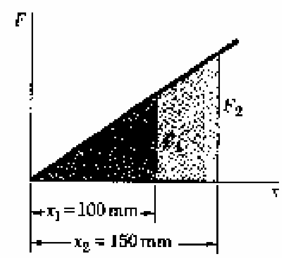
$$V_1 = (V_1)_g + (V_1)_e = 0 + \tfrac{1}{2}\, k\, (x - x_0)^2$$

$$= \tfrac{1}{2}\, 500\, (0.2 - 0.1)^2 = 2.5 \text{ J}$$

at point 2 $\quad T_2 = \tfrac{1}{2}\, m\, v_2^2$

$$V_2 = (V_2)_g + (V_2)_e = -\ 10\, g \times 0.15 + \tfrac{1}{2}\, 500 \times 0.15^2$$

$$= -9.09 \text{ J}$$



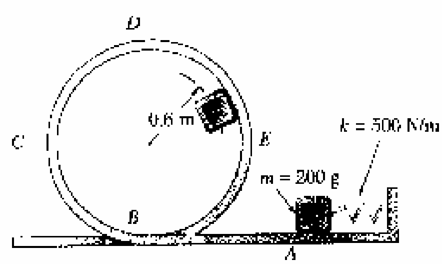
conservation of energy

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 2.5 = \tfrac{1}{2}\, m\, v_2^2 - 9.09$$

$$v_2^2 = 2.318 \qquad v_2 = \pm\ 1.52 \text{ m/s} \uparrow \text{or} \downarrow$$

Sample Problem 13-7



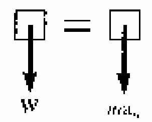
$m = 200$ g $\qquad k = 500$ N/m

determine the minimum deflection of the spring for the block always contact the loop at position D, V_g is maximum, then T is minimum

since the pellet must remain in contact with the loop

set $N = 0$

$\Sigma F_n = m\,a_n \qquad m\,g = m\,a_n \qquad a_n = g$



$a_n = v_D^2 / r$

$v_D^2 = r\,a_n = r\,g = 0.6 \times 9.81 = 5.89 \text{ m}^2/\text{s}^2$

at A $\qquad v_1 = 0 \qquad T_1 = 0$

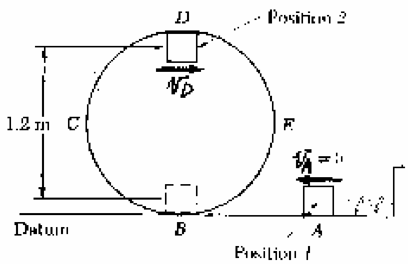
$V_1 = (V_1)_g + (V_1)_e = 0 + \frac{1}{2} k x^2$

$= \frac{1}{2}\, 500\, x^2 = 250\, x^2$

at D $\qquad T_2 = \frac{1}{2} m\, v_D^2 = \frac{1}{2}\, 0.2 \times 5.89 = 0.589 \text{ J}$

$V_2 = (V_2)_g + (V_2)_e = m\,g\,h + 0$

$= 0.2 \times 9.81 \times 1.2 = 2.35 \text{ J}$



conservation of energy

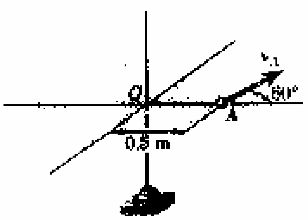
$T_1 + V_1 = T_2 + V_2$

$0 + 250\, x^2 = 0.589 + 2.35 = 2.939$

$x = 0.1084 \text{ m} = 108.4 \text{ mm}$

Sample Problem 13-8

$m = 0.6 \text{ kg} \qquad v_A = 20 \text{ m/s} \qquad \phi = 60^\circ$



sphere moves on frictionless surface

spring is unformed when sphere located at O

$k = 100 \text{ N/m}$

determine r_m, r_m', v_m and v_m'

central force : conservation of angular momentum

$r_A\, m\, v_A \sin 60^\circ = r_m\, m\, v_m$

$$0.5 \times 0.6 \times 20 \sin 60^{\circ} = r_m \times 0.6 \times v_m$$

$$v_m = 8.66 / r_m \qquad (1)$$

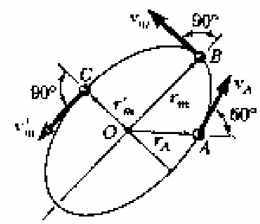
conservative force : conservation of energy

