

Chapter 12 Kinetics of Particles

12.1 Introduction

Newton's 1st and 3rd laws were used in Statics

Newton's 2nd law will be used in dynamics to relate the motion of bodies with forces acting on it

linear momentum : $\mathbf{L} = m \mathbf{v}$

m : mass of particle $\mathbf{v}$: velocity of particle

angular momentum : $\mathbf{H}_O = \mathbf{r} \times m \mathbf{v}$

$\mathbf{r}$: position vector with respect to O

O : reference point

12.2 Newton's Second Law of Motion

if the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force

$$F_1 / a_1 = F_2 / a_2 = \dots = \text{constant}$$

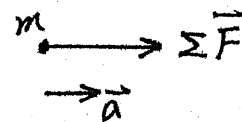
F : magnitude of force

a : magnitude of acceleration

$$\mathbf{F} = m \mathbf{a}$$

$\mathbf{F}$: resultant

$\mathbf{a}$: acceleration vector



$$\text{if } \mathbf{F} = 0 \Rightarrow \mathbf{a} = 0$$

i.e. if $\mathbf{v}_0 = 0$: particle is remain at rest

if $\mathbf{v}_0 \neq 0$: particle moves with the constant speed in a straight line

Newton's 1st law

another statement of Newton's 2nd law : a particle acted upon by a force moves so that the force vector is equal to the time rate of change of the linear momentum vector

Newtonian frame of reference (inertial frame) : constant orientation with respect to the stars, origin must be either attached to the sun or move with a constant velocity with respect to the sun

so that the axes attached to the earth do not constitute a Newtonian frame of reference, because the earth rotate with respect to stars and accelerated with respect to the sun

but in engineering application, this frame is used as the Newtonian frame without any appreciable error

Newton's 2nd law does not hold if the linear momentum or acceleration is measured with respect to moving axes, such as axes attached to an accelerated car

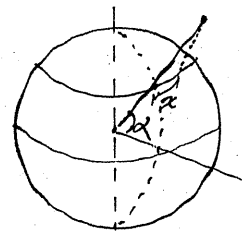
consider a freely falling body drop from sky

$$\omega = 7.29 \times 10^{-5} \text{ rad/s} \quad \text{for the earth}$$

$$x = \frac{2}{3} \omega (2h^3/g)^{1/2} \cos a$$

$$\text{for } a = 45^\circ \quad h = 200 \text{ m}$$

$$x = 43.9 \text{ mm}$$



12.3 Linear Momentum of Particle, Rate of Change of Linear Momentum

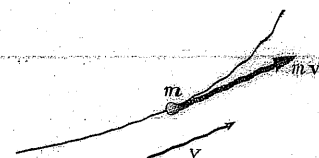
linear momentum is defined

$$L = m v$$

where m : mass of particle v : velocity of particle

Newton's 2nd law

$$\Sigma F = \dot{L} = d(mv) / dt$$



$$\text{if } m = \text{constant} \quad F = m a$$

$$\text{if } m \neq \text{constant} \quad F = m a + \dot{m} v$$

for $\dot{m} \neq 0$, gain or loss mass system, will discuss in Chapter 14

if $\Sigma F = 0$, then the linear momentum of the particle remains constant,
both in magnitude and direction

i.e. conservation of linear momentum

12.4 System of Units

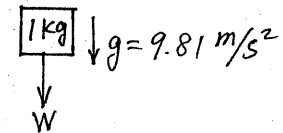
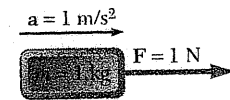
international system of units (SI units) : basic units are [kg, m, sec]

