Chapter 11 Kinematics of Particles

11.1 Introduction to Dynamics

dynamics : analysis of mechanics of bodies in motion

1. kinematics : study of geometry of motion, to relate displacement, velocity, acceleration and time

2. kinetics : to study the relation between the force acting on a body, the mass of body, and the motion of body

dynamics of particles : chapters 11-14

dynamics of rigid bodies : chapters 15-18

motion of particles

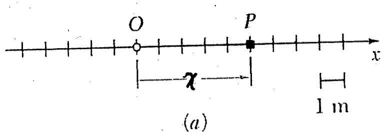
1. rectilinear motion : a particle moving along a straight line

2. curvilinear motion : a particle moving along a curve other than a straight line

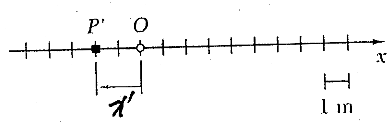
RECTILINEAR MOTION OF PARTICLES

11.2 Position, Velocity and Acceleration of Rectilinear Motion

a distance x, with the appropriate sign, completely defines the position of particles is called the position coordinate of the particle



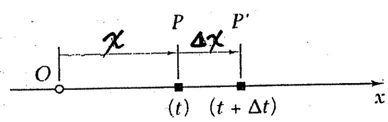
(a)



(b)

e.g. $x = +5$ m $\qquad x' = -2$ m

x may be a function of time, $x = x(t)$



e.g. $x(t) = 6t^2 - t^3$

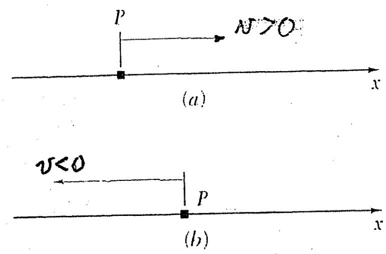
average velocity of the particle over the time interval Δt is defined

$$\text{average velocity} \equiv \Delta x / \Delta t$$

instantaneous velocity v at the instant t is defined by choosing a very short time interval $\Delta t \to 0$

$$\text{instantaneous velocity} \equiv v = \lim_{\Delta t \to 0} (\Delta x / \Delta t) = dx / dt$$

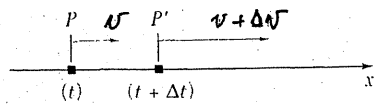
velocity is expressed in m/s



velocity is a vector quantity, may be positive or negative, a positive $\boldsymbol{v}$ indicates that x increases, and negative $\boldsymbol{v}$ indicates that x decreases

the magnitude of $\boldsymbol{v}$ is known as the speed of the particle

average acceleration of the particle over the time interval Δt is defined



$$\text{average acceleration} \equiv \Delta v / \Delta t$$

acceleration is expressed in m/s^2

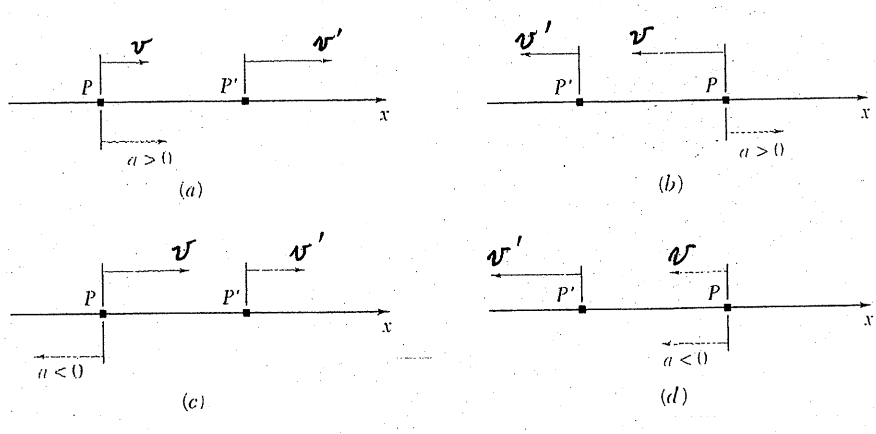
instantaneous acceleration a at the instant t is defined by choosing a very short time interval $\Delta t \to 0$

$$\text{instantaneous acceleration} \equiv \boldsymbol{a} = \lim_{\Delta t \to 0} (\Delta \boldsymbol{v} / \Delta t) = d\boldsymbol{v} / dt$$

$$\boldsymbol{a} = d\boldsymbol{v} / dt = d^2\mathbf{x} / dt^2$$

$\boldsymbol{a}$ may be positive or negative

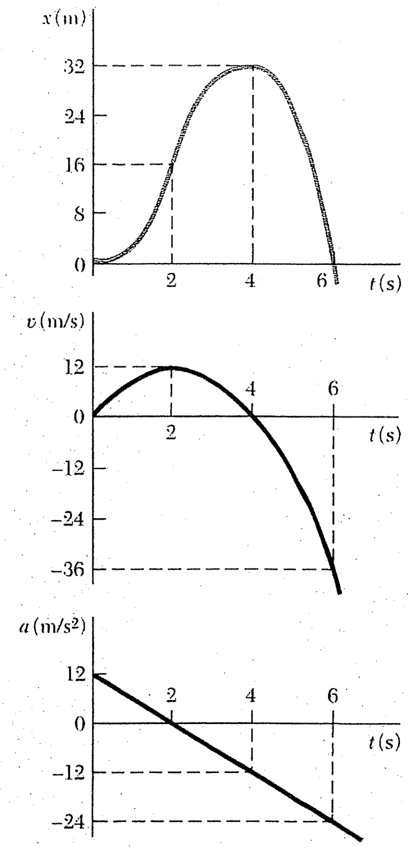
positive $\boldsymbol{a}$: velocity increase negative $\boldsymbol{a}$: velocity decrease



deceleration : speed of particle decrease (parts b and c of the above figure are deceleration)

another expression for the acceleration can be obtained

$$a = dv / dt = (dv/dx)(dx/dt) = v\,(dv/dx)$$



example

$x = 6\,t^2 - t^3$

$v = dx / dt = 12\,t - 3\,t^2$

$a = dv / dt = 12 - 6\,t$

1. at $t = 0$, $x = v = 0$, $a = 12$

   x, v, a are positive for $0 < t < 2$

2. at $t = 2$, $a = 0$, $v = v_{max}$

   x, v are positive, a is negative for $2 < t < 4$

3. at $t = 4$, $v = 0$, $x = x_{max}$

   v, a are negative, x is positive for $4 < t < 6$

4. at $t = 6$, v and a are negative

   x, v, a are negative for $t > 6$

11.3 Determination of the Motion of a Particles

for the freely falling body : $a = d^2x / dt^2 = \text{constant} = 9.81\ \text{m/s}^2$

for a spring : $a = (k/m)\,x$ proportional to the instantaneous elongation

in general, if $a = a(t, x, v)$ is known, $x(t)$ can be determined

$a = 0$ uniform motion

$a = \text{constant}$ uniformly acceleration motion

consider three common cases :

1. $a = f(t)$ $\quad dv = a\,dt = f(t)\,dt$

$$\int_{v_0}^{v} dv = v - v_0 = \int_0^t f(t)\,dt$$

$v_0 = v(0)$ initial velocity

and $\quad dx = v\, dt$

$$\int_{x_0}^{x} dx = x - x_0 = \int_0^t v(t)\, dt$$

$x_0 = x(0)$ initial position

2. $a = f(x)$ $\quad v\, dv = a\, dx = f(x)\, dx$

$$\int_{v_0}^{v} v\, dv = \int_0^x f(x)\, dx$$

$$(v^2 - v_0^2)/2 = \int_0^x f(x)\, dx$$

$$v = v(x) = dx / dt$$

$$dt = dx / v(x)$$

$$\int_0^t dt = \int_{x_0}^{x} f\, dx / v(x) \qquad x = x(t)$$

3. $a = f(v)$ $\quad a = f(v) = dv / dt$

$$dt = dv / f(v) \qquad v = v(t)$$

$$a = f(v) = v\,(dv/dx)$$

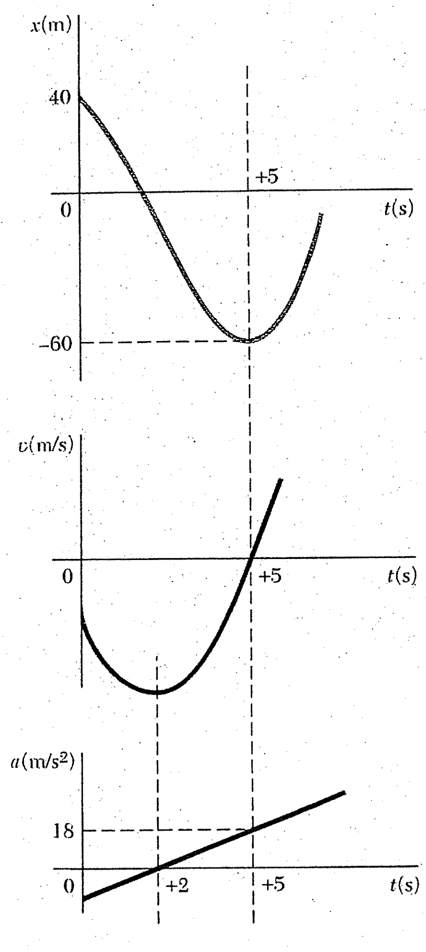
$$dx = [v/f(v)]\, dv$$

$$x = x(v) = x[v(t)] \qquad x = x(t)$$

Sample Problem 11.1

$x = t^3 - 6t^2 - 15t + 40$ $\quad x$: m $\quad t$: sec

determine t when $v = 0$, $x(t)$, $a(t)$ at that time, traveling distance from $t = 4 \sim 6$ sec