at A $\quad T_A = ½\, m\, v_A^2 = 120$ J

$\quad V_A = ½\, k\, r_A^2 = 12.5$ J

at B $\quad T_B = ½\, m\, v_m^2 = 0.3\, v_m^2$

$\quad V_B = ½\, k\, r_m^2 = 50\, r_m^2$



<u>conservation of energy</u>

$$T_1 + V_1 = T_2 + V_2$$

$$120 + 12.5 = 0.3\, v_m^2 + 50\, r_m^2 \qquad (2)$$

from equations (1) and (2), it is obtained

$$r_m^2 = 2.468 \text{ and } 0.1824$$

$$r_m = 1.571 \text{ m} \qquad r_m' = 0.427 \text{ m}$$

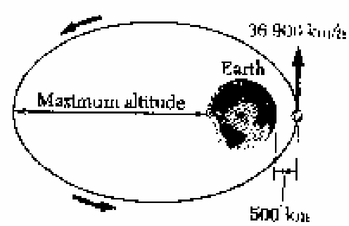
and the corresponding velocities are

$$v_m = 5.51 \text{ m/s} \qquad v_m' = 20.3 \text{ m/s}$$

Sample Problem 13-9

$v_0 = 36900$ km/h for altitude $= 500$ km

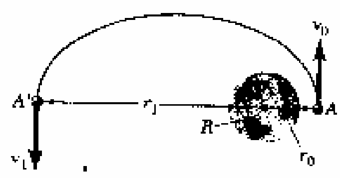
determine (a) the maximum altitude, (b) for minimum altitude $=$ 200 km, the maximum allowable error of launching



(a) conservation of energy

$$T_A + V_A = T_{A'} + V_{A'}$$

$$\frac{1}{2}\, m\, v_0^2 - \frac{G\,M\,m}{r_0} = \frac{1}{2}\, m\, v_1^2 - \frac{G\,M\,m}{r_1} \qquad (1)$$



conservation of angular momentum

$$r_0\, m\, v_0 \;=\; r_1\, m\, v_1 \quad => \quad v_1 \;=\; r_0\, v_0 \,/\, r_1 \qquad (2)$$

(2) into (1)

$$\frac{1}{2}\, v_0^2\, (1 \;-\; \frac{r_0^2}{r_0^2}) \;=\; \frac{G\,M}{r_0}(1 \;-\; \frac{r_0}{r_1}) \qquad \text{or} \qquad 1 \;+\; \frac{r_0}{r_1} \;=\; \frac{2\,G\,M}{r_0\, v_0^2}$$

for $\quad r_1 \;=\; 6370 \;+\; 500 \;=\; 6.87 \times 10^6 \text{ m}$

$\quad v_0 \;=\; 36900 \text{ km/h} \;=\; 1.025 \times 10^4 \text{ m/s}$

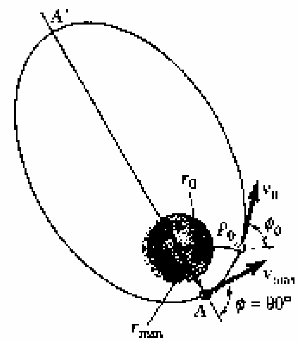
$\quad GM \;=\; gR^2 \;=\; 9.81 \times (6.37 \times 10^6)^2 \;=\; 3.98 \times 10^{14} \text{ m}^3/\text{s}^2$

it is obtained $\qquad \underline{r_0 \;=\; 66.8 \times 10^6 \text{ m} \;=\; 66800 \text{ km}}$

maximum altitude = 66800 - 6370 = 60430 km

(b) for $\quad r_{min} \;=\; 6370 \;+\; 200 \;=\; 6570 \text{ km}$

$\quad r_0 \;=\; 6870 \text{ km}$



conservation of energy and angular momentum are

$$\frac{1}{2}\, m\, v_0^2 \;-\; \frac{G\,M\,m}{r_0} \;=\; \frac{1}{2}\, m\, v_{max}^2 \;-\; \frac{G\,M\,m}{r_{min}} \qquad (4)$$

$$r_0\, m\, v_0 \sin\phi_0 \;=\; r_{min}\, m\, v_{max} \qquad (5)$$

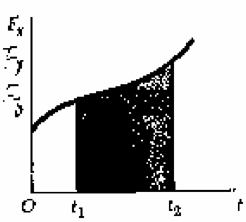
equations (4) and (5) can be solved for $\sin\phi_0$ and v_{max}

$\quad \sin\phi_0 \;=\; 0.9801 \qquad \phi_0 \;=\; 90 \;\pm\; 11.5°$

$\quad v_{max} \;=\; 40584 \text{ km/h}$

13.10 Principle of Impulse and Momentum

$$\boldsymbol{F} \;=\; m\,\boldsymbol{a} \;=\; \frac{d}{dt}(m\,\boldsymbol{v})$$



$$\boldsymbol{F}\, dt \;=\; d\,(m\,\boldsymbol{v})$$

$$\int_{t_1}^{t_2} \boldsymbol{F}\, dt \;=\; m\,\boldsymbol{v}_2 \;-\; m\,\boldsymbol{v}_1$$