$$1 \text{ N} = (1 \text{ kg}) (1 \text{ m/s}^2) = 1 \text{ kg-m/s}^2$$

for the weight of a 1 kg mass

$$W = (1 \text{ kg}) (9.81 \text{ m/s}^2) = 9.81 \text{ N}$$

$$W = m g$$



the unit of linear momentum is

$$L = m v \quad [\text{kg-m/s}]$$

12.5 Equations of Motion

$$\Sigma F = m a \quad \text{for } m = \text{constant}$$

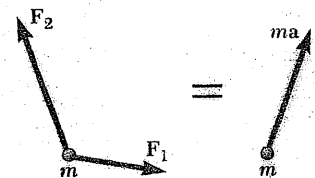
in rectangular components

$$\Sigma (F_x i + F_y j + F_z k) = m (a_x i + a_y j + a_z k)$$

$$\text{i.e. } \Sigma F_x = m a_x = m \ddot{x}$$

$$\Sigma F_y = m a_y = m \ddot{y}$$

$$\Sigma F_z = m a_z = m \ddot{z}$$



for the motion of projectile

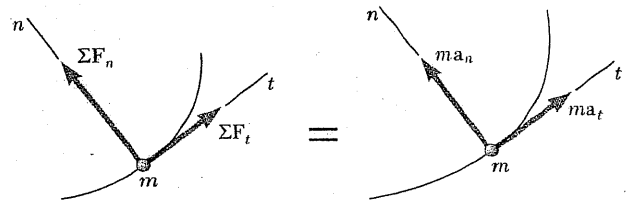
$$\Sigma F_x = \Sigma F_z = 0$$

$$\Sigma F_y = -W = -mg \Rightarrow \ddot{y} = -g$$

in tangential and normal components

$$\Sigma F_t = m a_t = m dv/dt$$

$$\Sigma F_n = m a_n = m v^2/\rho$$



in radial and transverse components

$$\Sigma F_r = m a_r = m (\ddot{r} - r \dot{\theta}^2)$$

$$\Sigma F_\theta = m a_\theta = m (r \ddot{\theta} + 2 \dot{r} \dot{\theta})$$

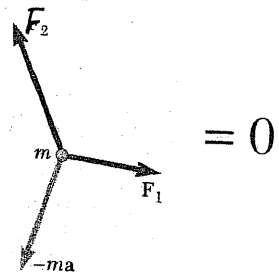
12.6 Dynamic Equilibrium

$$\Sigma \mathbf{F} - m \mathbf{a} = 0$$

the vector $-m\mathbf{a}$ is called inertia vector

then the particle is said to be dynamic equilibrium

if the inertia vector is included, it is obtained



rectangular components

$$\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$$

tangential and normal components

$$\Sigma F_t = \Sigma F_n = 0$$

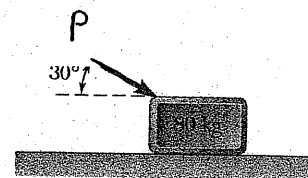
radial and transverse components

$$\Sigma F_r = \Sigma F_\theta = 0$$

Sample Problem 12-1

$$m = 80 \text{ kg} \quad a = 2.5 \text{ m/s}^2 \rightarrow$$

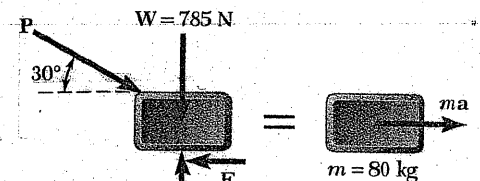
$$\mu_k = 0.25$$



determine the magnitude of the force P

$$W = mg = 80 \times 9.81 = 785 \text{ N}$$

$$F = \mu_k N = 0.25 N$$



$$\Sigma F_x = m a_x \quad P \cos 30^\circ - 0.25 N = 80 \times 0.25$$

$$\Sigma F_y = m a_y = 0 \quad N - P \sin 30^\circ - W = 0$$

2 equations to solve two unknowns P and N

$$P = 535 \text{ N} \quad N = 1052.5 \text{ N}$$

Sample Problem 12-1a

$$m = 80 \text{ kg} \quad P = 298 \text{ N}$$

$$v_0 = 0 \quad x = 12 \text{ m for } t = 4 \text{ s}$$

determine μ_k

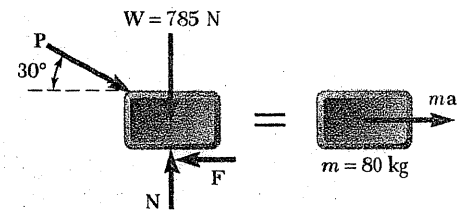
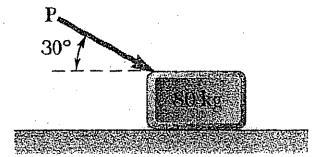
$$a = 2x/t^2 = 2 \times 12 / 4^2 = 1.5 \text{ m/s}^2$$

$$\Sigma F_y = ma_y = 0 \quad N - 298 \sin 30^\circ - 785 = 0$$

$$N = 934 \text{ N}$$

$$\Sigma F_x = ma_x \quad 298 \cos 30^\circ - 934 \mu_k = 80 \times 1.5$$

$$\mu_k = 0.148$$



Sample Problem 12-2

$$m_A = 100 \text{ kg} \quad m_B = 300 \text{ kg}$$

determine a_A , a_B and tensions in each cord

$$s_B = \frac{1}{2} s_A \Rightarrow a_B = \frac{1}{2} a_A$$

for block A

$$\Sigma F_x = m_A a_A \quad T_1 = 100 a_A$$

for block B

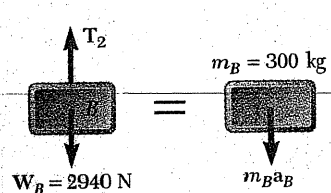
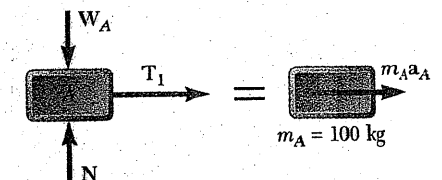
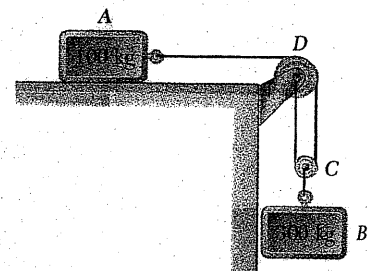
$$\Sigma F_y = m_B a_B \quad m_B g - T_2 = 300 a_B$$

