Chapter 12 Kinetics of Particles

12.1 Introduction

Newton's 1st and 3rd laws were used in Statics

Newton's 2nd law will be used in dynamics to relate the motion of bodies with forces acting on it

linear momentum : $\boldsymbol{L} = m\,\boldsymbol{v}$

m : mass of particle $\boldsymbol{v}$: velocity of particle

angular momentum : $\boldsymbol{H}_0 = \boldsymbol{r} \times m\,\boldsymbol{v}$

$\boldsymbol{r}$: position vector with respect to O

O : reference point

12.2 Newton's Second Law of Motion

if the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force

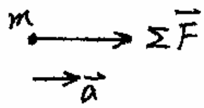
$F_1 / a_1 = F_2 / a_2 = \cdots\cdots = \text{constant}$

F : magnitude of force

a : magnitude of acceleration

$\boldsymbol{F} = m\,\boldsymbol{a}$ $\boldsymbol{F}$: resultant

$\boldsymbol{a}$: acceleration vector



if $\boldsymbol{F} = 0$ => $\boldsymbol{a} = 0$

i.e. if $v_0 = 0$: particle is remain at rest

if $v_0 \neq 0$: particle moves with the constant speed in a straight line

Newton's 1st law

another statement of Newton's 2^{nd} law : a particle acted upon by a force moves so that the force vector is equal to the time rate of change of the linear momentum vector

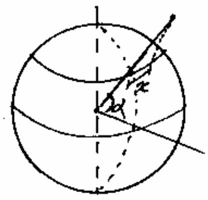
Newtonian frame of reference (inertial frame) : constant orientation with respect to the stars, origin must be either attached to the sun or move with a constant velocity with respect to the sun

so that the axes attached to the earth do not constitute a Newtonian frame of reference, because the earth rotate with respect to stars and accelerated with respect to the sun

but in engineering application, this frame is used as the Newtonian frame without any appreciable error

Newton's 2^{nd} law does not held if the linear momentum or acceleration is measured with respect to moving axes, such as axes attached to an accelerated car

consider a freely falling body drop from sky



$\omega = 7.29 \times 10^{-5}$ rad/s for the earth

$x = \frac{2}{3}\, \omega\, (2\, h^3 / g)^{1/2} \cos a$

for $a = 45°$ $h = 200$ m

$x = 43.9$ mm

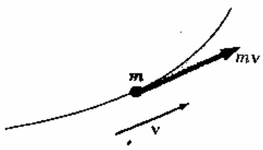
12.3 Linear Momentum of Particle, Rate of Change of Linear Momentum

linear momentum is defined

$$L = m\, v$$

where m : mass of particle v : velocity of particle

Newton's 2^{nd} law



$$\Sigma F = \dot{L} = d(mv) / dt$$

if m = constant $\qquad \boldsymbol{F} = m\,\boldsymbol{a}$

if $m \neq$ constant $\qquad \boldsymbol{F} = m\,\boldsymbol{a} + \dot{m}\,\boldsymbol{v}$

for $\dot{m} \neq 0$, grain or loss mass system, will discuss in Chapter 14

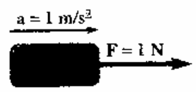
if $\Sigma\,\boldsymbol{F} = 0$, then the linear momentum of the particle remains constant, both in magnitude and direction

i.e. conservation of linear momentum

12.4 System of Units

international system of units (SI units) : basic units are [kg, m, sec]

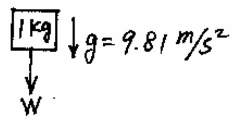
$$1\text{ N} = (1\text{ kg})\,(1\text{ m/s}^2) = 1\text{ kg-m/s}^2$$



for the weight of a 1 kg mass

$$W = (1\text{ kg})\,(9.81\text{ m/s}^2) = 9.81\text{ N}$$

$$W = m\,g$$

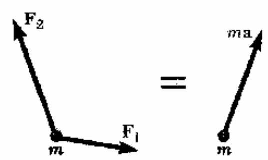


the unit of linear momentum is

$$\boldsymbol{L} = m\,\boldsymbol{v} \qquad [\text{kg-m/s}]$$

12.5 Equations of Motion

$$\Sigma\,\boldsymbol{F} = m\,\boldsymbol{a} \qquad \text{for } m = \text{constant}$$



in rectangular components

$$\Sigma\,(F_x\,\boldsymbol{i} + F_y\,\boldsymbol{j} + F_z\,\boldsymbol{k}) = m\,(a_x\,\boldsymbol{i} + a_y\,\boldsymbol{j} + a_z\,\boldsymbol{k})$$