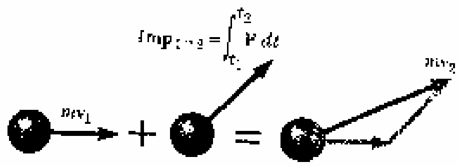
where $\int_{t_1}^{t_2} \boldsymbol{F}\, dt \;=\; \boldsymbol{Imp}_{1\to 2}$ is called linear impulse in rectangular component

$$\boldsymbol{Imp}_{1\to2} = \int_{t_1}^{t_2} \boldsymbol{F}\,dt = \int_{t_1}^{t_2} F_x\,dt\,\boldsymbol{i} + \int_{t_1}^{t_2} F_y\,dt\,\boldsymbol{j} + \int_{t_1}^{t_2} F_z\,dt\,\boldsymbol{k}$$

if $\boldsymbol{F}$ is independent of t, then

$$\boldsymbol{Imp}_{1\to2} = \boldsymbol{F}(t_2 - t_1)$$

or $\quad m\,\boldsymbol{v}_1 + \boldsymbol{Imp}_{1\to2} = m\,\boldsymbol{v}_2$



for several forces act on a particle

$$m\,\boldsymbol{v}_1 + \Sigma\,\boldsymbol{Imp}_{1\to2} = m\,\boldsymbol{v}_2$$

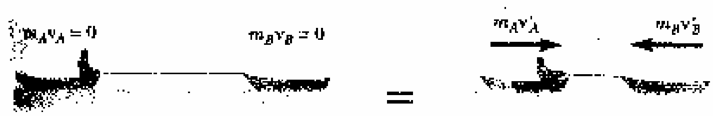
if a problem involves two particles or more

$$\Sigma\,m\,\boldsymbol{v}_1 + \Sigma\,\boldsymbol{Imp}_{1\to2} = \Sigma\,m\,\boldsymbol{v}_2$$

if no external force or $\boldsymbol{R} = 0$ on the particle

$$\Sigma\,m\,\boldsymbol{v}_1 = \Sigma\,m\,\boldsymbol{v}_2 \quad \text{[conservation of linear momentum]}$$

e.g.



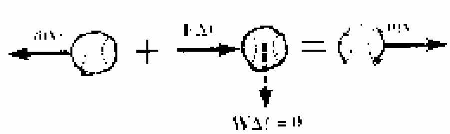
$$\Sigma\,m\,\boldsymbol{v}_1 = \Sigma\,m\,\boldsymbol{v}_2$$

$$0 = m_A\,\boldsymbol{v}_A' + m_B\,\boldsymbol{v}_B' \quad \Rightarrow \quad \boldsymbol{v}_B' = -(m_A/m_B)\,\boldsymbol{v}_A'$$

13.11 Impulsive Motion

very large force act during a very short time interval, this force is called impulsive force

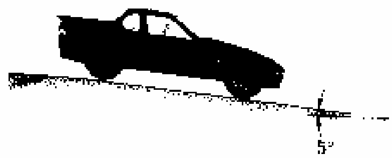
$$m\,\boldsymbol{v}_1 + \boldsymbol{F}\Delta t = m\,\boldsymbol{v}_2$$



$\because$ $\boldsymbol{F}$ is large, then the nonimpulsive force may be neglected during this short time interval for several particles considered, it can be written

$$\Sigma m v_1 + \Sigma F \Delta t = \Sigma m v_2$$

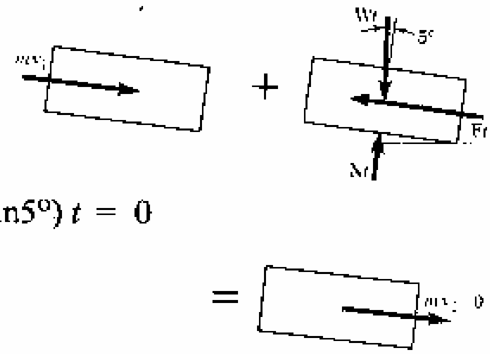
Sample Problem 13-10



$m = 2000$ kg $\qquad v_1 = 90$ km/h

breaking force $= 7.5$ kN

determine the time for stop



$$v_1 = 90 \text{ km/h} = 25 \text{ m/s}$$

$$m v_1 - (F - m g \sin 5^\circ)\, t = 0$$

$$2000 \times 25 - (7500 - 2000 \times 9.81 \sin 5^\circ)\, t = 0$$

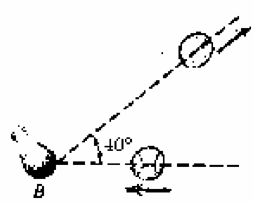
$$t = 8.64 \text{ sec}$$

for the automobile upgrade

$$m v_1 - (F + m g \sin 5^\circ)\, t = 0$$

$t = 5.43$ sec $\qquad$ the time required is shorter

Sample Problem 13.11



$m = 110$ g $\qquad v_1 = 24$ m/s $\leftarrow$

$\Delta t = 0.015$ s $\qquad v_2 = 36$ m/s $\angle 40^\circ$

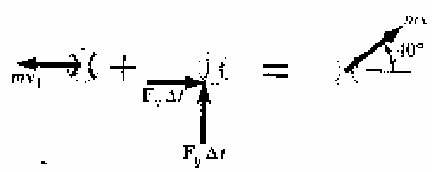
determine the impulsive force

$$m v_1 + \Sigma \boldsymbol{Imp}_{1\rightarrow 2} = m v_2$$

x-comp. $\qquad -m v_1 + F_x \Delta t = m v_2 \cos 40^\circ$

$$-0.11 \times 24 + F_x \times 0.015 = 0.11 \times 36 \cos 40^\circ$$

$$F_x = 378 \text{ N}$$



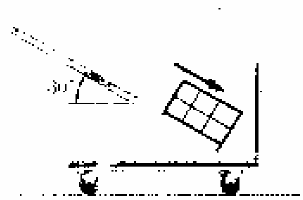
y-comp. $\qquad 0 + F_y \Delta t = m v_2 \sin 40^\circ$

$$F_y \times 0.015 = 0.11 \times 36 \sin 40^\circ$$

$$F_y = 170 \text{ N}$$

$$\boldsymbol{F} = F_x \boldsymbol{i} + F_y \boldsymbol{j} = 414 \text{ N} \measuredangle 24.2^\circ$$