for the pulley C

$$\Sigma F_y = m_C a_C = 0 \quad T_2 - 2 T_1 = 0$$

4 equations to solve 4 unknowns

$$a_A = 8.4 \text{ m/s}^2 \quad a_B = 4.2 \text{ m/s}^2$$



$$T_1 = 840 \text{ N}$$

$$T_2 = 1680 \text{ N}$$

Sample Problem 12-3

$$m_A = 10 \text{ kg} \quad m_B = 4 \text{ kg}$$

neglecting friction, determine

(a) a_A (b) $a_{B/A}$

for the wedge A

$$\begin{aligned} \Sigma F_x &= m_A a_A & N_1 \sin 30^\circ &= m_A a_A \\ 0.5 N_1 &= m_A a_A & (1) \end{aligned}$$

for the block B

$$a_B = a_A + a_{B/A}$$

$$\Sigma F_x = m_B a_x$$

$$-m_B g \sin 30^\circ = m_B a_A \cos 30^\circ - m_B a_{B/A}$$

$$a_{B/A} = a_A \cos 30^\circ + g \sin 30^\circ$$

$$\Sigma F_y = m_B a_y$$

$$N_1 - m_B g \cos 30^\circ = -m_B a_A \sin 30^\circ \quad (3)$$

(1) and (3)

$$a_A = \frac{m_B g \cos 30^\circ}{2 m_A + m_B \sin 30^\circ} = 1.54 \text{ m/s}^2 \rightarrow$$

(2) $\Rightarrow$

$$a_{B/A} = 6.24 \text{ m/s}^2 \swarrow 30^\circ$$

also, N_1 can be obtained

$$N_1 = 30.8 \text{ N}$$

[from eq. 1]

wrong method :

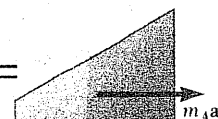
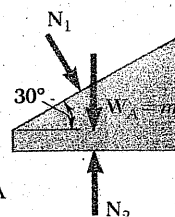
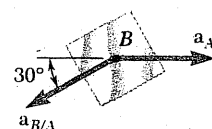
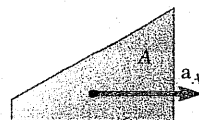
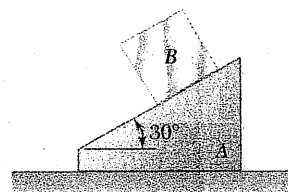
for the block B

$$\Sigma F_x = m_B a_x \quad m_B g \sin 30^\circ = m_B a_B$$

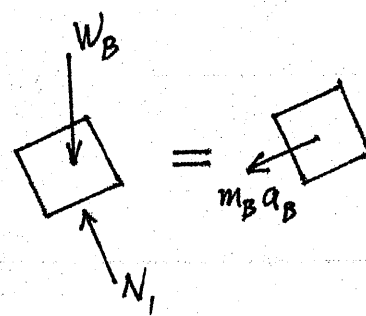
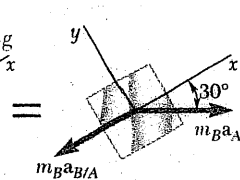
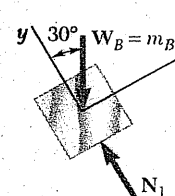
$$a_B = 0.5 g = 4.90 \text{ m/s}^2 \swarrow 30^\circ$$

$$\Sigma F_y = 0$$

$$N_1 = m_B g \cos 30^\circ$$



(2)



for the wedge A

$$\Sigma F_x = m_A a_A \quad m_A a_A = N_1 \sin 30^\circ$$

$$a_A = 1.697 \text{ m/s}^2 \rightarrow$$

Sample Problem 12-4

$$T = 2.5 m g \quad \text{for } \theta = 30^\circ$$

determine v and a

$$\Sigma F_t = m a_t \quad mg \sin 30^\circ = m a_t$$

$$a_t = 4.9 \text{ m/s}^2 \swarrow$$

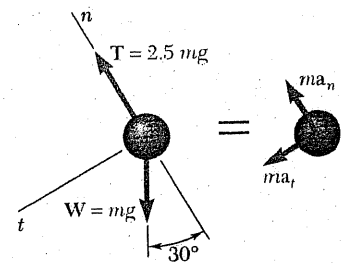
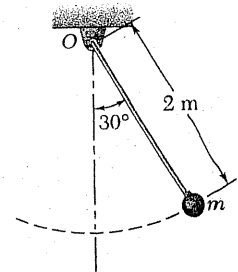
$$\Sigma F_n = m a_n \quad 2.5 mg - mg \cos 30^\circ = m a_n$$

