

Chapter 11 Columns

11.1 Introduction

Types of failure : fracture, buckling, fatigue, creep rupture etc., it depends on materials, kind of loads, condition of supports

another type of failure is buckling, consider specifically for columns

if a compression member is relatively slender, it may fail by bending or deflecting laterally rather than by direct compression of the material

buckling may occurs in column, cylindrical walls etc.

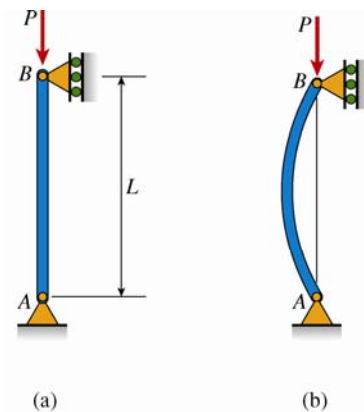


FIG. 11-1 Buckling of a slender column due to an axial compressive load P

11.2 Buckling and Stability

consider a structure consists of two rigid bars AB and BC , pinned at B with a rotational spring having stiffness β_k

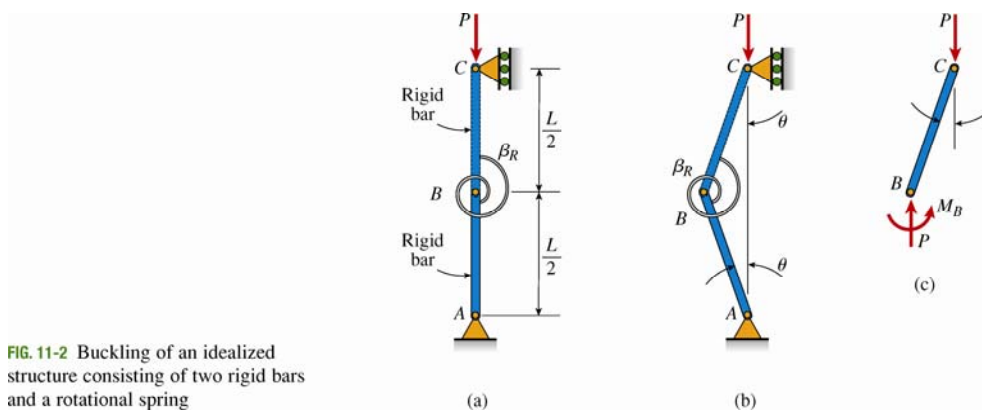


FIG. 11-2 Buckling of an idealized structure consisting of two rigid bars and a rotational spring

in the idealized structure, the two bars are perfectly aligned and the axial load P has its line of action along the longitudinal axis, the spring is unstressed and the bars are direct compression

now suppose the structure is disturbed by some external force, the rigid bars rotate a small angle θ and a moment develops in the spring

when the disturbing force is removed, if P is relatively small, the structure will return to its initial straight position, it is said to be **stable**, if P is large, the lateral displacement of point B will increase and the bars will rotate through larger and larger until structure collapses, the structure is said to be **unstable**

the transition between the stable and unstable conditions of the axial load known the **critical load** P_{cr}

consider the bar BC , the moment M_B is due to the rotation of the spring, then

$$M_B = 2 \beta_k \theta$$

equilibrium of moment at point B , (for small θ)

$$M_B - P \left(\frac{\theta L}{2} \right) = 0$$

$$(2 \beta_k - \frac{P L}{2}) \theta = 0$$

the solutions are $\theta = 0 \implies$ perfectly straight

$$P_{cr} = \frac{4 \beta_k}{L} \implies \text{critical load}$$

the stability of the structure is increased either by increasing its stiffness or by decreasing its length

if $P < P_{cr}$, the structure is **stable**
 if $P > P_{cr}$, the structure is **unstable**
 if $P = P_{cr}$, the structure is in **neutral equilibrium**

point B is called a **bifurcation point**

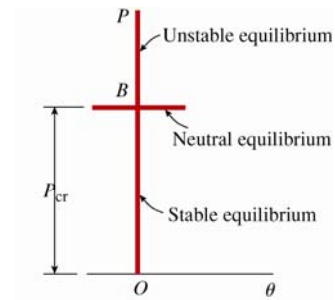
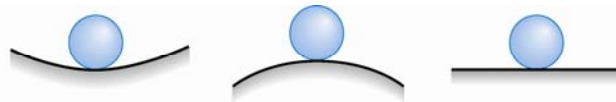