Sample Problem 13-12



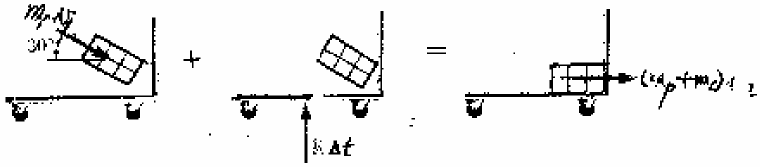
$m_p = 10$ kg $\qquad m_c = 25$ kg

$v_p = 3$ m/s

determine (a) v_c' (b) impulse exerted by the cart

(c) energy lost

(a) for the package-cart system

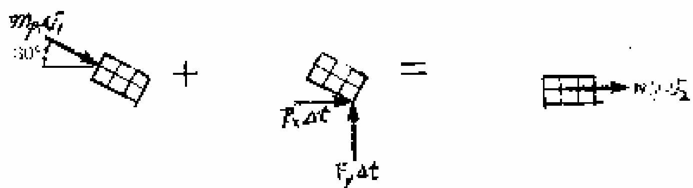


$$m_p \boldsymbol{v}_1 + \Sigma \boldsymbol{Imp}_{1\to2} = (m_p + m_c)\, \boldsymbol{v}_2$$

x-comp. $\qquad 10 \times 3 \cos 30^\circ = (10 + 25)\, v_2$

$$v_2 = 0.742 \text{ m/s}$$

(b) for the package



$$m_p \boldsymbol{v}_1 + \Sigma \boldsymbol{Imp}_{1\to2} = m_p \boldsymbol{v}_2$$

x-comp. $\qquad 10 \times 3 \cos 30^\circ + F_x \Delta t = 10 \times 0.742$

$$F_x \Delta t = -18.56 \text{ N-s}$$

y-comp. $\qquad -10 \times 3 \sin 30^\circ + F_y \Delta t = 0$

$$F_y \Delta t = 15 \text{ N-s}$$

thus $\qquad \boldsymbol{F} \Delta t = 23.9 \text{ N-s} \nwarrow 38.9^\circ$

(c) energy lost

$$T_1 = \tfrac{1}{2} m_p v_1^2 = \tfrac{1}{2}\, 10 \times 3^2 = 45 \text{ J}$$

$$T_2 = \tfrac{1}{2} (m_p + m_c) v_2^2 = \tfrac{1}{2} (10 + 25)\, 0.742^2 = 9.63 \text{ J}$$

$$\text{energy lost} = (T_1 - T_2) / T_1 = 0.786 = 78.6\%$$

energy lost is due to impact

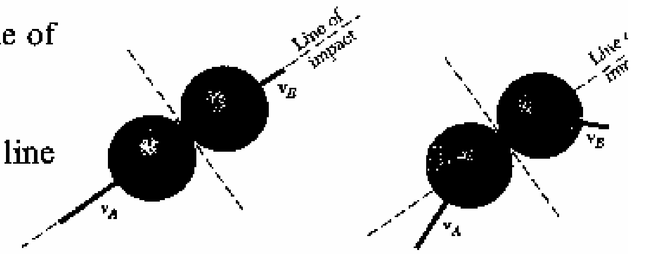
13.12 Impact

a collision between two bodies occurs in very small time interval and exert relatively large forces is called impact

the common normal to the surface in contact is called line of impact

direct impact : the velocity and line of impact in the same direction

oblique impact : the velocity and line of impact is not in the same direction



13.13 Direct Central Impact

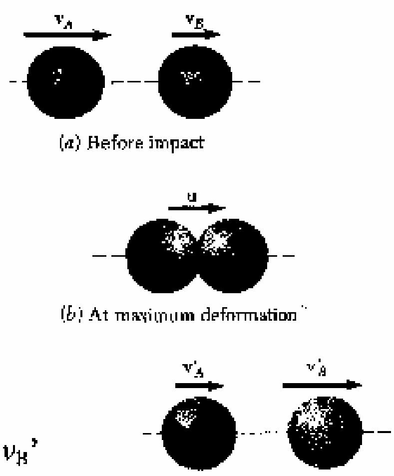
consider two particles A and B moving in the same straight line

if $v_A > v_B$, A will eventually strike B

under impact, two particles will deform

at the end of the period of deformation, they have the same velocity u

after impact, A and B has velocities v_A' and v_B'



(a) Before impact

(b) At maximum deformation

(c) After impact

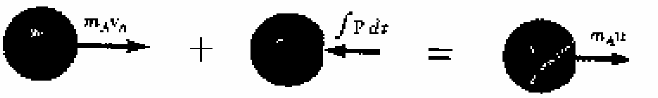
since no impulsive, no external force for the two particles, thus the total momentum is conserved