a. time for $v = 0$

$$v = dx/dt = 3t^2 - 12t - 15$$

$$a = dv/dt = 6t - 12$$

for $v = 0$, $t = -1,\ 5$ sec

b. at $t = 5$ sec $\quad x = -60$ m

c. at $t = 5$ sec $\quad a = 18\ \text{m/s}^2$

d. distance traveled from $t = 4$ to 6 sec

$x(4) = -52$ m $\quad$ $x(5) = -60$ m $= x_{min}$ $\quad$ $x(6) = -50$ m

$x_5 - x_4 = -8$ m $\quad$ $x_6 - x_5 = 10$ m

total traveled = 18 m

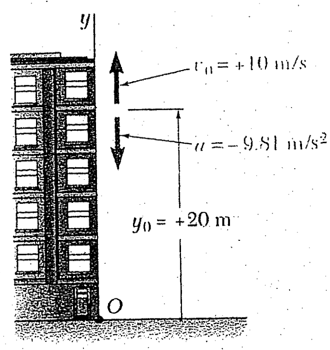
Sample Problem 11.2

for a freely falling body, $a = 9.81$ m/s^2 ↓ , $y(0) = 20$ m, $v(0) = 10$ m/s ↑

determine : (a) $v(t)$ and $y(t)$

(b) $y = y_{max}$, t = ?

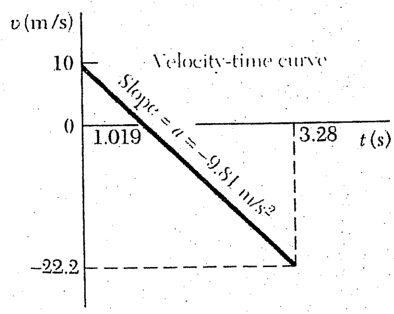
(c) $y = 0$, t = ?, v = ?



(a) $\quad dv / dt = a = -9.81$ m/s^2

$$\int_{v_0}^{v} dv = v - v_0 = -9.81 \int_0^t dt$$

$$v = 10 - 9.81\, t$$



$$dx / dt = v = 10 - 9.81\, t$$

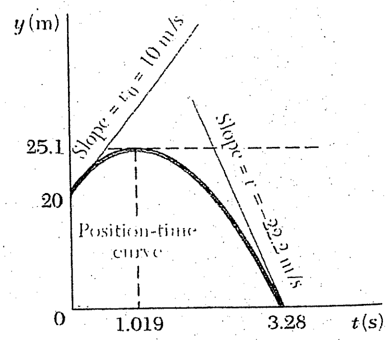
$$\int_{y_0}^{y} dy = y - y_0 = \int_0^t (10 - 9.81\, t)\, dt$$

$$y = 20 + 10\, t - 4.905\, t^2$$

(b) $\quad y = y_{max}$ occurs at $v = 0$

i.e. $\quad 10 - 9.81\, t = 0 \qquad t = 1.019$ sec

$y(1.019) = 25.1$ m



(c) $\quad$ for $y = 0$

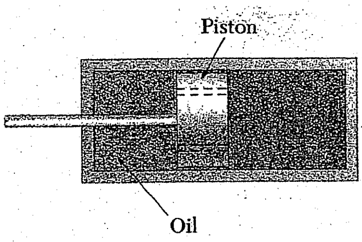
i.e. $\quad 20 + 10\, t - 4.905\, t^2 = 0$

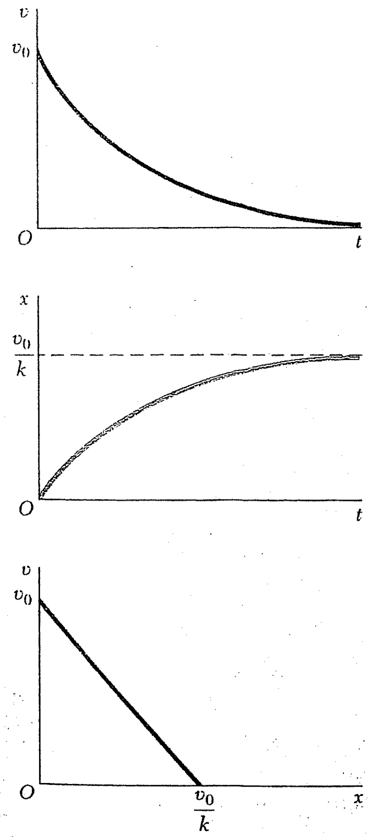
$t = -1.243$ and 3.28 sec

$v(3.28) = -22.2 = 22.2$ m/s ↓

Sample Problem 11.3

$a = -k\, v \qquad v(0) = 0 \qquad x(0) = 0$





determine $v(t)$, $x(t)$ and $v(x)$

(a) $a = -kv = dv/dt$

$$\int_{v_0}^{v} dv/v = -k\int_0^t dt$$

$$ln\,(v/v_0) = -kt \quad \text{or} \quad v = v_0\,e^{-kt}$$

(b) $v = e^{-kt} = dx/dt$

$$\int_0^x dx = v_0\int_0^t e^{-kt}\,dt$$

$$x = -v_0\,(e^{-kt} - 1)/k$$

(c) $a = v\,(dv/dx) = -kv$ or $dv = -k\,dx$

$$\int_{v_0}^{v} dv = -k\int_0^x dx \qquad v = v_0 - kx$$

or from (b)

$$x = v_0\,(1 - e^{-kt})/k = v_0\,(1 - v/v_0)/k = v_0\,(v_0 - v)/v_0\,k$$

$kx = v_0 - v$ same as above

11.4 Uniform Rectilinear Motion

$a = 0$ for all time

$dx/dt = v =$ constant

$$\int_{x_0}^{x} dx = v\int_0^t dt$$

$x = x_0 + vt$ be used only if $v =$ constant

11.5 Uniformly Accelerated Rectilinear Motion

$a = dv/dt =$ constant

$$\int_{v_0}^{v} dv = a\int_0^t dt$$

$v = v_0 + at$ for $a =$ constant only

$dx/dt = v(t) = v_0 + at$

$$\int_{x_0}^{x} dx = \int_{0}^{t} (v_0 + a\,t)\; dt$$

$$x(t) = x_0 + v_0\, t + ½\, a\, t^2$$

$$a = v\, dv \,/\, dx = \text{constant} \qquad v\, dv = a\, dx$$

$$\int_{v_0}^{v} v\, dv = a \int_{x_0}^{x} dx$$

$$½\,(v^2 - v_0^2) = a\,(x - x_0) \qquad \text{or} \qquad v^2 = v_0^2 + a\,(x - x_0)$$

important application : freely falling body

$$a = 9.81 \text{ m/s}^2 = \text{constant}$$

11.6 Motion of Several Particles

relative motion of two particles

$x_{B/A}$ is defined the relative position coordinate of B with respect to A

$x_{B/A} = x_B - x_A$ or $x_B = x_A + x_{B/A}$

$x_{B/A} = +$ means that B is to the right of A

$x_{B/A} = -$ means that B is to the left of A

$v_{B/A} = dx_{B/A} \,/\, dt$ is known as relative velocity of B with respect to A

$v_{B/A} = v_B - v_A$ or $v_B = v_A + v_{B/A}$

$v_{B/A} = +$ means that B is observed from A to move in the $+$ direction

$v_{B/A} = -$ means that B is observed from A to move in the $-$ direction

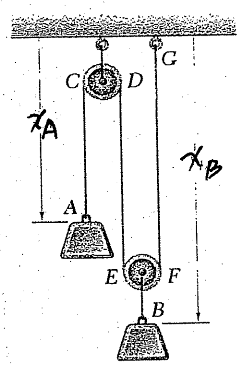
$a_{B/A} = dv_{B/A} \,/\, dt$ is known as relative acceleration of B with respect to A