FIG. 11-3 Equilibrium diagram for buckling of an idealized structure

the three equilibrium conditions are analogous to those of a ball placed upon a smooth surface as shown

FIG. 11-4 Ball in stable, unstable, and neutral equilibrium



11.3 Columns with Pinned Ends

consider a slender column with perfectly straight and is made of a linear elastic material (ideal column)

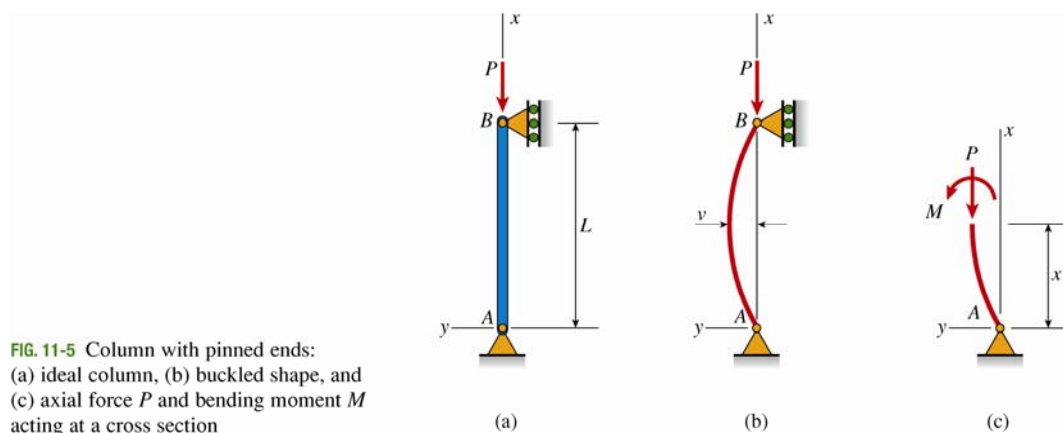


FIG. 11-5 Column with pinned ends: (a) ideal column, (b) buckled shape, and (c) axial force P and bending moment M acting at a cross section

if P small, $\sigma = P / A$
 it is in stable equilibrium

if P is gradually increased, it will reach a condition of neutral equilibrium, the corresponding load is called critical load P_{cr}

if P is higher, the column is unstable and may collapse by buckling

if $P < P_{cr}$, the column is in **stable equilibrium** in the straight position

if $P = P_{cr}$, the column is in **neutral equilibrium** either the straight or a slightly bent position

if $P > P_{cr}$, the structure is in **unstable equilibrium** in the straight position and will buckle under the slightest disturbance

actual columns do not behave in the idealized manner because imperfections always exist

to determine P_{cr} , we will use the second-order differential equation in terms of bending moment M , the moment-curvature equation is

$$EI v'' = M$$

moment equilibrium at point A, we obtain

$$M + P v = 0$$

or
$$M = -P v$$

the differential equation of deflection now becomes

$$EI v'' + P v = 0$$

it is a homogeneous linear differential equation with constant coefficients

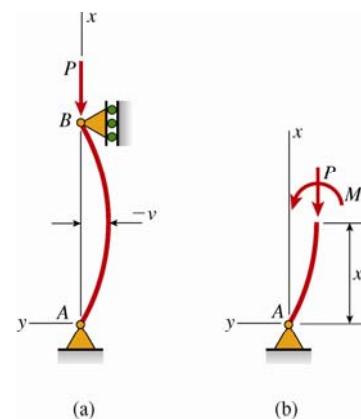


FIG. 11-6 Column with pinned ends (alternative direction of buckling)