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

now we have two unknowns v_A' and v_B' in a single scalar equation

we want to establish a second relation between this two unknowns

since the only impulsive force acting on A during the period of deformation (or restitution) is the force $\boldsymbol{P}$ (or $\boldsymbol{Q}$) exerted by B



(a) Period of deformation

(b) Period of restitution

$$m_A v_A - \int P\,dt = m_A u$$

$$m_A u - \int R\,dt = m_A v_A'$$

in general, R is differs from P, and the deformation impulse $\int P\,dt$ is always smaller than the restitution impulse $\int R\,dt$

define the coefficient of restitution

$$e = \frac{\int R\,dt}{\int P\,dt}$$

$0 \leqq e \leqq 1$, depends on the materials of A and B, and it is obtained

$$e = \frac{u - v_A'}{v_A - u}$$

similarly for particle B, we have

$$e = \frac{v_B' - u}{u - v_B}$$

from the above two relations, e can be expressed

$$e = \frac{v_B' - v_A'}{v_A - v_B}$$

or $\quad v_B' - v_A' = e\,(v_A - v_B)$

the relative velocity after impact may be obtained by multiplying their relative velocity before impact by the coefficient of restitution for $e = 0$, that is perfectly plastic impact, in this case, $v_B' = v_A'$, there is no period of restitution, both particles stay together after impact

the total momentum is

$$m_A v_A + m_B v_B = (m_A + m_B) v'$$

for $e = 1$, that is elastic impact, then

$$v_B' - v_A' = v_A - v_B \qquad \text{(a)}$$

together with the total momentum equation

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B' \qquad \text{(b)}$$

it can be solved the total energy of two particles is conserved

eq. (a) $\quad v_A + v_A' = v_B + v_B' \qquad$ (c)

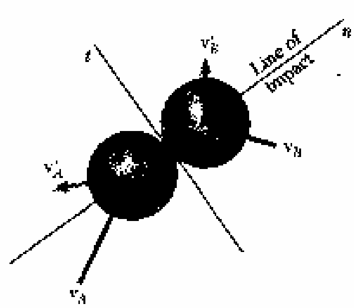
eq. (b) $\quad m_A (v_A - v_A') = m_B (v_B' - v_B) \qquad$ (d)

eq. (c) x (d) $\quad m_A (v_A^2 - v_A'^2) = m_B (v_B'^2 - v_B^2)$

or $\quad \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} m_A v_A'^2 + \frac{1}{2} m_B v_B'^2$

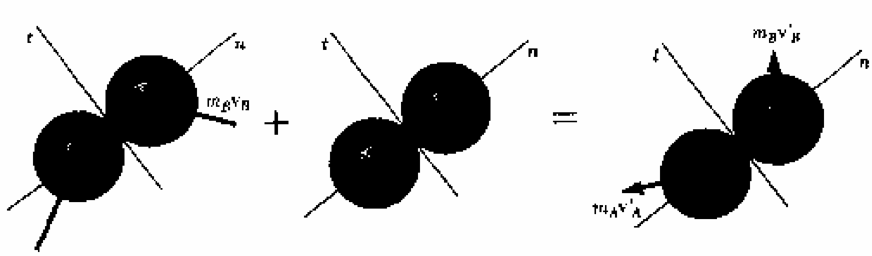
that is the total energy before impact is equal the total energy after impact, in general case of impact, $e < 1$, the total energy is not conserved, the energy lost it transformed into heat and plastic deformation

13.14 Oblique Central Impact



velocities of the two particles are not directed along the line of impact

v_A' and v_B' are unknowns in direction as well as in magnitude, so we need four equations to solve



assume that the particles are perfectly smooth and frictionless, choosing n and t axes along the line of impact and common tangent to the surface of impact, respectively

1. in t direction, the component of the momentum of particles A and B are conserved separately, we have

$$(v_A)_t = (v_A')_t \qquad \text{and} \qquad (v_B)_t = (v_B')_t$$

2. in n direction, the total momentum of two particles is conserved

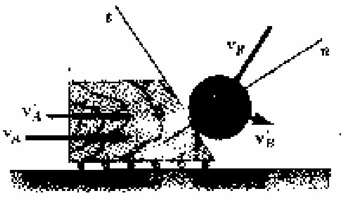
$$m_A (v_A)_n + m_B (v_B)_n = m_A (v_A')_n + m_B (v_B')_n$$