$a_{B/A} = a_B - a_A$ or $a_B = a_A + a_{B/A}$

dependent motion : position of a particle depends upon the position of another or of several other particles

rope has constant length

$$x_A + 2\, x_B = \text{constant}$$



one coordinate may be chosen arbitrary

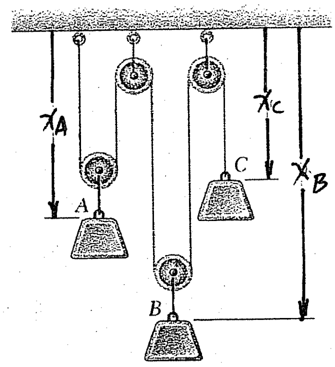
i.e. one degree of freedom, differentiating

once and twice, it is obtained

$$v_A + 2\, v_B = 0$$

and $$a_A + 2\, a_B = 0$$

in the case of three blocks of in figure



$$2\, x_A + 2\, x_B + x_C = \text{constant}$$

two coordinates may be chosen arbitrary

i.e. two degree of freedom

differentiating once and twice and obtained

$$2\, v_A + 2\, v_B + v_C = 0$$

and $$2\, a_A + 2\, a_B + a_C = 0$$

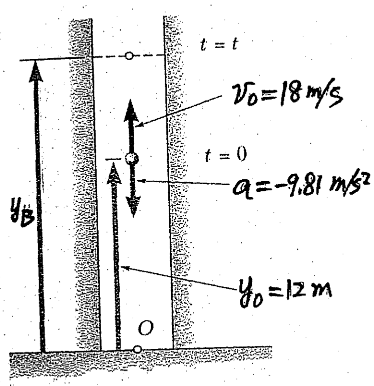
Sample Problem 11-4

ball : $a = -9.81$ m/s^2 $\quad (v_B)_0 = 18$ m/s $\quad (y_B)_0 = 12$ m

elevator : $a = 0 \quad v_E = 2$ m/s $=$ constant $\quad (y_E)_0 = 5$ m

a. when and where the ball hits the elevator

b. $v_{B/E}$ at that time



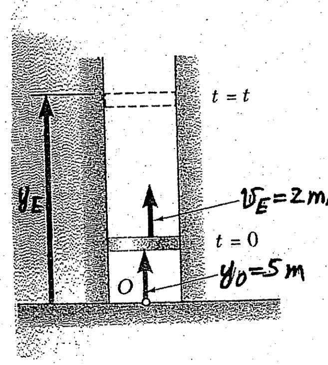
a. ball is a uniform accelerated motion

$$v_B = v_0 + a\, t = 18 - 9.81\, t$$

$$y_B = y_0 + y_0\, t + \tfrac{1}{2}\, a\, t^2$$

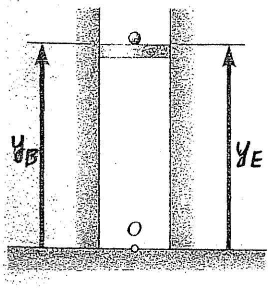
$$= 12 + 18\, t - 9.81\, t^2$$

elevator is uniformly motion



$$v_E = 2 \text{ m/s} = \text{constant}$$

$$y_E = y_0 + v_E\, t = 5 + 2\, t$$



ball hits elevator when $y_B = y_E$

$$12 + 18\,t - 9.81\,t^2 = 5 + 2\,t$$

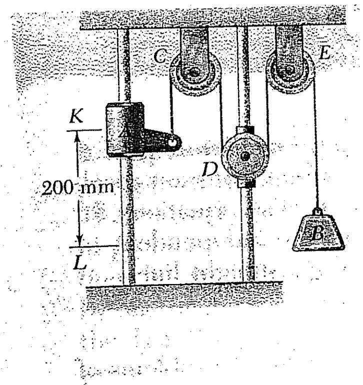
$$t = -0.39 \text{ and } 3.65 \text{ sec}$$

at $t = 3.65$ s $\quad y_B = y_E = 12.3$ m

b. $v_{B/E} = v_B - v_E = (18 - 9.81\,t) - 2$

at $t = 3.65$ s $\quad v_{B/E} = -19.81 = 19.81$ m/s $\downarrow$

Sample Problem 11-5



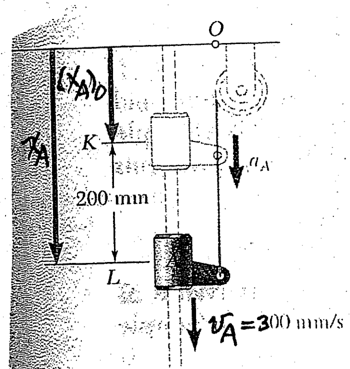
$v_D = 75$ mm/s $=$ constant

$a_A =$ constant $\qquad v_A = 0$ at $t = 0$

$\qquad v_A(\text{at L}) = 300$ mm/s

determine $\triangle x_B$, v_B, a_B when A pass L

for particle A



$$v_A^2 = (v_A)_0^2 + 2\,a_A\,[x_A - (x_A)_0]$$

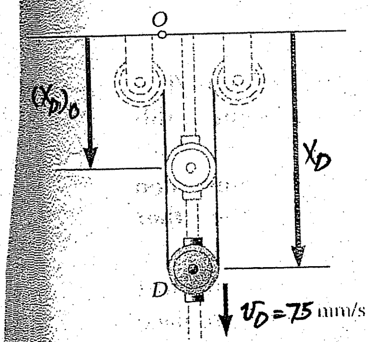
$$300^2 = 0 + 2\,a_A \times 200$$

$$a_A = 225 \text{ mm/s}^2$$

$$v_A = (v_A)_0 + a\,t$$

$$300 = 0 + 225\,t \qquad t = 1.333 \text{ sec}$$

for particle D

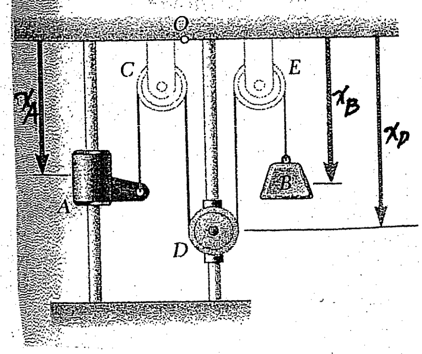


$$v_D = 75 \text{ mm/s} = \text{constant}$$

$$x_D = (x_D)_0 + v_D\,t = (x_D)_0 + 75 \times 1.333$$

$$x_D - (x_D)_0 = 100 \text{ mm}$$

for particle B



$$x_A + 2\,x_D + x_B = \text{constant}$$

$$v_A + 2\,v_D + v_B = 0$$

$$a_A + 2\,a_D + a_B = 0$$

for distance x

$$[x_A - (x_A)_0] + 2\,[x_D - (x_D)_0] + [x_B - (x_B)_0] = 0$$

$$200 + 2 \times 100 + [x_B - (x_B)_0] = 0$$

$$[x_B - (x_B)_0] = -400 \text{ mm} \uparrow$$

for the velocity v

$$300 + 2 \times 75 + v_B = 0 \qquad v_B = -450 \text{ mm/s} \uparrow$$

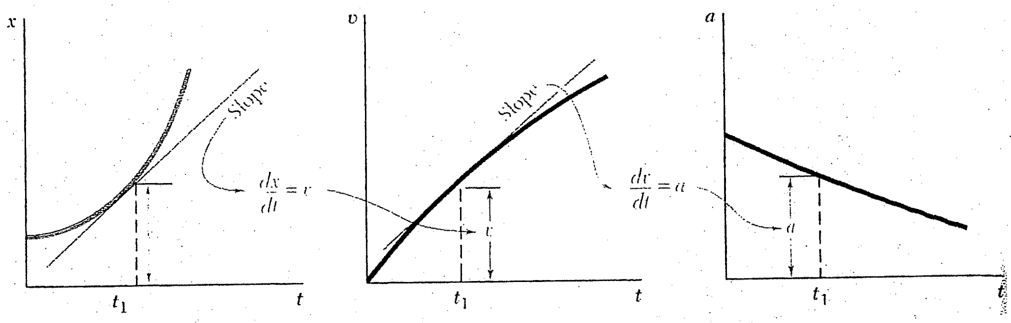
for the acceleration a

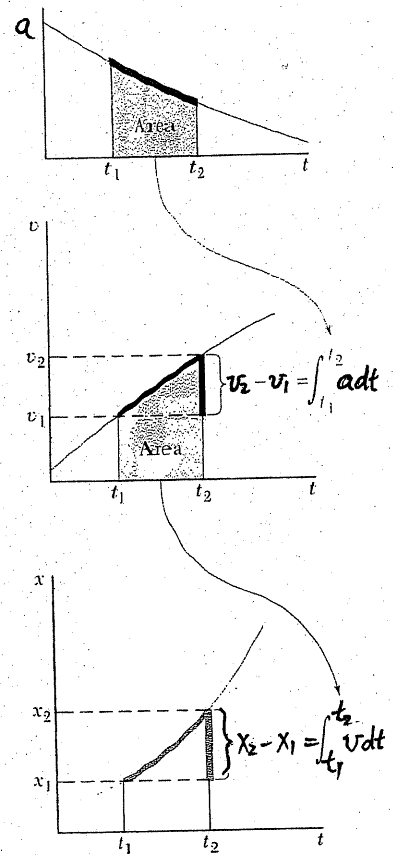
$$225 + 0 + a_B = 0 \qquad a_B = -225 \text{ mm/s}^2 \uparrow$$