i.e. $\quad \Sigma\,F_x = m\,a_x = m\,\ddot{x}$

$$\Sigma\,F_y = m\,a_y = m\,\ddot{y}$$

$$\Sigma\,F_z = m\,a_z = m\,\ddot{z}$$

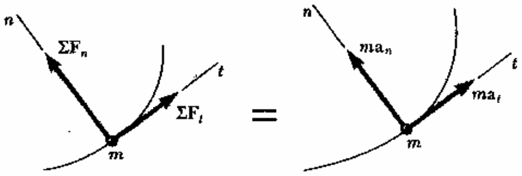
for the motion of projectile

$$\Sigma\,F_x = \Sigma\,F_z = 0$$

$$\Sigma F_y = = - W = - m g \qquad \Rightarrow \qquad \ddot{y} = - g$$

in tangential and normal components

$$\Sigma F_t = m\, a_t = m\, dv/dt$$
$$\Sigma F_n = m\, a_n = m\, v^2/\rho$$

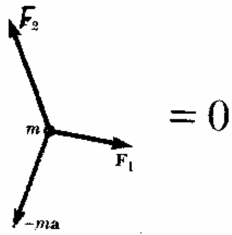


in radial and transverse components

$$\Sigma F_r = m\, a_r = m\,(\ddot{r} - r\,\dot{\theta}^2)$$
$$\Sigma F_\theta = m\, a_\theta = m\,(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})$$

12.6 Dynamic Equilibrium

$$\Sigma \boldsymbol{F} - m\,\boldsymbol{a} = 0$$



the vector - ***ma*** is called inertia vector

then the particle is said to be dynamic equilibrium

if the inertia vector is included, it is obtained

rectangular components $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$

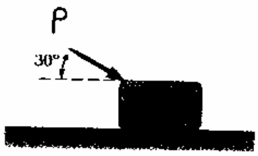
tangential and normal components $\Sigma F_t = \Sigma F_n = 0$

radial and transverse components $\Sigma F_r = \Sigma F_\theta = 0$

Sample Problem 12-1

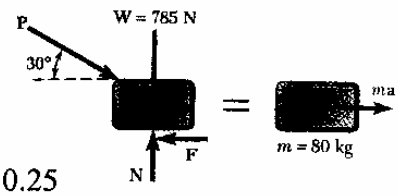
$m = 80$ kg $\qquad a = 2.5$ m/s$^2 \rightarrow$

$\mu_k = 0.25$



determine the magnitude of the force ***P***

$$W = m\,g = 80 \times 9.81 = 785 \text{ N}$$
$$F = \mu_k N = 0.25\, N$$



$\Sigma F_x = m\, a_x \qquad P\cos 30^\circ - 0.25\, N = 80 \times 0.25$

$\Sigma F_y = m a_y = 0 \qquad N - P\sin 30^\circ - W = 0$

2 equations to solve two unknowns P and N

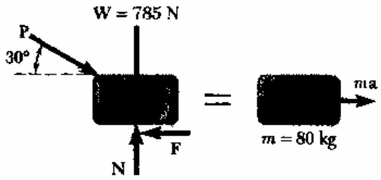
$$P = 535\ \text{N} \qquad N = 1052.5\ \text{N}$$

Sample Problem 12-2

$$m = 80\ \text{kg} \qquad P = 298\ \text{N}$$

$$v_0 = 0 \qquad x = 12\ \text{m for } t = 4\ \text{s}$$

determine μ_k



$$a = 2x/t^2 = 2 \times 12/4^2 = 1.5\ \text{m/s}^2$$

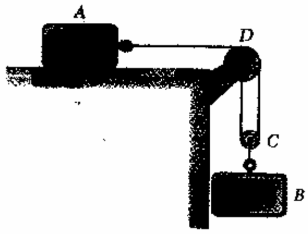
$$\Sigma F_y = ma_y = 0 \qquad N - 298 \sin 30^\circ - 785 = 0$$

$$N = 934\ \text{N}$$

$$\Sigma F_x = m\,a_x \qquad 298 \cos 30^\circ - 934\,\mu_k = 80 \times 1.5$$

$$\mu_k = 0.148$$

Sample Problem 12-3

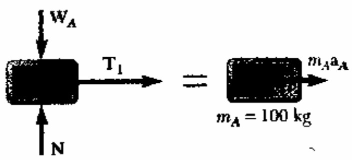


$$m_A = 100\ \text{kg} \qquad m_B = 300\ \text{kg}$$

determine a_A, a_B and tensions in each cord

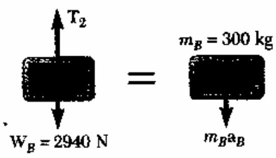
$$s_B = \tfrac{1}{2} s_A \quad \Rightarrow \quad a_B = \tfrac{1}{2} a_A$$