3. in n direction, the relative velocity of two particles after impact is obtained by multiplying the relative velocity of two particles before impact by e

$$(v_B')_n - (v_A')_n = e\,[(v_A)_n - (v_B)_n]$$

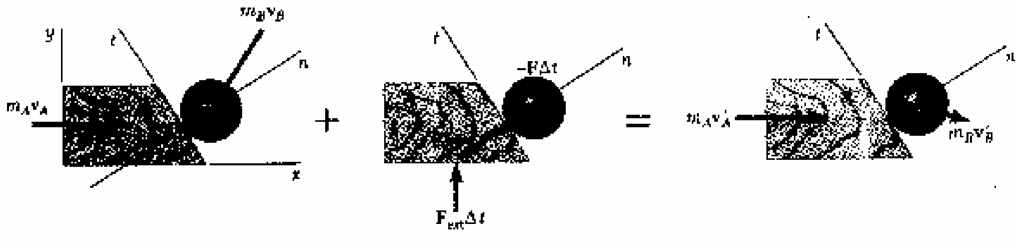
now, we have 4 equations to solve 4 unknowns $(v_A')_t$, $(v_A')_n$, $(v_B')_t$ and $(v_B')_n$

consider some problem for constrained motion



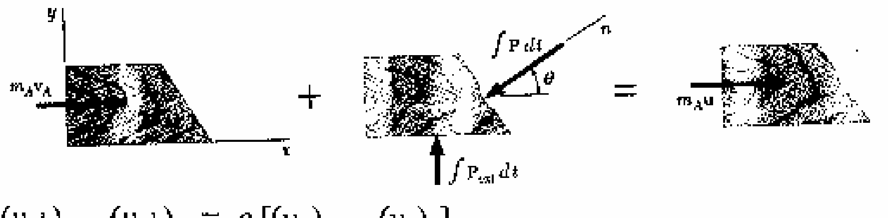
1. $(v_B)_t = (v_B')_t$

2. conservation of momentum in x- direction



$$m_A v_A + m_B (v_B)_x = m_A v_A' + m_B (v_B')_x$$

3. in normal direction



$$(v_B')_n - (v_A')_n = e\,[(v_A)_n - (v_B)_n]$$

3 equations to solve 3 unknowns $(v_B')_t$, $(v_B')_n$ and v_B'

to prove statement (3) is still valid $[\because\ R\,\Delta t\ \neq 0]$

for block A, in x-direction

$$m_A v_A - \int P\,dt \cos\theta = m_A u$$

$$m_A u - \int R\,dt \cos\theta = m_A v_A'$$

$$e = \frac{\int R\,dt}{\int P\,dt} = \frac{u - v_A'}{v_A - u} = \frac{(u - v_A')\cos\theta}{(v_A - u)\cos\theta}$$

$$= \frac{u_n - (v_A')_n}{(v_A)_n - u_n}$$

similarly for ball B

$$e = \frac{(v_B')_n - u_n}{u_n - (v_B)_n}$$

thus, we have $\qquad (v_B')_n - (v_A')_n = e\,[(v_A)_n - (v_B)_n]$

13.15 Problems Involving Energy and Momentum

three different methods

1. Newton' s second law
2. method of work and energy
3. method of impulse and momentum

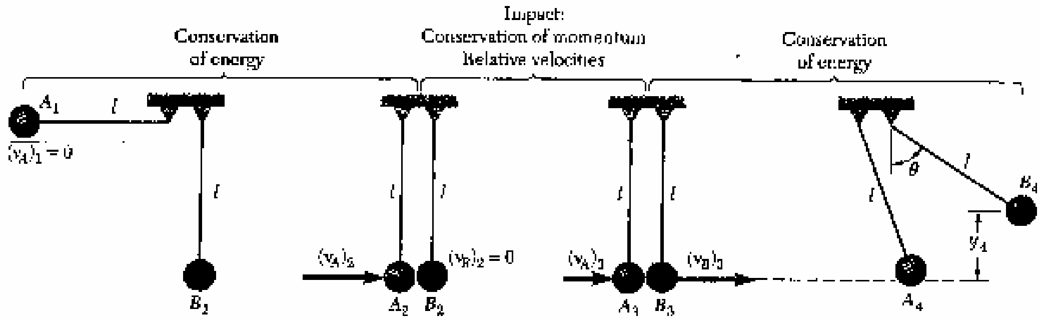
different methods should be used to solve the different part of a problem

there are some limitation in each method :

method of work and energy can directly calculate velocity, however, it must sometimes be supplemented by used of $\Sigma \boldsymbol{F} = m\,\boldsymbol{a}$ to determine acceleration or normal force

method of impulse and momentum to solve problem involving impulsive force, in this kind of problem, method of work and energy cannot be used, since impact involves a lost of energy (plastic deformation) many problems involve only conservative forces except for a short impact phase during which impulsive force act, only the part corresponding to the impact phase calls for the method of impulse and momentum, the other parts may usually be solved by method of work and energy, unless the problem involves the determination of acceleration or normal force

e.g. consider the pendulums A and B



Σ $\boldsymbol{F}$ = m $\boldsymbol{a}$ will be necessary if the tension in the cords holding the pendulums are to be determined