11.7 Graphical Solution of Rectilinear-Motion Problems

$v = dx/dt$ slope of the x-t curve

$a = dv/dt$ slope of the v-t curve





integrating those curves from t_1 to t_2

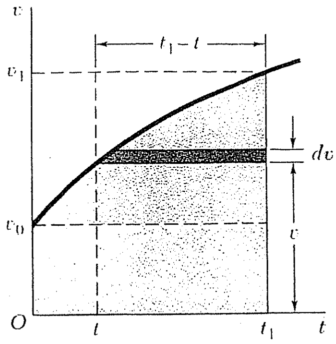
$$x_2 - x_1 = \int_{t_1}^{t_2} v\,dt$$

(area under the v-t curve between t_1 to t_2)

$$v_2 - v_1 = \int_{t_1}^{t_2} a\,dt$$

(area under the a-t curve between t_1 to t_2)

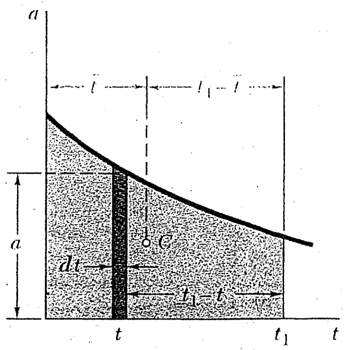
11.8 Other Graphical Methods



consider the v-t curve

$$x_1 - x_0 = \text{area under } v\text{-}t \text{ curve}$$

$$= v_0\, t + \int_{v_0}^{v_1} (t_1 - t)\, dv$$



but $dv = a\, dt$

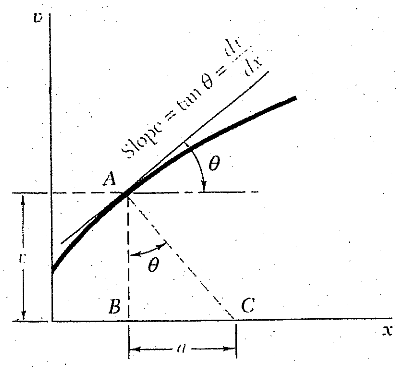
$$x_1 - x_0 = v_0\, t + \int_0^{t_1} (t_1 - t)\, a\, dt$$

the integral represents the first moment

of the area under the a-t curve

$$x_1 - x_0 = v_0\, t + (t_1 - \underline{t})\, A$$

A : area under a-t curve



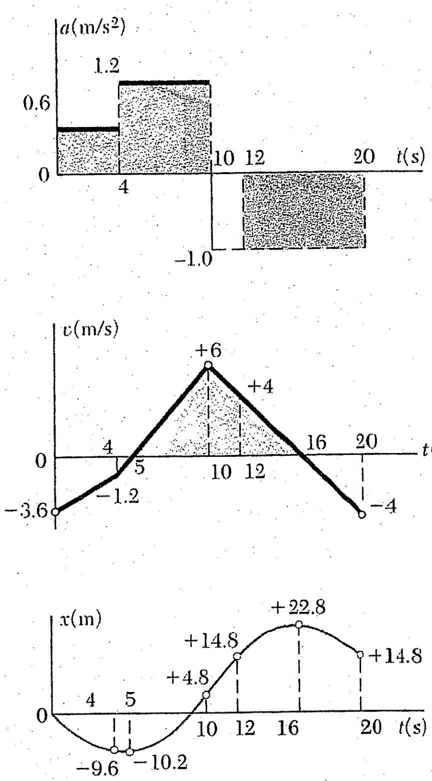
consider the v-x curve

$$BC = AB \tan\theta = v\, (dv/dx)$$

$$= a$$

a : acceleration of the particle at that time

Sample Problem 11-6



initial conditions: $t = 0$, $v_0 = -3.6$ m/s, $x_0 = 0$

$t = 0 \sim 4 \quad a = 0.6 \text{ m/s}^2$

$t = 4 \sim 10 \quad a = 1.2 \text{ m/s}^2$

$t = 12 \sim 20 \quad a = -1 \text{ m/s}^2$

$v_4 - v_0 = 0.6 \times 4 = 2.4$ m/s $\qquad v_4 = -1.2$ m/s

$v_{10} - v_4 = 1.2 \times 6 = 7.2$ m/s $\qquad v_{10} = 6$ m/s

$v_{12} - v_{10} = -1 \times 2 = -2$ m/s $\qquad v_{12} = 4$ m/s

$v_{20} - v_{12} = -1 \times 8 = -8$ m/s $\qquad v_{10} = -4$ m/s

$0 < t < 4\text{s}$	$x_4 - x_0 = ½(-3.6 - 1.2) \times 4 = -9.6$ m	$x_4 = -9.6$ m
$4\text{s} < t < 5\text{s}$	$x_5 - x_4 = ½(-1.2) \times 1 = -0.6$ m	$x_5 = -10.2$ m
$5\text{s} < t < 10\text{s}$	$x_{10} - x_5 = ½(+6) \times 5 = 15$ m	$x_{10} = 4.8$ m
$10\text{s} < t < 12\text{s}$	$x_{12} - x_{10} = ½(+6 + 4) \times 2 = 10$ m	$x_{12} = 14.8$ m
$12\text{s} < t < 16\text{s}$	$x_{16} - x_{12} = ½(+4) \times 4 = 8$ m	$x_{16} = 22.8$ m
$16\text{s} < t < 20\text{s}$	$x_{20} - x_{16} = ½(-4) \times 4 = -8$ m	$x_{20} = 14.8$ m

for $t = 12$s $\quad v_{12} = 4$ m/s $\quad x_{12} = 14.8$ m

distance traveled for $t = 0 \sim 12$s

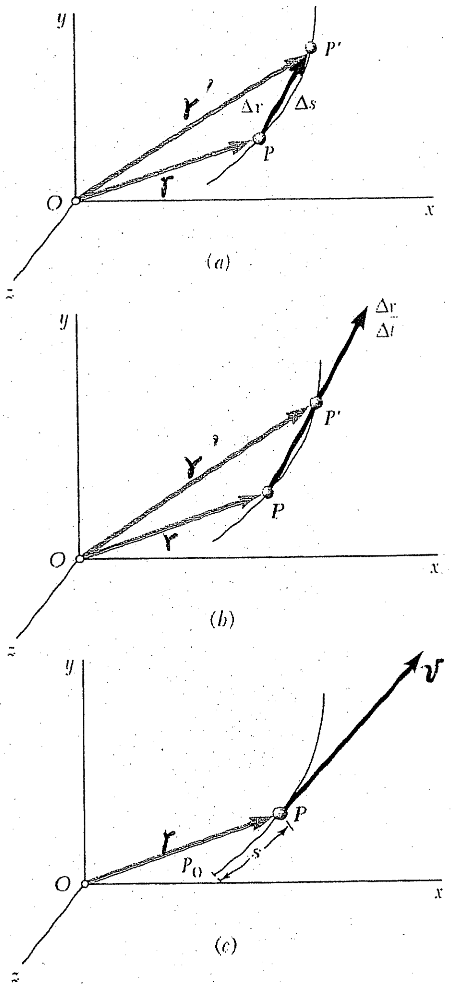
for $t = 0 \sim 5$s $\quad$ distance traveled = 10.2 m

for $t = 5 \sim 12$s $\quad$ distance traveled = (10.2 + 14.8) = 25 m

total traveled = 35.2 m

CURVILINEAR MOTION OF PARTICLES

11.9 Position Vector, Velocity, and Acceleration



defined a fixed coordinate system *Oxyz*

consider a particle moving in space, the position *P* occupied at time *t*, the vector $\boldsymbol{r}$ joint *O* and *P* is called the position vector of the particle at time *t*

consider the vector $\boldsymbol{r}'$ defining the position *P'* occupied by the particle at time $t + \Delta t$

$$\boldsymbol{PP'} = \Delta \boldsymbol{r}$$

the average velocity over the time internal Δt is

$$\boldsymbol{v}_{\text{ave}} = \Delta \boldsymbol{r} / \Delta t$$

and the instantaneous velocity is defined

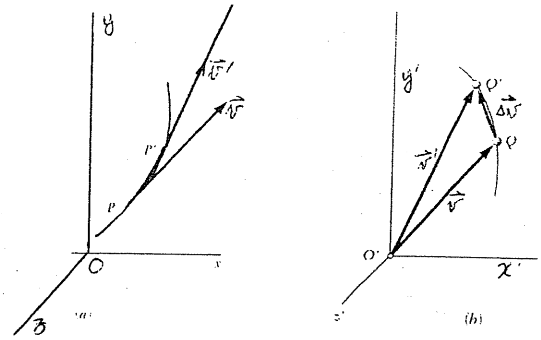
$$\boldsymbol{v} = \lim_{t \to 0} (\Delta \boldsymbol{r} / \Delta t) = d\boldsymbol{r} / dt$$

the magnitude of velocity $\boldsymbol{v}$ is called speed

$$v = |\boldsymbol{v}| = \lim_{t \to 0} (\Delta r/\Delta t) = \lim_{t \to 0} (\Delta s/\Delta t)$$

$$v = ds/dt$$

the velocity vector $\boldsymbol{v}$ is tangent to the path of the moving particle