$$a_n = 16.03 \text{ m/s}^2 \nwarrow$$

$$a_n = v^2 / \rho$$

$$v^2 = a_n \rho = 16.03 \times 2 = 32.06$$

$$v = \pm 5.66 \text{ m/s} \swarrow$$



Sample Problem 12-5

$$\rho = 120 \text{ m} \quad \theta = 18^\circ$$

determine v if no lateral friction force on the wheel is known

$$\Sigma F_y = 0 \quad R \cos \theta - W = 0$$

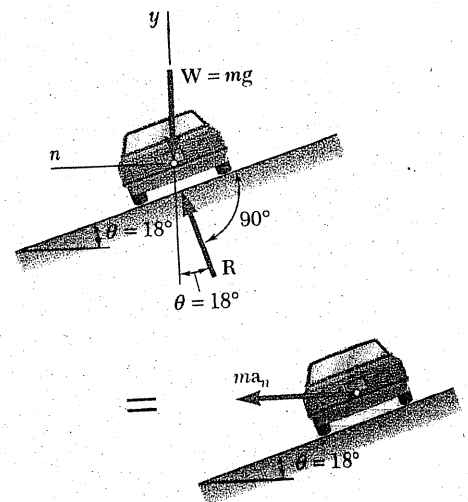
$$R = mg / \cos \theta$$

$$\Sigma F_n = m a_n \quad R \sin \theta = m a_n = m v^2 / \rho$$

$$(mg / \cos \theta) \sin \theta = m v^2 / \rho$$

$$v^2 = \rho g \tan \theta = 120 \times 9.81 \times \tan 18^\circ$$

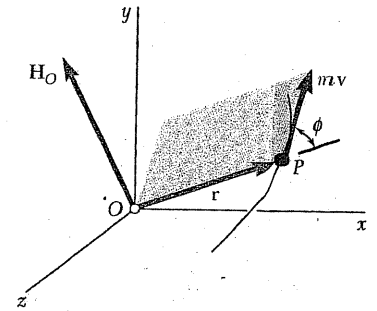
$$v = 19.56 \text{ m/s} = 70.4 \text{ km/h}$$



12.7 Angular Momentum of Particle, Rate of Change of Angular Momentum

12.7 Angular Momentum of Particle, Rate of Change of Angular Momentum

the moment of the linear momentum vector $m\mathbf{v}$ about a point O is called the moment of momentum, or angular momentum, of the particle, and is denoted by H_0



$$\mathbf{H}_0 = \mathbf{r} \times m\mathbf{v} \quad H_0 = r m v \sin \phi$$

SI unit for H_0 is $\text{m (kg-m/s)} = \text{kg-m}^2/\text{s}$

$$\mathbf{H}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix} = H_x \mathbf{i} + H_y \mathbf{j} + H_z \mathbf{k}$$

$$H_x = m (y v_z - z v_y)$$

$$H_y = m (z v_x - x v_z)$$

$$H_z = m (x v_y - y v_x)$$

$$H_0 = (H_x^2 + H_y^2 + H_z^2)^{1/2}$$

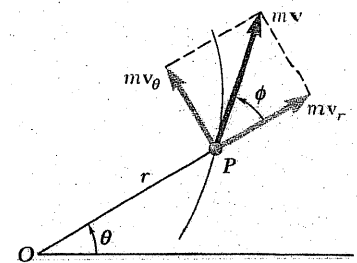
if particle moving in xy plane, then $z = v_z = 0$

$$H_x = H_y = 0$$

$$H_0 = H_z = m (x v_y - y v_x)$$

for polar coordinate

$$\begin{aligned} H_0 &= r m v \sin \phi = r m v_\theta \\ &= m r^2 \dot{\theta} \quad [v_\theta = r \dot{\theta}] \end{aligned}$$



rate of change of angular momentum

$$\dot{\mathbf{H}}_0 = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}}$$

$$= \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\mathbf{a}$$

$$= \mathbf{r} \times \Sigma \mathbf{F} = \Sigma \mathbf{M}_0$$

$$\dot{\mathbf{H}}_0 = \Sigma \mathbf{M}_0$$

the sum of moments about O of the forces acting on the particle is equal to rate of change of angular momentum

12.8 Equations of Motion in Terms of Radial and Transverse Components

in polar coordinate system

$$\Sigma F_r = m a_r = m (\ddot{r} - r \dot{\theta}^2)$$

$$\Sigma F_\theta = m a_\theta = m (r \ddot{\theta} + 2 \dot{r} \dot{\theta})$$

$$\therefore \quad \mathbf{r} \perp \Sigma F_\theta \mathbf{e}_\theta$$

$$\therefore \quad \Sigma \mathbf{M}_0 = \mathbf{r} \times \Sigma F_\theta \mathbf{e}_\theta$$

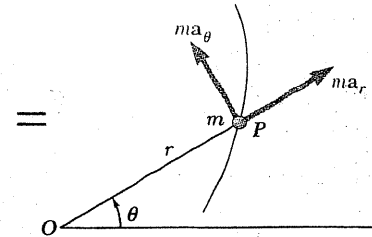
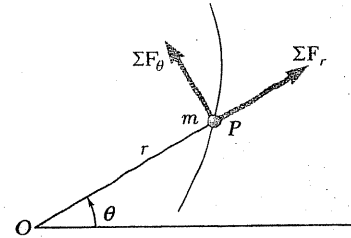
$$\text{or} \quad \Sigma M_0 = r \Sigma F_\theta$$

$$\Sigma M_0 = \dot{H}_0 = \frac{d}{dt} (m r^2 \dot{\theta})$$

$$r \Sigma F_\theta = m (r^2 \ddot{\theta} + 2 \dot{r} \dot{\theta})$$

$$\text{or} \quad \Sigma F_\theta = m (r \ddot{\theta} + 2 \dot{r} \dot{\theta})$$