for block A



$$\Sigma F_x = m_A\,a_A \qquad T_1 = 100\,a_A$$

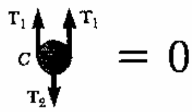
for block B



$$\Sigma F_y = m_B\,a_B \qquad m_B\,g - T_2 = 300\,a_B$$

for the pulley C

$$\Sigma F_y = m_C\,a_C = 0 \qquad T_2 - 2\,T_1 = 0$$

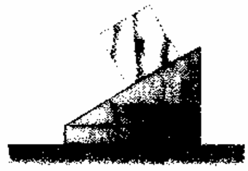


4 equations to solve 4 unknowns

$$a_A = 8.4\ \text{m/s}^2 \qquad a_B = 4.2\ \text{m/s}^2$$

$T_1 = 840$ N $\qquad\qquad T_2 = 1680$ N

Sample Problem 12-4

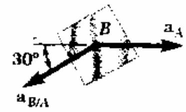


$m_A = 10$ kg $\qquad m_B = 4$ kg

neglecting friction, determine

(a) $\boldsymbol{a}_A$ $\qquad$ (b) $\boldsymbol{a}_{B/A}$

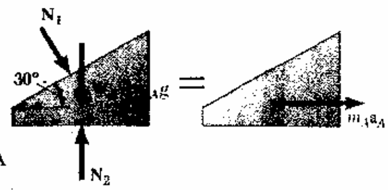
for the wedge A



$\Sigma F_x = m_A a_A \qquad N_1 \sin 30^\circ = m_A a_A$

$0.5 N_1 = m_A a_A \quad (1)$

for the block B

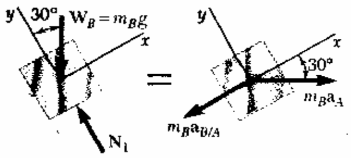


$\boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A}$

$\Sigma F_x = m_B a_x$

$- m_B g \sin 30^\circ = m_B a_A \cos 30^\circ - m_B a_{B/A}$

$a_{B/A} = a_A \cos 30^\circ + g \sin 30^\circ \quad (2)$



$\Sigma F_y = m_B a_y$

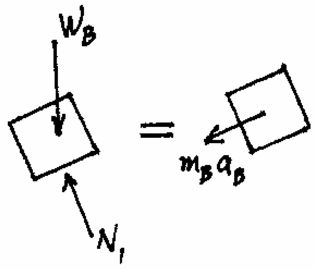
$N_1 - m_B g \cos 30^\circ = - m_B a_A \sin 30^\circ \quad (3)$

(1) and (3) $\qquad a_A = \dfrac{m_B g \cos 30^\circ}{2 m_A + m_B \sin 30^\circ} = 1.54 \text{ m/s}^2 \rightarrow$

(2) => $\qquad a_{B/A} = 6.24 \text{ m/s}^2 \measuredangle 30^\circ$

also, N_1 can be obtained $\qquad N_1 = 30.8$ N $\qquad$ [from eq. 1]

wrong method :



for the block B

$\Sigma F_x = m_B a_x \qquad m_B g \sin 30^\circ = m_B a_B$

$a_B = 0.5 g = 4.90 \text{ m/s}^2 \measuredangle 30^\circ$

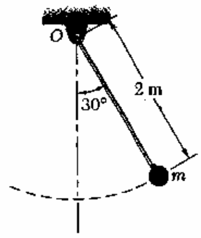
$\Sigma F_y = 0 \qquad N_1 = m_B g \text{ con } 30^\circ$

for the wedge A

$\Sigma F_x = m_A a_A \qquad m_A a_A = N_1 \sin 30^\circ$

$a_A = 1.697 \text{ m/s}^2 \rightarrow$

Sample Problem 12-5



$T = 2.5\, m\, g \quad \text{for} \quad \theta = 30^\circ$

determine $\boldsymbol{v}$ and $\boldsymbol{a}$

$\Sigma F_t = m\, a_t \qquad mg \sin 30^\circ = m\, a_t$

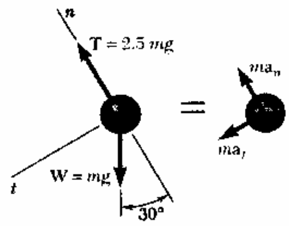
$a_t = 4.9 \text{ m/s}^2 \swarrow$

$\Sigma F_n = m\, a_n \qquad 2.5\, mg - mg \cos 30^\circ = m\, a_n$

$a_n = 16.03 \text{ m/s}^2 \nwarrow$

$a_n = v^2 / \rho \qquad v^2 = a_n \rho = 16.03 \times 2 = 32.06$

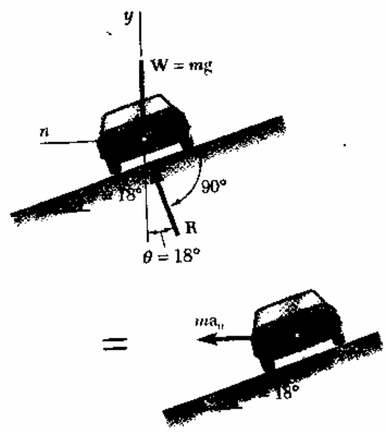
$\text{v} = \pm\ 5.66 \text{ m/s} \swarrow\!\nearrow$