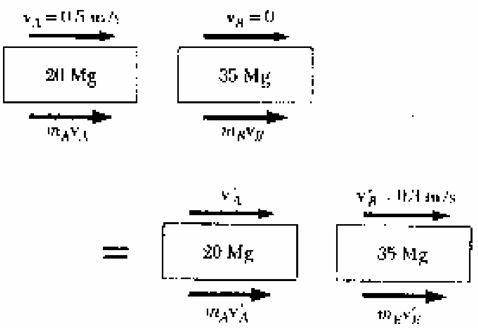
Sample Problem 13-13

m_A = 20 Mg $\quad$ m_B = 35 Mg

v_A = 0.5 m/s $\quad$ v_B = 0 $\quad$ v_B' = 0.3 m/s

determine e and v_A'

conservation of total momentum



$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

$$20 \times 0.5 + 0 = 20 v_A' + 35 \times 0.3$$

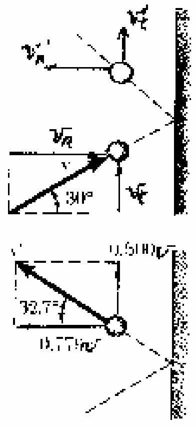
$$v_A' = -0.025 \text{ m/s} \leftarrow$$

$$e = \frac{v_B' - v_A'}{v_A - v_B} = \frac{0.3 + 0.025}{0.5} = 0.65$$

Sample Problem 13-14

$e = 0.9$ wall is frictionless

determine the rebound velocity $\mathbf{v}'$



$$v_n = v \cos 30^\circ \qquad v_t = v \sin 30^\circ$$

conservation of momentum in vertical direction

$$v_t' = v_t = v \sin 30^\circ = 0.5\, v$$

horizontal motion : since the mass of the wall is essentially infinite, and ~~$(v_n)_{wall}$ is considered to be zero~~ before and after impact

$$0 - v_n' = e\,(v_n - 0)$$

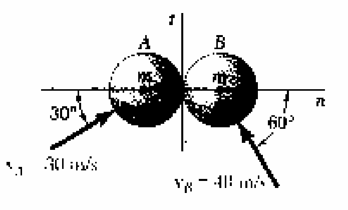
$$v_n' = -0.9\, v_n = -0.9 \times 0.866\, v = -0.779\, v \leftarrow$$

and $\qquad \mathbf{v}' = 0.926\, v \;\nwarrow 32.7^\circ$

Sample Problem 13-15

$m_A = m_B = m \qquad e = 0.9$

determine $\mathbf{v}_A'$ and $\mathbf{v}_B'$



$$(v_A)_n = 30 \cos 30^\circ = 26 \text{ m/s}$$

$$(v_A)_t = 30 \sin 30^\circ = 15 \text{ m/s}$$

$$(v_B)_n = -40 \cos 60^\circ = -20 \text{ m/s}$$

$$(v_B)_t = 40 \sin 60^\circ = 34.6 \text{ m/s}$$

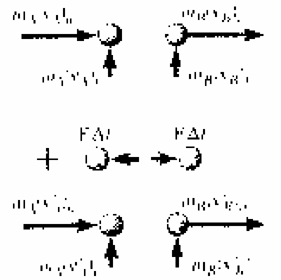
the vertical component of velocity for each ball is unchanged

$$(v_A)_t' = 15 \text{ m/s} \uparrow \qquad (v_B)_t' = 34.6 \text{ m/s} \uparrow$$

in normal direction

$$m_A\,(v_A)_n + m_B\,(v_B)_n = m_A\,(v_A)_n' + m_B\,(v_B)_n'$$

$$26\, m - 20\, m = (v_A)_n'\, m + (v_B)_n'\, m$$



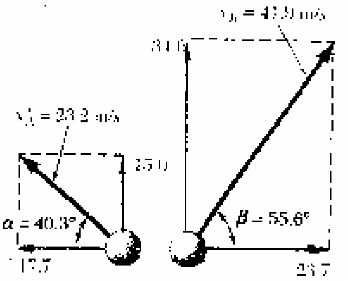
$$(v_A)_n' + (v_B)_n' = 6 \qquad (1)$$

relative velocity

$$(v_B)_n' - (v_A)_n' = e\,[(v_A)_n - (v_B)_n]$$
$$= 0.9\,[26 - (-20)] = 41.4 \qquad (2)$$

solve for equations (1) and (2), it is obtained

$$(v_A)_n' = -17.7 \text{ m/s} \leftarrow \qquad (v_B)_n' = 23.7 \text{ m/s} \rightarrow$$



then $\quad \mathbf{v}_A' = 23.2$ m/s $\nwarrow$ 40.3°

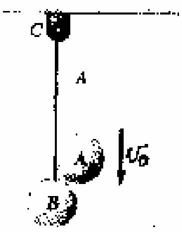
$\mathbf{v}_B' = 41.9$ m/s $\nearrow$ 55.6°