the same form of a_θ is obtained



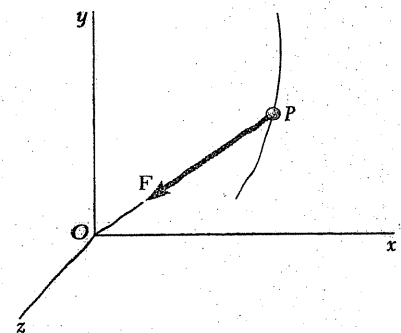
12.9 Motion Under a Central Force, Conservation of Angular Momentum

when $\Sigma \mathbf{F}$ directed toward or away from a fixed point, the particle is said to be moving under central force

$$\text{then} \quad \Sigma \mathbf{M}_0 = 0$$

$$\text{hence} \quad \dot{H}_0 = 0$$

$$H_0 = \text{constant}$$



thus the angular momentum of a particle moving under central force is constant, both in magnitude and direction

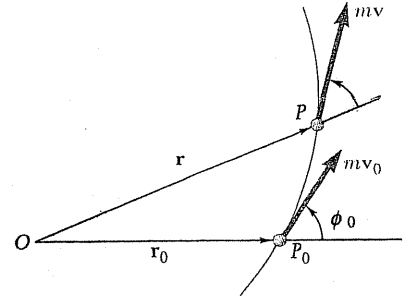
$$H_0 = \mathbf{r} \times m \mathbf{v} = \text{constant}$$

$\mathbf{r}$ and $\mathbf{v}$ must $\perp \mathbf{H}_0$, $\therefore$ the direction of $\mathbf{H}_0$ is constant, $\therefore$ the particle moves in a fixed plane $\perp \mathbf{H}_0$

$\therefore$ the magnitude of $\mathbf{H}_0$ is also constant

$$\therefore H_0 = r m v \sin \phi = r_0 m v_0 \sin \phi_0$$

$$\text{or } H_0 = m r^2 \dot{\theta}$$



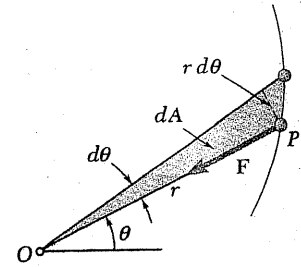
denoting $h = H_0/m$, the angular momentum per unit mass, then

$$h = r^2 \dot{\theta} = \text{constant}$$

$$\text{and } dA = \frac{1}{2} r^2 d\theta$$

denoting the areal velocity $= dA / dt$, then

$$dA / dt = \frac{1}{2} r^2 \dot{\theta} = h / 2 = \text{constant}$$

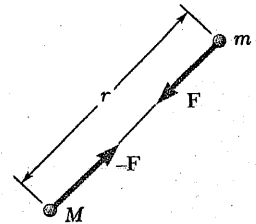


thus if a particle moves under central force, its areal velocity is constant

12.10 Newton's Law of Gravitation

law of universal gravitation between two particles

$$F = G M m / r^2$$



where $G = 6.673 \times 10^{-11} \text{ m}^3/\text{kg-s}^2$ is the universal constant (constant of gravitation), and r is the distance between center to center of two particles

consider a particle of mass m on the surface of earth

$$W = m g = G M m / R^2$$

where M is the mass of earth ($M \simeq 6 \times 10^{24} \text{ kg}$), and R is the radius of earth ($R = 6.37 \times 10^6 \text{ m}$), it is obtained

$$G M = g R^2$$

$$\text{or } g = G M / R^2$$

thus W and g will vary with altitude and latitude of the point considered

$$g = 9.781 \text{ m/s}^2 \text{ (at equator)} \sim 9.833 \text{ m/s}^2 \text{ (at poles)}$$

$$\text{in average} \quad g = 9.81 \text{ m/s}^2 \quad R = 6.37 \times 10^6 \text{ m}$$

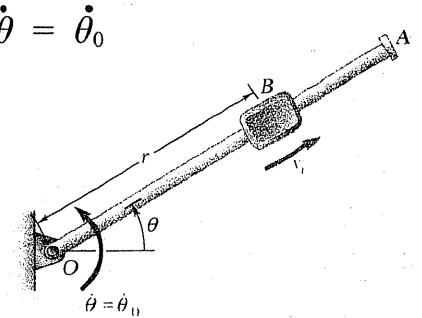
Sample Problem 12.6

$$\text{at } t = 0, \quad r = r_0 \quad v_r = 0 \quad \dot{\theta} = \dot{\theta}_0$$

determine v_r and F

$$\Sigma F_r = m a_r \quad m(\ddot{r} - r\dot{\theta}^2) = 0 \quad (1)$$

$$\Sigma F_\theta = m a_\theta \quad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F \quad (2)$$



$$(1) \Rightarrow \quad r\dot{\theta}^2 = \ddot{r} = \frac{dv_r}{dt} = \frac{dv_r}{dr} \frac{dr}{dt} = \dot{r} \frac{dv_r}{dr} = v_r \frac{dv_r}{dr}$$