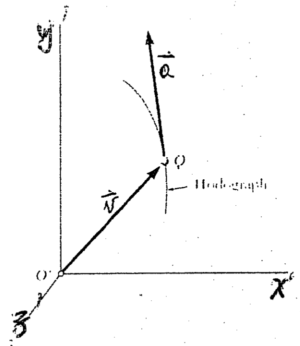


similarly for the acceleration vector

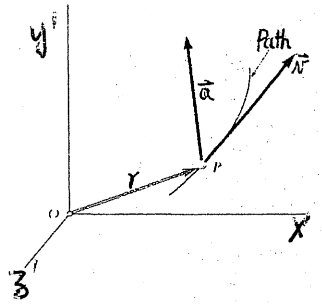
$$\boldsymbol{a} = d\boldsymbol{v}/dt$$

also $\boldsymbol{a}$ is tangent to the curve described by the tip Q of the vector $\boldsymbol{v}$ when the latter is drawn from a fixed origin O'

this path is called hodograph



Note that in the moving path of the particle, $\boldsymbol{a}$ is not a tangent



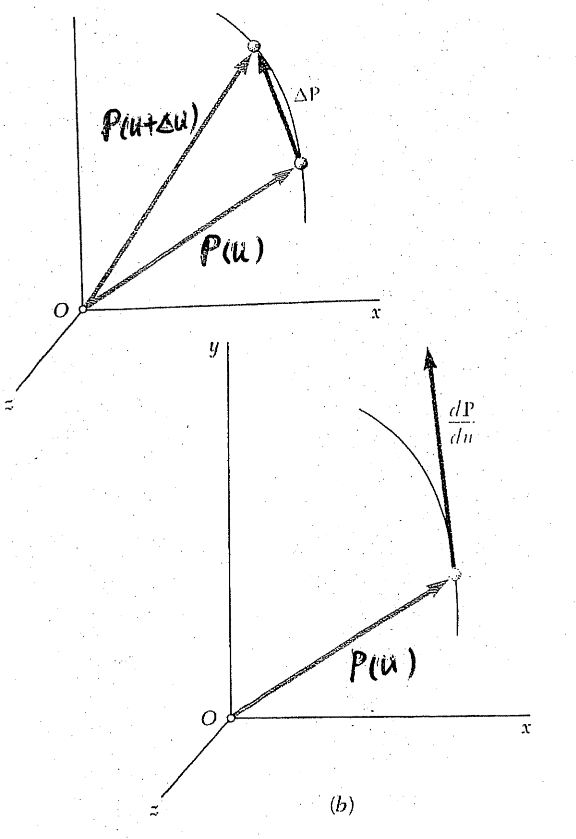
11.10 Derivatives of Vector Functions

$$\frac{d(\boldsymbol{P}+\boldsymbol{Q})}{du} = \frac{d\boldsymbol{P}}{du} + \frac{d\boldsymbol{Q}}{du}$$

$$\frac{d(f\boldsymbol{P})}{du} = \frac{df}{du}\boldsymbol{P} + f\frac{d\boldsymbol{P}}{du}$$

$$\frac{d(\boldsymbol{P}\cdot\boldsymbol{Q})}{du} = \frac{d\boldsymbol{P}}{du}\cdot\boldsymbol{Q} + \boldsymbol{P}\cdot\frac{d\boldsymbol{Q}}{du}$$

$$\frac{d(\boldsymbol{P}\times\boldsymbol{Q})}{du} = \frac{d\boldsymbol{P}}{du}\times\boldsymbol{Q} + \boldsymbol{P}\times\frac{d\boldsymbol{Q}}{du}$$



for a vector function in component form

$$\boldsymbol{P} = P_x \boldsymbol{i} + P_y \boldsymbol{j} + P_z \boldsymbol{k}$$

$$\frac{d\boldsymbol{P}}{du} = \frac{dP_x}{du}\boldsymbol{i} + \frac{dP_y}{du}\boldsymbol{j} + \frac{dP_z}{du}\boldsymbol{k}$$

the rate of change of the vector is

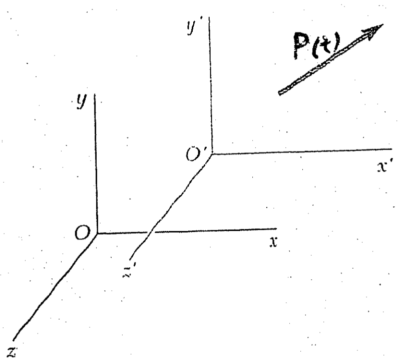
$$\frac{d\boldsymbol{P}}{dt} = \frac{dP_x}{dt}\boldsymbol{i} + \frac{dP_y}{dt}\boldsymbol{j} + \frac{dP_z}{dt}\boldsymbol{k}$$

$$\dot{\boldsymbol{P}} = \dot{P}_x \boldsymbol{i} + \dot{P}_y \boldsymbol{j} + \dot{P}_z \boldsymbol{k}$$

the rate of change of a vector, as observed from a moving frame of reference, is, in general, different from its rate of change as observed from a fixed frame of reference

but if two frames only in translation, then $\boldsymbol{P}$ is the same in both frames at any given instant, thus $\dot{\boldsymbol{P}}$ is also the same

Hence, the rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation



$$\boldsymbol{P} = P_x \boldsymbol{i} + P_y \boldsymbol{j} + P_z \boldsymbol{k}$$

$$\boldsymbol{P'} = P_x \boldsymbol{i'} + P_y \boldsymbol{j'} + P_z \boldsymbol{k'}$$

but $\quad \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} = \boldsymbol{i'}, \boldsymbol{j'}, \boldsymbol{k'}$

then $\quad \dot{\boldsymbol{P}} = \dot{\boldsymbol{P}}'$

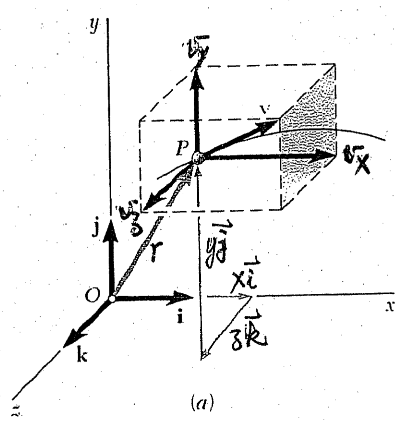
11.11 Rectangular Components of Velocity and Acceleration

consider the position vector into rectangular component form

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\boldsymbol{j} + z\,\boldsymbol{k}$$

then the velocity and acceleration vectors are

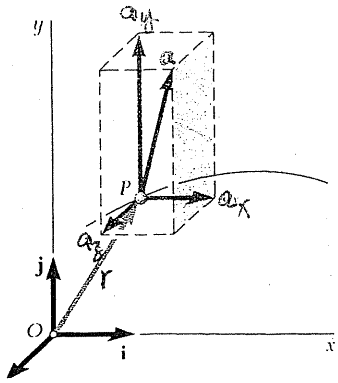
$$\boldsymbol{v} = d\boldsymbol{r} \,/\, dt = \dot{x}\,\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\,\boldsymbol{k}$$



(a)

$$\boldsymbol{a} = d\boldsymbol{v} / dt = \ddot{x}\,\boldsymbol{i} + \ddot{y}\,\boldsymbol{j} + \ddot{z}\,\boldsymbol{k}$$

the scalar components of velocity and acceleration are



$$v_x = \dot{x} \qquad v_y = \dot{y} \qquad v_z = \dot{z}$$

$$a_x = \ddot{x} \qquad a_y = \ddot{y} \qquad a_z = \ddot{z}$$

in general a_x depends only upon t, x and/or v_x

a_y depends only upon t, y and/or v_y

a_z depends only upon t, z and/or v_z

in the case of motion of a projectile

$$a_x = \ddot{x} = 0 \qquad a_z = \ddot{z} = 0$$

$$a_y = \ddot{y} = -g$$

let (x_0, y_0, z_0) be the position of the gun and

$$v_0 = [(v_x)_0, (v_y)_0, (v_z)_0]$$

then $v_x = \dot{x} = (v_x)_0 \qquad v_z = \dot{z} = (v_z)_0$ uniform motion

$v_y = \dot{y} = (v_y)_0 - g\,t$ uniformly acceleration motion

and $x = x_0 + (v_x)_0\, t \qquad z = z_0 + (v_z)_0\, t$

$y = y_0 + (v_y)_0\, t - ½\, g\, t^2$

if $(x_0, y_0, z_0) = (0, 0, 0)$ and $(v_z)_0 = 0$

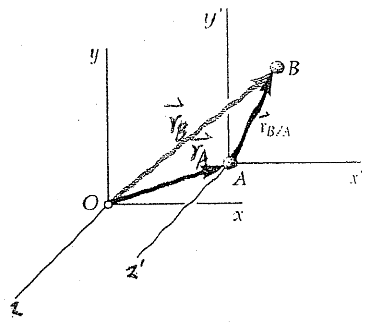
then $v_x = (v_x)_0 \qquad v_y = (v_y)_0 - g\,t \qquad v_z = 0$