Sample Problem 12-6

$\rho = 125 \text{ m} \qquad \theta = 18^\circ$

determine v if no lateral friction force on the wheel is known



$\Sigma F_y = 0 \qquad R \cos \theta - W = 0$

$R = mg / \cos \theta$

$\Sigma F_n = m\, a_n \qquad R \sin \theta = m\, a_n = m\, v^2 / \rho$

$(mg/\cos \theta) \sin \theta = m\, v^2 / \rho$

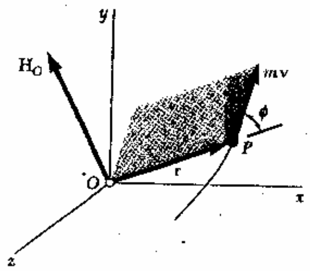
$v^2 = \rho g \tan \theta = 125 \times 9.81 \times \tan 18^\circ$

$v = 19.96 \text{ m/s} = 71.9 \text{ km/h}$

12.7 Angular Momentum of Particle, Rate of Change of Angular Momentum

12.7 Angular Momentum of Particle, Rate of Change of Angular Momentum

the moment of the linear momentum vector $\boldsymbol{mv}$ about a point O is called the moment of momentum, or angular momentum, of the particle, and is denoted by $\boldsymbol{H}_0$



$$\boldsymbol{H}_0 = \boldsymbol{r} \times \boldsymbol{mv} \qquad H_0 = r\, m\, v \sin\phi$$

SI unit for H_0 is m (kg-m/s) = kg-m^2/s

$$\boldsymbol{H}_0 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix} = H_x \boldsymbol{i} + H_y \boldsymbol{j} + H_z \boldsymbol{k}$$

$$H_x = m\,(y\, v_z - z\, v_y)$$

$$H_y = m\,(z\, v_x - x\, v_z)$$

$$H_z = m\,(x\, v_y - y\, v_x)$$

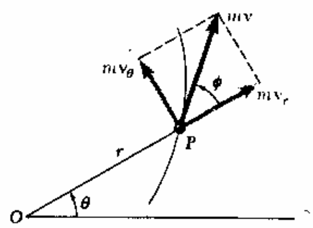
$$H_0 = (H_x^2 + H_y^2 + H_z^2)^{1/2}$$

if particle moving in xy plane, then $z = v_z = 0$

$$H_x = H_y = 0$$

$$H_0 = H_z = m\,(x\, v_y - y\, v_x)$$

for polar coordinate



$$H_0 = r\, m\, v \sin\phi = r\, m\, v_\theta$$

$$= m\, r^2\, \dot{\theta} \qquad [v_\theta = r\,\dot{\theta}]$$

rate of change of angular momentum

$$\dot{\boldsymbol{H}}_0 = \dot{\boldsymbol{r}} \times m\, \boldsymbol{v} + \boldsymbol{r} \times m\, \dot{\boldsymbol{v}}$$

$$= \boldsymbol{v} \times m\, \boldsymbol{v} + \boldsymbol{r} \times m\, \boldsymbol{a}$$

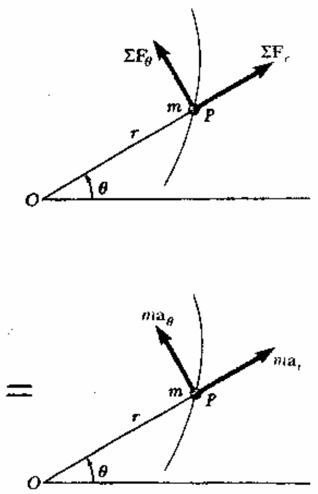
$$= \boldsymbol{r} \times \Sigma \boldsymbol{F} = \Sigma \boldsymbol{M}_0$$

$$\dot{\boldsymbol{H}}_0 = \Sigma \boldsymbol{M}_0$$

the sum of moments about O of the forces acting on the particle is equal to rate of change of angular momentum

12.8 Equations of Motion in Terms of Radial and Transverse Components

in polar coordinate system



$$\Sigma F_r = m\, a_r = m\,(\ddot{r} - r\,\dot{\theta}^2)$$

$$\Sigma F_\theta = m\, a_\theta = m\,(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})$$