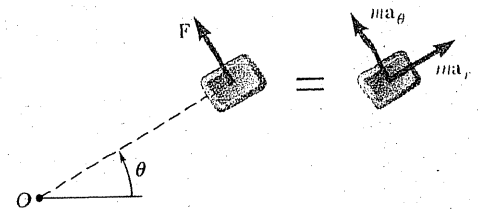
$$\dot{\theta} = \dot{\theta}_0 = \text{constant} \quad \ddot{\theta} = 0$$

$$\text{thus} \quad r\dot{\theta}_0^2 dr = v_r dv_r$$

$$\dot{\theta}_0^2 (r^2 - r_0^2) = (v_r^2 - 0)$$

$$\text{or} \quad \dot{r} = v_r = \dot{\theta}_0 (r^2 - r_0^2)^{1/2}$$

$$(2) \Rightarrow \quad F = 2m\dot{r}\dot{\theta} = 2m v_r \dot{\theta}_0 = 2m \dot{\theta}_0^2 (r^2 - r_0^2)^{1/2}$$



Sample Problem 12-7

$$v_A = 30.3 \times 10^3 \text{ km/h}$$

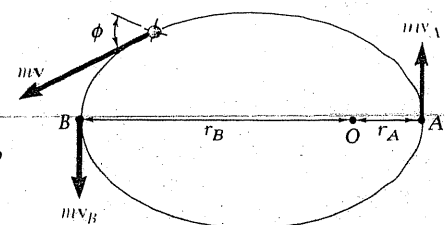
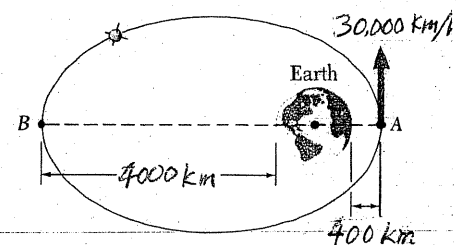
determine v_B

$$R = 6370 \text{ km}$$

$$r_A = OA = 6370 + 400 = 6770 \text{ km}$$

$$r_B = OB = 6370 + 4000 = 10370 \text{ km}$$

because the satellite is moving under central force,
thus its angular momentum is constant



$$H_0 = r m v \sin \phi = \text{constant}$$

$$\phi = 90^\circ \quad \text{at } A \text{ and } B$$

$$r_A m v_A = r_B m v_B$$

$$v_B = r_A v_A / r_B = 6770 \times 30,000 / 10370 = 19,590 \text{ km/h}$$

12.11 Trajectory of a Particle Under a Central Force

in this section, it is proposed to obtain the differential equation which defines the trajectory of the particle moving under central force

$$\begin{aligned} \Sigma \mathbf{F} &= m \mathbf{a} & m(\ddot{r} - r\dot{\theta}^2) &= -F \\ & & m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= 0 \end{aligned}$$

central force $\Rightarrow$ conservation of angular momentum

$$r^2 \dot{\theta} = h = \text{constant} \quad \Rightarrow \quad \dot{\theta} = h / r^2$$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta} \dot{\theta} = \frac{h}{r^2} \frac{dr}{d\theta} = -h \frac{d}{d\theta} \left(\frac{1}{r} \right)$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta} \dot{\theta} = \frac{h}{r^2} \frac{d}{d\theta} \left[-h \frac{d}{d\theta} \left(\frac{1}{r} \right) \right] = -\frac{h^2}{r^2} \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right)$$

let $u = 1 / r$

$$\ddot{r} = -h^2 u^2 \frac{d^2 u}{d\theta^2}$$

$$\text{but } \ddot{r} = r\dot{\theta}^2 - \frac{F}{m} = \frac{1}{u} h^2 u^4 - \frac{F}{m} = -h^2 u^2 \frac{d^2 u}{d\theta^2}$$

$$\frac{d^2 u}{d\theta^2} + u = \frac{F}{m h^2 u^2}$$

this is the differential equation to describe the relation between r and θ
(trajectory of the particle)

12.12 Application to Space Mechanics

consider a space vehicle moving around the earth under central force, the gravitational attraction between the earth and vehicle is

$$F = G M m / r^2 = G M m u^2$$

where M : mass of the earth m : mass of the vehicle

r : distance from center of earth to vehicle $u : 1/r$

the differential equation becomes

$$\frac{d^2 u}{d\theta^2} + u = \frac{G M}{h^2} = \text{constant}$$

to solve this differential equation, and obtain the result

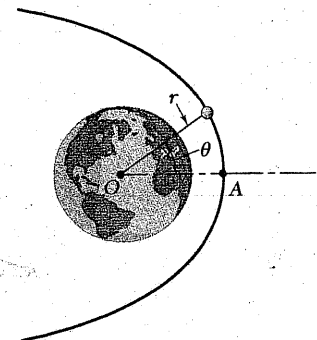
$$u = \frac{1}{r} = \frac{G M}{h^2} + C \cos \theta$$

where C is a constant, this equation is a polar equation of a conic section (elliptic, parabola, or hyperbola)