$x = (v_x)_0\, t \qquad y = (v_y)_0\, t - ½\, g\, t^2 \qquad z = 0$

the projectile is moving in xy plane, the motion of vertical direction is uniformly accelerated

the motion of projectile may thus be replaced by two independent rectilinear motions

11.12 Motion Relative to a Frame in Translation



fixed frame of reference : attached to earth

moving frame of reference : other frame

consider two particles A and B moving in space, the vectors $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ define their positions

the vector $\boldsymbol{r}_{B/A}$ jointing A and B defines the position of B relative to the moving frame $Ax'y'z'$ is called relative position vector

$$\boldsymbol{r}_{B/A} = \boldsymbol{r}_B - \boldsymbol{r}_A$$

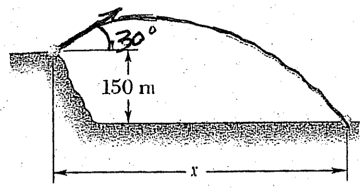
or $$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A}$$

similarly, the relative velocity and acceleration of B relative to the moving frame $Ax'y'z'$ are

$$\boldsymbol{v}_{B/A} = \boldsymbol{v}_B - \boldsymbol{v}_A \quad \text{or} \quad \boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$$

$$\boldsymbol{a}_{B/A} = \boldsymbol{a}_B - \boldsymbol{a}_A \quad \text{or} \quad \boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A}$$

Sample Problem 11-7

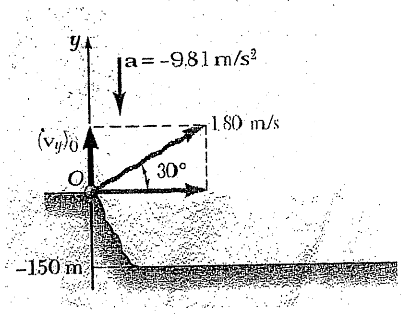


$v_0 = 180$ m/s $\angle 30°$

determine a. the horizontal distance x

b. the greatest elevation

vertical motion : uniformly accelerated motion



$$(v_y)_0 = 180 \sin 30° = 90 \text{ m/s}$$

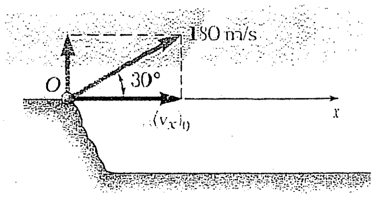
$$a_y = -9.81 \text{ m/s}^2$$

$$\therefore \quad v_y = (v_y)_0 + a_y t = 90 - 9.81\, t$$

$$y = y_0 + (v_y)_0 t + \tfrac{1}{2} a_y t^2$$

$$= 90\, t - 4.90\, t^2$$

$$v_y^2 = (v_y)_0^2 + 2\, a_y\, y = 8100 - 19.62\, y$$



horizontal motion : uniform motion

$$(v_x)_0 = v_x = 180 \cos 30^{o} = 155.9 \text{ m/s}$$

$$x = x_0 + (v_x)_0\, t = 155.9\, t$$

a. horizontal distance at $y = -150$ m

$$-150 = 90\, t - 4.90\, t^2 \quad => \quad t = 19.91 \text{ sec}$$

then $x = 155.9 \text{ x } 19.91 = 3100$ m

b. greatest elevation

$$dy / dt = v_y = 90 - 9.81\, t = 0 \quad => \quad t = 9.174 \text{ sec}$$

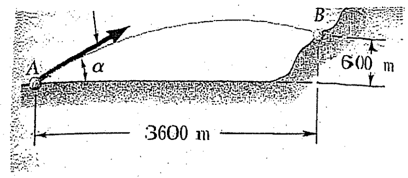
$$y = 90 \text{ x } 9.174 - 4.90 \text{ x } 9.174^2 = 413 \text{ m}$$

or $\quad v_y^2 = (v_y)_0^2 + 2\, a_y\, y$

$$0 = 90^2 - 19.62\, y \quad => \quad y = 413 \text{ m}$$

greatest elevation $= 150 + 413 = 563$ m

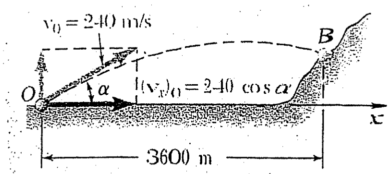
Sample Problem 11-8



the projectile is fired to hit a target at B

$$v_0 = 240 \text{ m/s} \angle\, \alpha \qquad a_y = -9.81 \text{ m/s}^2$$

determine the angle α

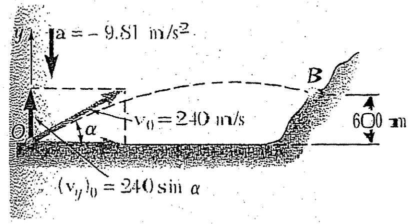


$$(v_x)_0 = 240 \cos \alpha \qquad (v_y)_0 = 240 \sin \alpha$$

$$v_x = (v_x)_0 = 240 \cos \alpha$$

$$x = (v_x)_0\, t = 240 \cos \alpha \text{ x } t$$

$$v_y = (v_y)_0 + a\, t = 240 \sin \alpha - 9.81\, t$$



$$y = (v_x)_0\, t + \tfrac{1}{2}\, a\, t^2 = 240 \sin \alpha \text{ x } t - 4.90\, t^2$$

for $x = 3600$ m

$$3600 = 240 \cos \alpha \text{ x } t \quad => \quad t = 15 / \cos \alpha$$

for $y = 600$ m

$600 = 240 \sin \alpha \times t - 4.90\, t^2$

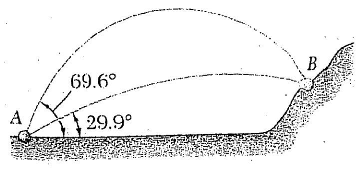
$= 240 \sin \alpha\, (15/\cos \alpha) - 4.90\, (15/\cos \alpha)^2$

$= 3600 \tan \alpha - 4.90 \times 15^2\, (1 + \tan^2 \alpha)$

$1103 \tan^2 \alpha - 3600 \tan \alpha + 1703 = 0$

$\tan \alpha = 0.574$ and 2.69

$\alpha = 29.9^{\circ}$ and 69.6°



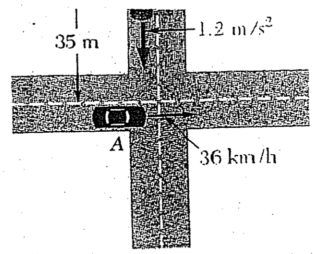
the corresponding times are 17.3 and 43.03 sec

Sample Problem 11-9

$\boldsymbol{v}_A = 36$ km/h $\rightarrow$ = constant

$\boldsymbol{a}_B = 1.2$ m/s^2 $\downarrow$ $\quad (v_B)_0 = 0 \quad (y_B)_0 = 35$ m

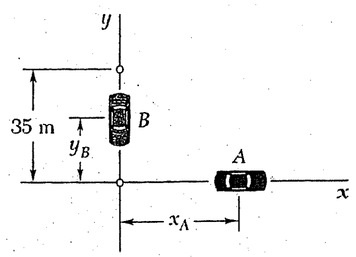
determine $\boldsymbol{r}_{B/A}$, $\boldsymbol{v}_{B/A}$, and $\boldsymbol{a}_{B/A}$ at $t = 5$ sec



$v_A = 36$ km/h $= 36{,}000$ m / 3600 s

$= 10$ m/s = constant

A : uniform motion

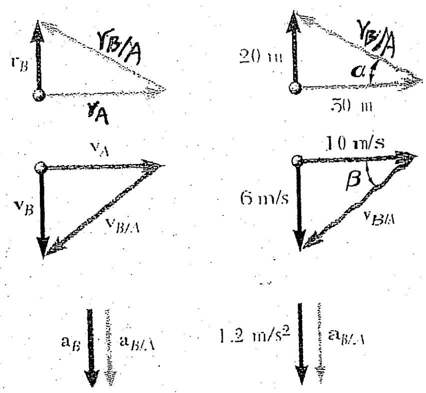


$\boldsymbol{a}_A = 0 \qquad \boldsymbol{v}_A = 10$ m/s $\rightarrow$

$x_A = (x_A)_0 + v_A\, t = 50$ m (at $t = 5$ sec)