$\because \quad \boldsymbol{r} \perp \Sigma F_\theta\, \boldsymbol{e}_\theta$

$\therefore \quad \Sigma \boldsymbol{M}_0 = \boldsymbol{r} \times \Sigma F_\theta\, \boldsymbol{e}_\theta$

or $\quad \Sigma M_0 = r\, \Sigma F_\theta$

$$\Sigma M_0 = \dot{H}_0 = \frac{d}{dt}(m\, r^2\, \dot{\theta})$$

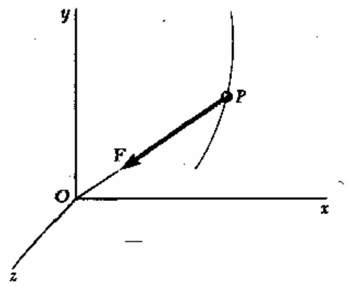
$$r\, \Sigma F_\theta = m\,(r^2\, \ddot{\theta} + 2\, r\, \dot{r}\, \dot{\theta})$$

or $\quad \Sigma F_\theta = m\,(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})$

the same form of a_θ is obtained

12.9 Motion Under a Central Force, Conservation of Angular Momentum

when $\Sigma\, \boldsymbol{F}$ directed toward or away from a fixed point, the particle is said to be moving under central force



then $\quad \Sigma \boldsymbol{M}_0 = 0$

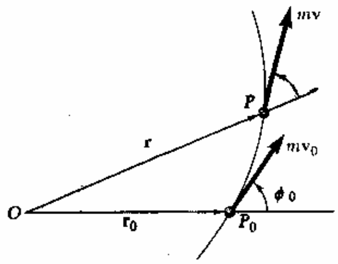
hence $\quad \dot{\boldsymbol{H}}_0 = 0$

$\quad \boldsymbol{H}_0 = \text{constant}$

thus the angular momentum of a particle moving under central force is constant, both in magnitude and direction

$$\boldsymbol{H}_0 = \boldsymbol{r} \times m\, \boldsymbol{v} = \text{constant}$$

$\boldsymbol{r}$ and $\boldsymbol{v}$ must $\perp$ $\boldsymbol{H}_0$, $\because$ the direction of $\boldsymbol{H}_0$ is constant, $\therefore$ the particle moves in a fixed plane $\perp$ $\boldsymbol{H}_0$



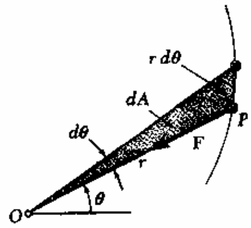
$\because$ the magnitude of $\boldsymbol{H}_0$ is also constant

$\therefore \quad H_0 = r\, m\, v \sin\phi = r_0\, m\, v_0 \sin\phi_0$

or $\quad H_0 = m\, r^2\, \dot{\theta}$

denoting $h = H_0/m$, the angular momentum per unit mass, then

$$h = r^2\, \dot{\theta} = \text{constant}$$



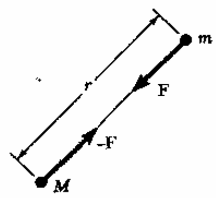
and $\quad dA = \frac{1}{2} r^2\, d\theta$

denoting the areal velocity $= dA / dt$, then

$$dA / dt = \tfrac{1}{2} r^2\, \dot{\theta} = h / 2 = \text{constant}$$

thus if a particle moves under central force, its areal velocity is constant

12.10 Newton' s Law of Gravitation



law of universal gravitation between two particles

$$F = G\, M\, m / r^2$$

where $G = 6.673 \times 10^{-11}$ m^3/kg-s^2 is the universal constant (constant of gravitation), and r is the distance between center to center of two particles

consider a particle of mass m on the surface of earth

$$W = m\, g = G\, M\, m / R^2$$

where M is the mass of earth ($M \simeq 6 \times 10^{24}$ kg), and R is the radius of earth ($R = 6.37 \times 10^6$ m), it is obtained

$$G\, M = g\, R^2$$

or $\quad g = G\, M / R^2$

thus W and g will vary with altitude and latitude of the point considered

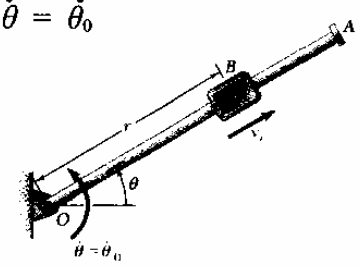
$$g = 9.781 \text{ m/s}^2 \text{ (at equator)} \sim 9.833 \text{ m/s}^2 \text{ (at poles)}$$