define the eccentricity ε

$$\varepsilon = \frac{C}{G M / h^2} = \frac{C h^2}{G M}$$

$$\text{then } \frac{1}{r} = \frac{G M}{h^2} (1 + \varepsilon \cos \theta)$$



this equation represents three possible trajectories

$$1. \varepsilon > 1 \text{ or } C > G M / h^2$$

$$\text{when } \cos (\pm \theta) = -1/\varepsilon = -G M / C h^2$$

$$\Rightarrow 1/r = 0 \Rightarrow r \rightarrow \infty$$

that is a hyperbola

$$2. \varepsilon = 1 \text{ or } C = G M / h^2$$

$$\text{when } \theta = \pm 180^\circ \quad \cos (\pm \theta) = -1$$

$$r \rightarrow \infty$$

that is a parabola

3. $\varepsilon < 1$ or $C < GM/h^2$

for any value of θ , $1/r \neq 0$

the radius remains finite, that is an ellipse

for $\varepsilon = 0$, then $1/r = GM/h^2 = \text{constant}$

that is a circle

consider a rocket launching on the earth surface, and burnout at point A , it is free flight at stage A (central force only)

at point A $h = r_0 v_0 = r_0^2 \dot{\theta}$

then h can be determined

$$W = mg = GMm/R^2$$

then $GM = gR^2$ is also determined

the equation of trajectory is

$$\frac{1}{r} = \frac{GM}{h^2} (1 + \varepsilon \cos \theta) = \frac{GM}{h^2} + C \cos \theta$$

for $\theta = 0^\circ$ $r = r_0$

$$C = 1/r_0 - GM/h^2$$

we know for $\varepsilon = 1$ or $C = GM/h^2$, the trajectory is a parabola

$$GM/h^2 = 1/r_0 - GM/h^2$$

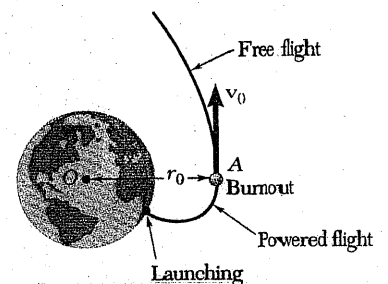
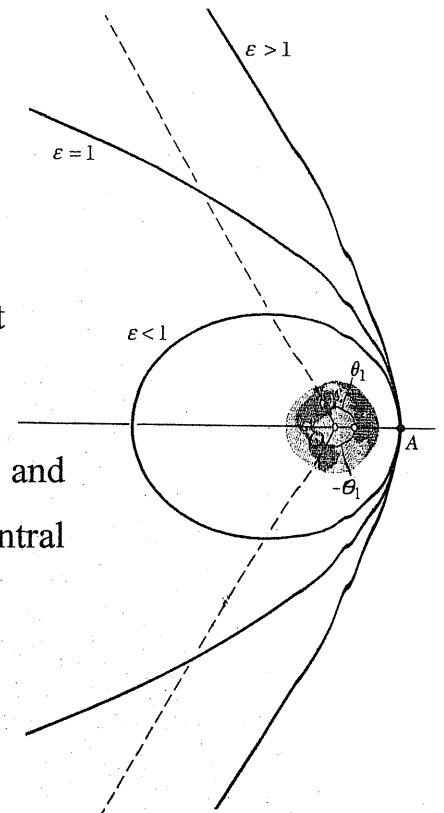
i.e. $1/r_0 = 2 GM/h^2$

or $h^2 = 2 GM r_0$

but $h = r_0 v_0$

$\therefore 2 GM r_0 = r_0^2 v_0^2$

or $v_0 = (2 GM / r_0)^{1/2} = (2 g R^2 / r_0)^{1/2}$



define $v_{\text{esc}} = (2 GM/r_0)^{1/2} = (2 gR^2/r_0)^{1/2}$ is the escape velocity

then the trajectory curve will be

hyperbolic if $v_0 > v_{\text{esc}}$ i.e. $\varepsilon > 1$

parabola if $v_0 = v_{\text{esc}}$ i.e. $\varepsilon = 1$

ellipse if $v_0 < v_{\text{esc}}$ i.e. $\varepsilon < 1$

for $C = 0$ ($\varepsilon = 0$), it is a circular orbit

$$1/r_0 = GM/h^2$$

$$\text{i.e. } h^2 = r_0^2 v_0^2 = GM r_0$$

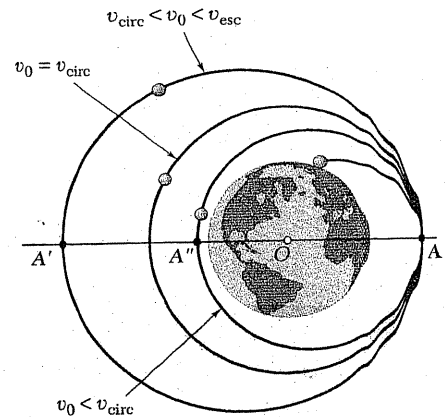
$$\therefore v_0 = (GM/r_0)^{1/2} = (gR^2/r_0)^{1/2}$$

$$\text{define } v_{\text{circ}} = (GM/r_0)^{1/2} = (gR^2/r_0)^{1/2}$$

if $v_0 = v_{\text{circ}}$, the trajectory is a circular orbit

for $v_{\text{circ}} < v_0 < v_{\text{esc}}$, the trajectory is an ellipse, A is the closest point to the earth, called perigee; A' is the farthest point to the earth, called apogee

for $v_0 < v_{\text{circ}}$, then A becomes an apogee, it can find a point A'' at which the $v > v_{\text{circ}}$