$\boldsymbol{r}_A = 50$ m $\rightarrow$

B : uniformly accelerated motion (at $t = 5$ sec)



$\boldsymbol{a}_B = 1.2$ m/s^2 $\downarrow$

$\boldsymbol{v}_B = (v_B)_0 + \boldsymbol{a}_B\, t = -1.2\, t = -6$ m/s^2 $\downarrow$

$y_B = (y_B)_0 + (v_B)_0\, t + \frac{1}{2} a_B\, t^2 = 20$ m

$\boldsymbol{r}_B = 20$ m $\uparrow$

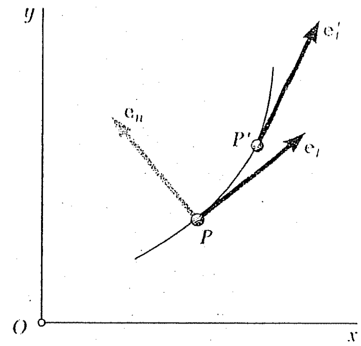
then the motion of *B* relative to *A* are

$\boldsymbol{r}_{B/A} = \boldsymbol{r}_B - \boldsymbol{r}_A = 20\boldsymbol{j} - 50\boldsymbol{i} = 53.9$ m $\nwarrow 21.8^{\circ}$

$\boldsymbol{v}_{B/A} = \boldsymbol{v}_B - \boldsymbol{v}_A = -6\boldsymbol{j} - 10\boldsymbol{i} = 11.66$ m/s $\swarrow 31^{\circ}$

$$\boldsymbol{a}_{B/A} = \boldsymbol{a}_B - \boldsymbol{a}_A = -1.2\boldsymbol{j} = 1.2 \text{ m/s}^2 \downarrow$$

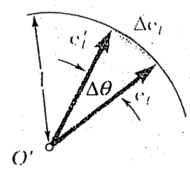
11.13 Tangential and Normal Components



(a)

plane motion of a particle

consider a moving path in xy plane, let the unit vectors $\boldsymbol{e}_t$ and $\boldsymbol{e}_t'$ be the tangent vectors at P (time = t) and P' (time = $t + \Delta t$)

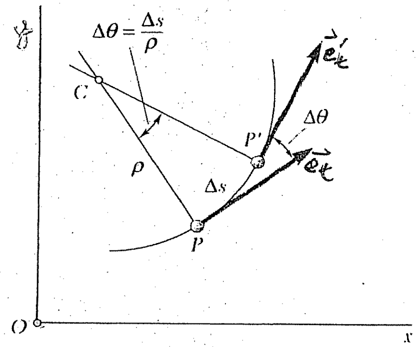


$$\Delta\boldsymbol{e}_t = \boldsymbol{e}_t' - \boldsymbol{e}_t$$

$$|\Delta\boldsymbol{e}_t| = 2\sin(\Delta\theta/2)$$

$\because$ $\Delta\boldsymbol{e}_t \perp \boldsymbol{e}_t$ for $\Delta\theta \to 0$ then $\Delta\boldsymbol{e}_t \,//\, \boldsymbol{e}_n$

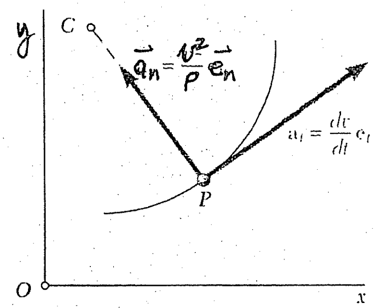
$$\lim_{\Delta\theta\to 0} \left|\frac{\Delta\boldsymbol{e}_t}{\Delta\theta}\right| = \lim_{\Delta\theta\to 0} \frac{2\sin(\Delta\theta/2)}{\Delta\theta} = \lim_{\Delta\theta\to 0} \frac{\sin(\Delta\theta/2)}{\Delta\theta/2} = 1$$



$\therefore$ $$\boldsymbol{e}_n = \lim_{\Delta\theta\to 0} \frac{\Delta\boldsymbol{e}_t}{\Delta\theta} = \frac{d\boldsymbol{e}_t}{d\theta}$$

$\because$ $\boldsymbol{v}$ is tangent to the path, then $\boldsymbol{v} = v\,\boldsymbol{e}_t$

and $$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{dv}{dt}\boldsymbol{e}_t + v\frac{d\boldsymbol{e}_t}{dt}$$



but $$\frac{d\boldsymbol{e}_t}{dt} = \frac{d\boldsymbol{e}_t}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt} = \frac{v}{\rho}\boldsymbol{e}_n$$

ρ : radius of curvature of the moving path

$\therefore$ $$\boldsymbol{a} = \frac{dv}{dt}\boldsymbol{e}_t + \frac{v^2}{\rho}\boldsymbol{e}_n = a_t\,\boldsymbol{e}_t + a_n\,\boldsymbol{e}_n$$

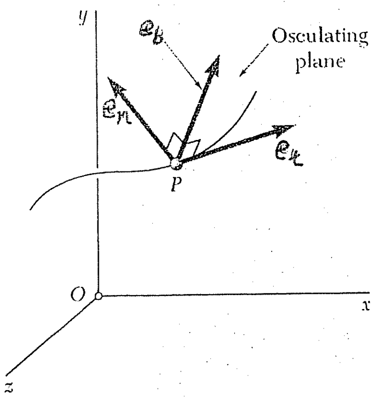
a_t = dv/dt is the tangent component of acceleration (rate of change of the speed of particle), may be positive or negative (speed increase or decrease)

a_n = v^2/ρ is the normal component of acceleration, always directed toward the center of curvature C of the moving path

$\boldsymbol{a} = 0$ only if $a_t = a_n = 0$

for a particle move with constant speed on a curve, $\boldsymbol{a} \neq 0$ ($\because a_n \neq 0$), unless at the point of inflection or when the curve is a straight line ($\rho = \infty$)

motion of particle in space



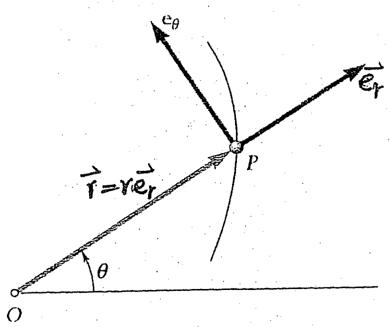
$\boldsymbol{e}_n = d\boldsymbol{e}_t / d\theta$ principal normal at P

$\boldsymbol{e}_b = \boldsymbol{e}_t \times \boldsymbol{e}_n$ binormal at P

$\boldsymbol{e}_b \perp$ the osculating plane

the acceleration can be resolved into two components, one along the tangent and the other one along the principal normal, but no component along the binormal

11.14 Radial and Transverse Components

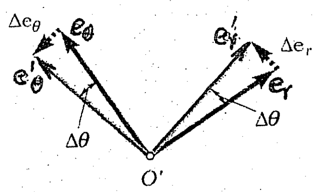


in some problem of plane motion, the position of P may be defined by polar coordinates r and θ, it is convenient to resolve the velocity and acceleration into radial and transverse components

$\boldsymbol{e}_r$ unit vector along radial direction

$\boldsymbol{e}_\theta$ unit vector along transverse direction

note that $\boldsymbol{e}_r \perp \boldsymbol{e}_\theta$



similarly as in the previous section

$$\frac{d\boldsymbol{e}_r}{d\theta} = \boldsymbol{e}_\theta \qquad \frac{d\boldsymbol{e}_\theta}{d\theta} = -\boldsymbol{e}_r$$

$$\frac{d\boldsymbol{e}_r}{dt} = \frac{d\boldsymbol{e}_r}{d\theta}\frac{d\theta}{dt} = \boldsymbol{e}_\theta \frac{d\theta}{dt} \qquad \frac{d\boldsymbol{e}_\theta}{dt} = \frac{d\boldsymbol{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\boldsymbol{e}_r \frac{d\theta}{dt}$$

or $\dot{\boldsymbol{e}}_r = \dot{\theta}\,\boldsymbol{e}_\theta \qquad \dot{\boldsymbol{e}}_\theta = -\dot{\theta}\,\boldsymbol{e}_r$

consider a particle in plane motion described by polar coordinate

$$\boldsymbol{r} = r\,\boldsymbol{e}_r$$

then $$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\,\boldsymbol{e}_r + r\,\dot{\boldsymbol{e}}_r = \dot{r}\,\boldsymbol{e}_r + r\,\dot{\theta}\,\boldsymbol{e}_\theta$$