in average $\quad g = 9.81 \text{ m/s}^2 \qquad R = 6.37 \times 10^6 \text{ m}$

Sample Problem 12.7

at $t = 0, \quad r = r_0 \qquad v_r = 0 \qquad \dot{\theta} = \dot{\theta}_0$

determine v_r and F

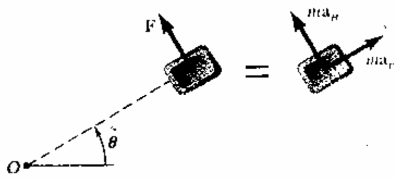


$$\Sigma F_r = m\, a_r \qquad m\,(\ddot{r} - r\,\dot{\theta}^2) = 0 \qquad (1)$$

$$\Sigma F_\theta = m\, a_\theta \qquad m\,(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}) = F \qquad (2)$$

$$(1) \Rightarrow \qquad r\,\dot{\theta}^2 = \ddot{r} = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = \dot{r}\,\frac{dv_r}{dr} = v_r\,\frac{dv_r}{dr}$$

$\dot{\theta} = \dot{\theta}_0 = \text{constant} \qquad \ddot{\theta} = 0$



thus $\qquad r\,\dot{\theta}_0^2\, dr = v_r\, dv_r$

$$\dot{\theta}_0^2\,(r^2 - r_0^2) = (v_r^2 - 0)$$

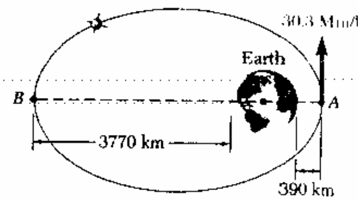
or $\qquad \dot{r} = v_r = \dot{\theta}_0\,(r^2 - r_0^2)^{1/2}$

$$(2) \Rightarrow \qquad F = 2\,m\,\dot{r}\,\dot{\theta} = 2\,m\,v_r\,\dot{\theta}_0 = 2\,m\,\dot{\theta}_0^2\,(r^2 - r_0^2)^{1/2}$$

Sample Problem 12-8

$v_A = 30.3 \times 10^3$ km/h

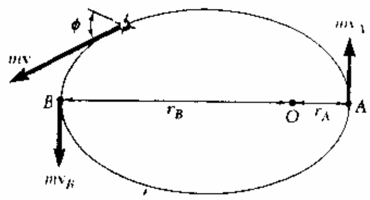
determine v_B



$R = 6370$ km

$$r_A = OA = 6370 + 390 = 6760 \text{ km}$$

$$r_B = OB = 6370 + 3770 = 10140 \text{ km}$$



because the satellite is moving under central force, thus its angular momentum is constant

$$H_0 = r\, m\, v \sin\phi = \text{constant}$$

$$\phi = 90^\circ \quad \text{at } A \text{ and } B$$

$$r_A\, m\, v_A = r_B\, m\, v_B$$

$$v_B = r_A\, v_A / r_B = 6760 \times 30.3 \times 10^3 / 10140 = 20.2 \times 10^3 \text{ km/h}$$

12.11 Trajectory of a Particle Under a Central Force

in this section, it is proposed to obtain the differential equation which defines the trajectory of the particle moving under central force

$$\Sigma \boldsymbol{F} = m\boldsymbol{a} \qquad m(\ddot{r} - r\dot{\theta}^2) = -F$$

$$m(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}) = 0$$

central force => conservation of angular momentum

$$r^2\dot{\theta} = h = \text{constant} \qquad \Rightarrow \qquad \dot{\theta} = h / r^2$$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right)$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\dot{\theta} = \frac{h}{r^2}\frac{d}{d\theta}\left[-h\frac{d}{d\theta}\left(\frac{1}{r}\right)\right] = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$$

let $u = 1 / r$

$$\ddot{r} = -h^2 u^2 \frac{d^2u}{d\theta^2}$$

but $\quad \ddot{r} = r\dot{\theta}^2 - \dfrac{F}{m} = \dfrac{1}{u}h^2u^4 - \dfrac{F}{m} = -h^2u^2\dfrac{d^2u}{d\theta^2}$

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{m\,h^2u^2}$$

this is the differential equation to describe the relation between r and θ (trajectory of the particle)

12.12 Application to Space Mechanics

consider a space vehicle moving around the earth under central force, the gravitational attraction between the earth and vehicle is

$$F = GMm / r^2 = GMm u^2$$

where M : mass of the earth m : mass of the vehicle

r : distance from center of earth to vehicle u : $1/r$

the differential equation becomes

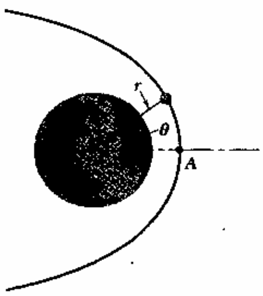
$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2} = \text{constant}$$

to solve this differential equation, and obtain the result

$$u = \frac{1}{r} = \frac{GM}{h^2} + C\cos\theta$$

where C is a constant, this equation is a polar equation of a conic section (elliptic, parabola, or hyperbola)