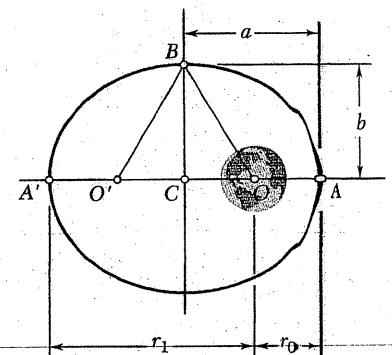
periodic time (for a closed orbit)

the areal velocity $dA/dt = h/2$

$$\text{or } dA = \frac{1}{2} h dt$$

$$\pi a b = A = \int dA = \frac{1}{2} h \int dt = \frac{1}{2} h \tau$$

$$\tau = 2 \pi a b / h$$



τ : periodic time

a, b : semiaxes of the ellipse, (but how to determine)

$$r_0 + r_1 = 2a$$

$$\therefore a = (r_0 + r_1) / 2$$

because the sum of distances from each of the foci to any point of the ellipse is constant, then

$$OB + O'B = OA + O'A = 2a$$

then $OB = a$

and $OC = a - r_0$

$$b^2 = BC^2 = a^2 - OC^2 = a^2 - (a - r_0)^2$$

$$= r_0 (2a - r_0) = r_0 r_1$$

$$\therefore b = (r_0 r_1)^{1/2}$$

then
$$\tau = \frac{2\pi [(r_0 + r_1)/2] (r_0 r_1)^{1/2}}{h}$$

$$= \frac{\pi (r_0 + r_1) (r_0 r_1)^{1/2}}{h}$$

12.13 Kepler's Law of Planetary Motion

1. each planet describes an ellipse, with the sun located at one of its foci
2. the radius vector drawn from the sun to a planet sweeps equal areas in equal time, i.e. $dA/dt = \text{constant}$
3. the square of the periodic times of the planets are proportional to the cubes of the semimajor axes of their orbits, i.e. $\tau^2 \sim a^3$

Sample Problem 12-8

$v_0 = 36900 \text{ km/h}$ at altitude of 500 km

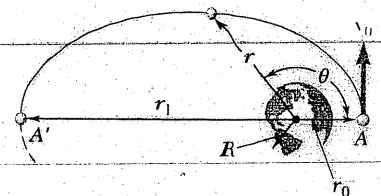
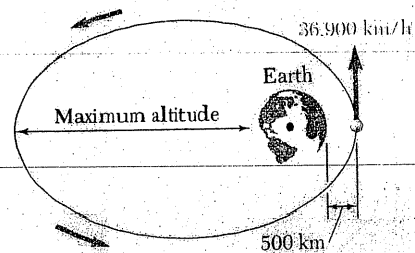
determine (1) maximum altitude, (2) periodic time

$$\frac{1}{r} = \frac{GM}{h^2} + C \cos \theta$$

(1) $r_0 = 500 + 6370 = 6870 \text{ km} = 6.87 \times 10^6 \text{ m}$

$$v_0 = 36900 \text{ km/h} = 10.25 \times 10^4 \text{ m/s}$$

$$\therefore h = r_0 v_0 = 7.04 \times 10^9 \text{ m}^2/\text{s}$$



$$h^2 = 4.96 \times 10^{21} \text{ m}^4/\text{s}^2$$

$$\begin{aligned} GM &= gR^2 = 9.81 \text{ (m/s}^2\text{)} (6.37 \times 10^6 \text{ m})^2 \\ &= 3.98 \times 10^{14} \text{ m}^3/\text{s}^2 \end{aligned}$$

$$\text{then } GM/h^2 = 8.03 \times 10^{-8} \text{ m}^{-1}$$

$$\frac{1}{r} = 8.03 \times 10^{-8} + C \cos \theta$$

$$\text{for } \theta = 0^\circ \quad r = r_0 = 6.87 \times 10^6 \text{ m}$$

$$\frac{1}{6.87 \times 10^6} = 8.03 \times 10^{-8} + C$$

$$C = 6.526 \times 10^{-8} \text{ m}^{-1}$$

$$\text{for } \theta = 180^\circ \quad r = r_1$$

$$\frac{1}{r_1} = 8.03 \times 10^{-8} - 6.526 \times 10^{-8} = 1.504 \times 10^{-8} \text{ m}^{-1}$$

$$\therefore r_1 = 0.665 \times 10^8 = 66500 \text{ km}$$

$$\text{then maximum altitude} = 66500 - 6370 = 60130 \text{ km}$$

$$(2) \quad a = \frac{1}{2}(r_0 + r_1) = \frac{1}{2}(6.87 + 66.7) \times 10^6 = 36.8 \times 10^6 \text{ m}$$

$$b = (r_0 r_1)^{\frac{1}{2}} = (6.87 \times 66.7)^{\frac{1}{2}} \times 10^6 = 21.4 \times 10^6 \text{ m}$$

$$\begin{aligned} \tau &= \frac{2\pi ab}{h} = \frac{2\pi \times 36.8 \times 21.4 \times 10^{12}}{7.04 \times 10^{10}} \\ &= 7.03 \times 10^4 \text{ s} = 19 \text{ h } 31 \text{ min} \end{aligned}$$