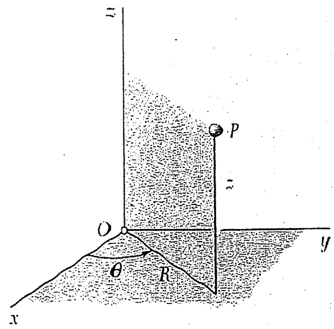
$$\boldsymbol{a} = \dot{\boldsymbol{v}} = (\ddot{r}\,\boldsymbol{e}_r + \dot{r}\,\dot{\boldsymbol{e}}_r) + (\dot{r}\,\dot{\theta}\,\boldsymbol{e}_\theta + r\,\ddot{\theta}\,\boldsymbol{e}_\theta + r\,\dot{\theta}\,\dot{\boldsymbol{e}}_\theta)$$

$$= (\ddot{r} - r\,\dot{\theta}^2)\,\boldsymbol{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\boldsymbol{e}_\theta$$

$$v_r = \dot{r} \qquad v_\theta = r\,\dot{\theta}$$

$$a_r = \ddot{r} - r\,\dot{\theta}^2 \qquad a_\theta = r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}$$

note that $a_r \neq \dot{v}_r$ $\qquad$ $a_\theta \neq \dot{v}_\theta$



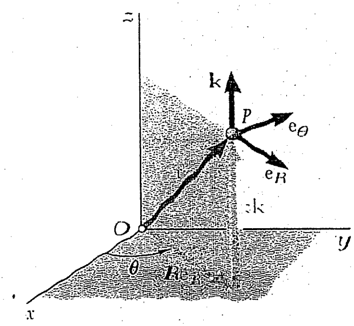
(a)

for a particle moving along a circle

$$r = \text{constant} \qquad \dot{r} = \ddot{r} = 0$$

$$\boldsymbol{v} = r\,\dot{\theta}\,\boldsymbol{e}_\theta = v\,\boldsymbol{e}_\theta$$

$$\boldsymbol{a} = -\,r\,\dot{\theta}^2\,\boldsymbol{e}_r + r\,\ddot{\theta}\,\boldsymbol{e}_\theta = -\,(v^2/r)\,\boldsymbol{e}_r + \dot{v}\,\boldsymbol{e}_\theta$$



motion in space

$$\boldsymbol{r} = R\,\boldsymbol{e}_r + z\,\boldsymbol{k}$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{R}\,\boldsymbol{e}_r + R\,\dot{\theta}\,\boldsymbol{e}_\theta + \dot{z}\,\boldsymbol{k}$$

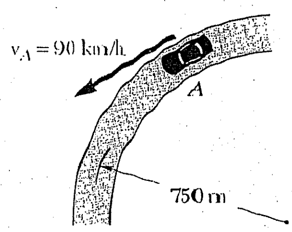
$$\boldsymbol{a} = \dot{\boldsymbol{v}} = (\ddot{r} - r\,\dot{\theta}^2)\,\boldsymbol{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\boldsymbol{e}_\theta + \ddot{z}\,\boldsymbol{k}$$

Sample Problem 11-10

$\rho = 750$ m $\qquad (v_A)_0 = 90$ km/h

$v_A = 72$ km/h after 8 sec

determine $\boldsymbol{a}$ for break applied immediately



$90 \text{ km/h} = 90{,}000 \text{ m/ } 3600 \text{ s} = 25 \text{ m/s}$

$72 \text{ km/h} = 20 \text{ m/s}$

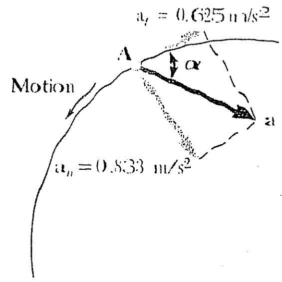
$a_t = \Delta v / \Delta t = (20 - 25) / 8 = - 0.625 \text{ m/s}^2$

$a_n = v^2 / \rho = 27^2 / 750 = 0.833 \text{ m/s}^2$

$\tan \alpha = a_n / a_t = 0.833 / 0.625 = 1.3328$

$\alpha = 53.1^o$

$a = (a_n^2 + a_t^2)^{½} = 1.041 \text{ m/s}^2$



Sample Problem 11-11

for the projectile in sample problem 11-7

determine the minimum radius of the trajectory

$\boldsymbol{v}_0 = 180 \text{ m/s} \angle 30^o$

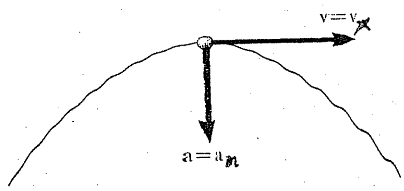
$(v_x)_0 = v_x = 155.9 \text{ m/s} = \text{constant}$

at the top of the trajectory, $v_y = 0$

$| \boldsymbol{v} | = (v_x^2 + v_y^2)^{½} = v_x$

v has a minimum value at the top

$| \boldsymbol{a} | = (a_n^2 + a_t^2)^{½} = a_n = g$ $\qquad$ ($a_t = 0$ at the top)



a_n has a maximum value at the top

$\therefore \quad \rho = \dfrac{v^2(\text{min})}{a_n(\text{max})}$ $\quad$ has a minimum value at the top

$= 155.92 / 9.81 = 2480 \text{ m}$

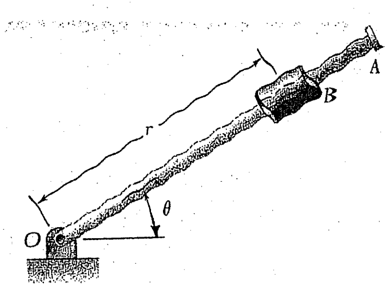
Sample Problem 11-12

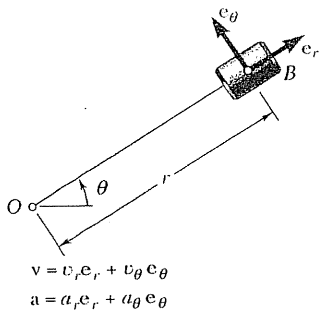
a collar B slides along a rotation arm

$OA = 0.9 \text{ m}$

$\theta = 0.15\, t^2$ (radius) $\qquad r = 0.9 - 0.12\, t^2$ (m)

determine v_B, $\boldsymbol{a}_B$ and $a_{B/OA}$ at $\theta = 30^o$

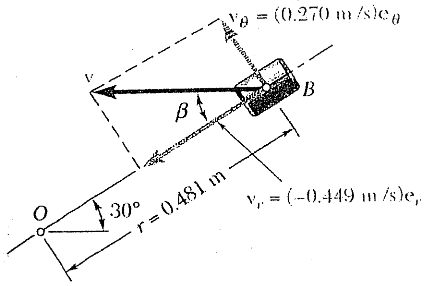




for $\theta = 30^o = 0.524$ rad

$t = (0.524 / 0.15)^{½} = 1.869$ sec

for the block B at $t = 1.869$ sec



$r = 0.9 - 0.12\, t^2 = 0.481$ m

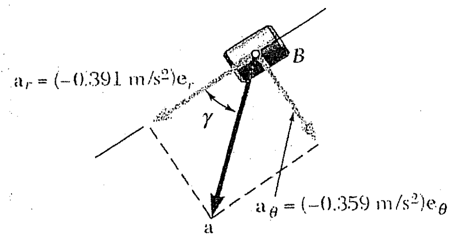
$\dot{r} = -0.24\, t = -0.449$ m/s

$\ddot{r} = -0.24$ m/s^2

$\theta = 0.15\, t^2 = 0.524$ rad

$\dot{\theta} = 0.3\, t = 0.561$ rad/s

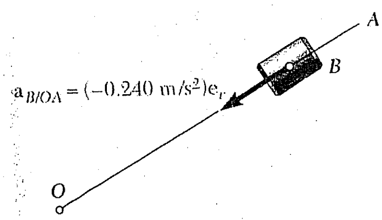
$\ddot{\theta} = 0.3$ rad/s^2



$v_r = \dot{r} = -0.449$ m/s

$v_\theta = r\,\dot{\theta} = 0.481 \times 0.561 = 0.27$ m/s

$v = 0.524$ m/s $\quad \beta = 31^o$



$a_r = \ddot{r} - r\,\dot{\theta}^2 = -0.391$ m/s^2

$a_\theta = r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} = -0.359$ m/s2

$a = 0.531$ m/s^2 $\quad \gamma = 42.6^o$

$a_{B/OA} = \ddot{r} = -0.24 = 0.24$ m/s^2 ↙ toward O

$r = 0.9 - 0.12\, t^2 = 0.9 - 0.12\,(\theta/0.15)^2 = 0.9 - 0.8\,\theta$

$dr / d\theta = -0.8$ m/rad $\quad$ radius variation due to unit angle

for $r = 0$ $\quad \theta = 9/8$ rad $= 64.46^o$