define the eccentricity ε



$$\varepsilon = \frac{C}{GM/h^2} = \frac{Ch^2}{GM}$$

then $\dfrac{1}{r} = \dfrac{GM}{h^2}(1 + \varepsilon\cos\theta)$

this equation represents three possible trajectories

1. $\varepsilon > 1$ or $C > GM/h^2$

   when $\cos(\pm\theta) = -1/\varepsilon = -GM/Ch^2$

   => $1/r = 0$ => $r \rightarrow \infty$

   that is a hyperbola

2. $\varepsilon = 1$ or $C = GM/h^2$

   when $\theta = \pm 180°$ $\cos(\pm\theta) = -1$

   $r \rightarrow \infty$

that is a parabola

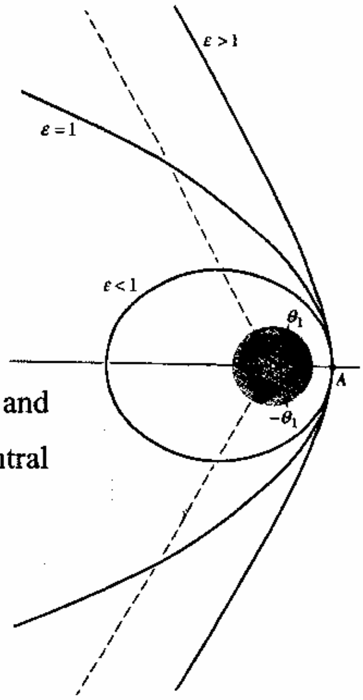
3. $\varepsilon < 1$ or $C < GM/h^2$

for any value of θ, $1/r \neq 0$

the radius remains finite, that is an ellipse

for $\varepsilon = 0$, then $1/r = GM/h^2 =$ constant

that is a circle



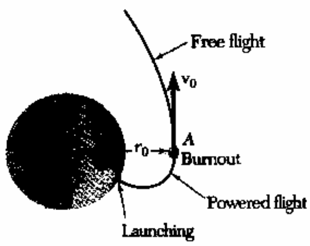
consider a rocket launching on the earth surface, and burnout at point A, it is free flight at stage A (central force only)

at point A $h = r_0 v_0 = r_0^2 \dot{\theta}$

then h can be determined

$W = m g = G M m / R^2$

then $G M = g R^2$ is also determined



the equation of trajectory is

$$\frac{1}{r} = \frac{GM}{h^2}(1 + \varepsilon \cos\theta) = \frac{GM}{h^2} + C\cos\theta$$

for $\theta = 0^\circ$ $r = r_0$

$C = 1/r_0 - GM/h^2$

we know for $\varepsilon = 1$ or $C = GM/h^2$, the trajectory is a parabola

$GM/h^2 = 1/r_0 - GM/h^2$

i.e. $1/r_0 = 2\, GM/h^2$

or $h^2 = 2\, G M r_0$

but $h = r_0 v_0$

$\therefore$ $2\, G M r_0 = r_0^2 v_0^2$

or $v_0 = (2\, G M / r_0)^{1/2} = (2\, g R^2 / r_0)^{1/2}$

define $\qquad v_{esc} = (2\,GM/r_0)^{½} = (2\,gR^2/r_0)^{½}$ $\quad$ is the escape velocity

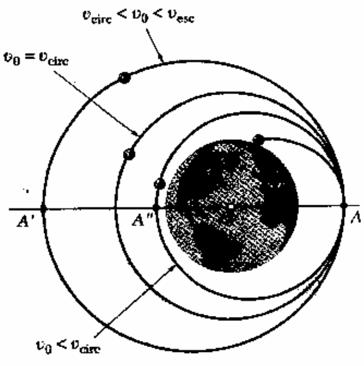
then the trajectory curve will be

hyperbolic $\quad$ if $\quad v_0 > v_{esc} \qquad$ i.e. $\quad \varepsilon > 1$

parabola $\quad$ if $\quad v_0 = v_{esc} \qquad$ i.e. $\quad \varepsilon = 1$

ellipse $\quad$ if $\quad v_0 < v_{esc} \qquad$ i.e. $\quad \varepsilon < 1$

for $C = 0$ ($\varepsilon = 0$), it is a circular orbit



$1 / r_0 = G\,M / h^2$

i.e. $\qquad h^2 = r_0^{\,2}\, v_0^{\,2} = G\,M\,r_0$

$\therefore \qquad v_0 = (G\,M / r_0)^{½} = (g\,R^2 / r_0)^{½}$

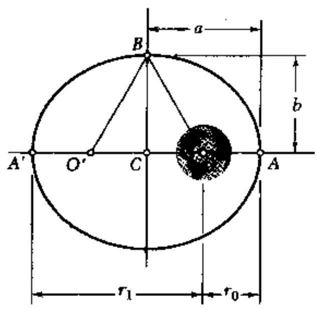
define $\qquad v_{circ} = (GM/r_0)^{½} = (gR^2/r_0)^{½}$