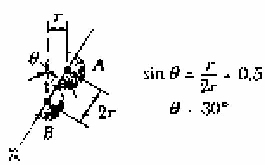
Sample Problem 13-16

$m_A = m_B = m \qquad e = 1 \qquad$ no friction

determine $\mathbf{v}'$ for each ball after impact

$$\sin\theta = r / 2r = 0.5 \qquad \theta = 30^\circ$$



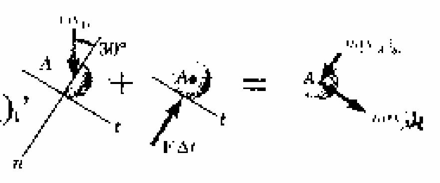


impulse-momentum for ball A

$$m\,\mathbf{v}_A + \mathbf{F}\,\Delta t = m\,\mathbf{v}_A'$$

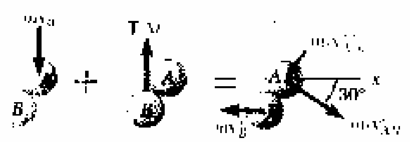
in t components : $\qquad m\,v_0 \sin 30^\circ + 0 = m\,(v_A)_t'$

$$(v_A)_t' = 0.5\,v_0 \qquad (1)$$



impulse-momentum for balls A and B

$$m\,\mathbf{v}_A + \mathbf{T}\,\Delta t = m\,\mathbf{v}_A' + m\,\mathbf{v}_B'$$



in x components :

$$0 = m\,(v_A)_t' \cos 30^\circ - m\,(v_A)_n' \sin 30^\circ - m\,v_B'$$

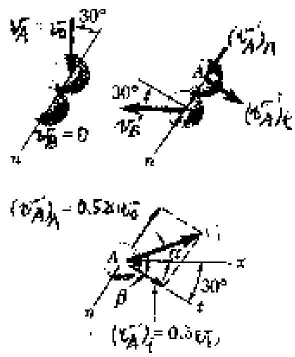
$$0.5\,(v_A)_n' + v_B' = 0.433\,v_0 \qquad (2)$$

relative velocities along the line of impact

$(v_B)_n' - (v_A)_n' = (v_A)_n - (v_B)_n \qquad [e = 1]$

$v_B' \sin 30^\circ - (v_A)_n' = v_0 \cos 30^\circ$

$0.5\, v_B' - (v_A)_n' = 0.866\, v_0 \qquad (3)$



solve for equations (2) and (3)

$(v_A)_n' = -0.52\, v_0 \qquad v_B' = 0.693\, v_0$

or $\quad v_A' = 0.721\, v_0 \qquad \beta = 46.1^\circ$

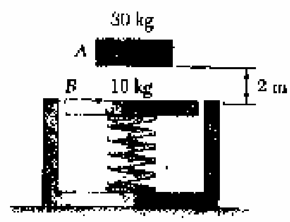
i.e. $\quad v_A' = 0.721\, v_0 \angle 16.1^\circ \qquad v_B' = 0.693\, v_0 \leftarrow$

note : check the total energy

$$v_A'^2 + v_B'^2 = (0.721^2 + 0.693^2)\, v_0^2$$

$$= (0.52 + 0.48)\, v_0^2 = v_0^2 \qquad [\text{O. K.}]$$

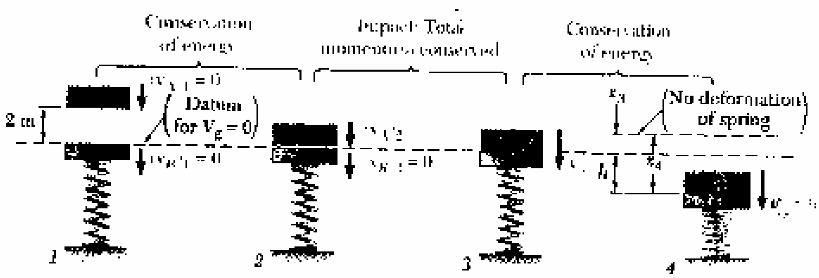
Sample Problem 13-17



$m_A = 30$ kg $\quad m_B = 10$ kg $\quad k = 20$ kN/m

$e = 0 \qquad$ [perfectly plastic]

determine the maximum deflection of the pan



consider the energy for block A

$$T_1 + V_1 = T_2 + V_2$$

$$0 + mg \times 2 = \tfrac{1}{2} m (v_A)_2^2$$

$$(v_A)_2 = 6.26 \text{ m/s}$$

during impact, conservation of linear momentum

$$m_A (v_A)_2 + m_B (v_B)_2 = (m_A + m_B)\, v_3 \qquad [e = 0,\ (v_A)_3 = (v_B)_3]$$

$$30 \times 6.26 + 0 = 40\, v_3$$

$$v_3 = 4.7 \text{ m/s}$$

after impact, conservation of energy can be used again

$$W_B = k\, x_3$$

$$10 \times 9.81 = 20 \times 10^3\, x_3$$

$$x_3 = 4.91 \times 10^{-3} \text{ m}$$

$$T_3 = ½\,(m_A + m_B)\, v_3^2 = 442 \text{ J}$$

$$V_3 = (V_g)_3 + (V_e)_3 = 0 + ½\, k\, x_3^2 = 0.241 \text{ J}$$

$$T_4 = 0$$

$$V_4 = (V_g)_4 + (V_e)_4 = (W_A + W_B)(-h) + ½\, k\, x_4^2$$

$$T_3 + V_3 = T_4 + V_4$$

$$442 + 0 = -(30 + 10) \times 9.81\,(x_4 - 4.91 \times 10^{-3}) + 10^4\, x_4^2$$

$$x_4 = 0.230 \text{ m}$$

$$h = x_4 - x_3 = 0.225 \text{ m} = 225 \text{ mm}$$