if $v_0 = v_{circ}$, the trajectory is a circular orbit

for $v_{circ} < v_0 < v_{esc}$, the trajectory is an ellipse, A is the closest point to the earth, called perigee; A' is the fastest point to the earth, called apogee

for $v_0 < v_{circ}$, then A becomes an apogee, it can find a point A'' at which the $v > v_{circ}$

periodic time (for a closed orbit)



the areal velocity $\qquad dA / dt = h / 2$

or $\qquad dA = ½\, h\, dt$

$\pi\, a\, b = A = \int dA = ½\, h \int dt = ½\, h\, \tau$

$\tau = 2\,\pi\, a\, b / h$

τ : periodic time

a, b : semiaxes of the ellipse, (but how to determine)

$r_0 + r_1 = 2\,a$

$\therefore \qquad a = (r_0 + r_1) / 2$

because the sum of distances from each of the foci to any point of the ellipse is constant, then

$$OB + O'B = OA + O'A = 2a$$

then $\qquad OB = a$

and $\qquad OC = a - r_0$

$$b^2 = BC^2 = a^2 - OC^2 = a^2 - (a - r_0)^2$$

$$= r_0 (2a - r_0) = r_0 r_1$$

$\therefore \qquad b = (r_0 r_1)^{1/2}$

then
$$\tau = \frac{2\pi\,[(r_0 + r_1)/2]\,(r_0 r_1)^{1/2}}{h}$$

$$= \frac{\pi\,(r_0 + r_1)\,(r_0 r_1)^{1/2}}{h}$$

12.13 Kepler' s Law of Planetary Motion

1. each planet describes an ellipse, with the sun located at one of its foci

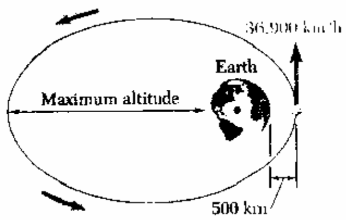
2. the radius vector drawn from the sum to a planet sweeps equal areas in equal time, i.e. dA / dt = constant

3. the square of the periodic times of the planets are proportional to the cubes of the semimajor axes of their orbits, i.e. $\tau^2 \sim a^3$

Sample Problem 12-9

v_0 = 36900 km/h at altitude of 500 km

determine (1) maximum altitude, (2) periodic time

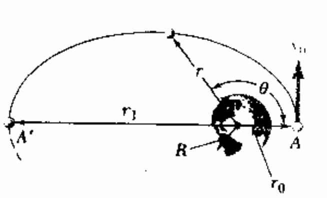


$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta$$

(1) $\quad r_0 = 500 + 6370 = 6870\text{ km} = 6.87 \times 10^6\text{ m}$

$\qquad v_0 = 39600\text{ km/h} = 10.25 \times 10^4\text{ m/s}$

$\therefore \quad h = r_0 v_0 = 7.04 \times 10^9\ \text{m}^2/\text{s}$



$h^2 = 4.96 \times 10^{21}\ m^4/s^2$

$GM = gR^2 = 9.81\ (m/s^2)\ (6.37 \times 10^6\ m)^2$

$= 3.98 \times 10^{14}\ m^3/s^2$

then $GM/h^2 = 8.03 \times 10^{-8}\ m^{-1}$

$$\frac{1}{r} = 8.03 \times 10^{-8} + C\cos\theta$$

for $\theta = 0^o$ $\qquad r = r_0 = 6.87 \times 10^6\ m$

$$\frac{1}{6.87 \times 10^6} = 8.03 \times 10^{-8} + C$$

$C = 6.526 \times 10^{-8}\ m^{-1}$

for $\theta = 180^o$ $\qquad r = r_1$

$$\frac{1}{r_1} = 8.03 \times 10^{-8} - 6.526 \times 10^{-8} = 1.504 \times 10^{-8}\ m^{-1}$$

$\therefore \quad r_1 = 0.665 \times 10^8 = 66500\ km$

then maximum altitude $= 66500 - 6370 = 60130$ km

(2) $a = ½\,(r_0 + r_1) = ½\,(6.87 + 66.7) \times 10^6 = 36.8 \times 10^6\ m$

$b = (r_0\, r_1)^{½} = (6.87 \times 66.7)^{½} \times 10^6 = 21.4 \times 10^6\ m$

$$\tau = \frac{2\pi a b}{h} = \frac{2\pi \times 36.8 \times 21.4 \times 10^{12}}{7.04 \times 10^{10}}$$

$= 7.03 \times 10^4\ s = 19\ h\ 31\ min$