let $k^2 = P / EI$

then $v'' + k^2 v = 0$

the general solution of this equation is

$$v = C_1 \sin kx + C_2 \cos kx$$

with boundary conditions

$$v(0) = v(L) = 0$$

$$v(0) = 0 \Rightarrow C_2 = 0$$

then $v = C_1 \sin kx$

$$v(L) = 0 \Rightarrow C_1 \sin kL = 0$$

case 1. $C_1 = 0$ that means the column remains straight for any value of kL , that is P may also have any value (this is known as the trivial solution)

case 2. $\sin kL = 0 \Rightarrow kL = 0, 2\pi, \dots, n\pi$

$$\therefore k^2 = P / EI$$

$$kL = 0 \Rightarrow P = 0 \quad \text{it is not of interest}$$

for $kL = n\pi$ with $n = 1, 2, 3, \dots$

the corresponding values of P are

$$P = \frac{n^2 \pi^2 EI}{L^2} \quad n = 1, 2, 3, \dots$$

for this values of P , the column may have bent shape, for other

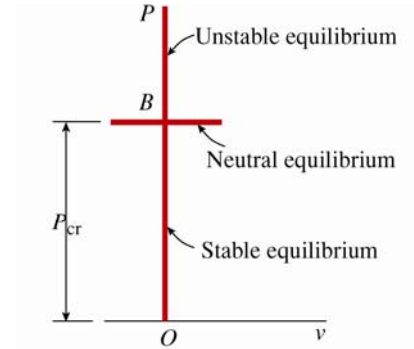


FIG. 11-7 Load-deflection diagram for an ideal, linearly elastic column

value of P , C_1 must equal 0 (column remains straight)

the equation of the deflection curve now becomes

$$v = C_1 \sin kx = C_1 \sin \frac{n\pi x}{L} \quad n = 1, 2, 3, \dots$$

the lowest critical load for a column with pinned ends is obtained when $n = 1$

$$P_{cr} = \frac{\pi^2 E I}{L^2}$$

the corresponding buckled shape (mode shape) is

$$v = C_1 \sin \pi x/L \quad (1^{\text{st}} \text{ mode})$$

buckling of a pinned-end column in the first mode ($n = 1$) is called fundamental case

the critical load for an ideal elastic column is also known as the **Euler**

Load, the bifurcation point B occurs at the critical load

$$\text{for } x = L/2 \sin \pi x/L = 1 \quad v = C_1$$

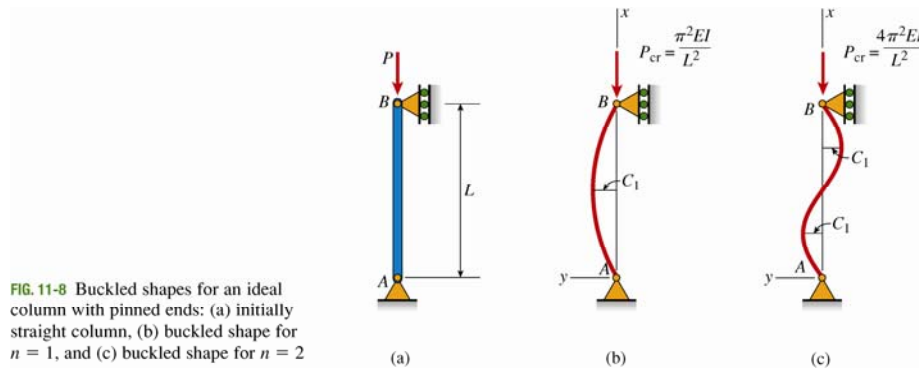
thus $C_1 = v(L/2)$ is the midpoint deflection

$$\text{for } n = 2 \quad P_{cr}(n=2) = 4 P_{cr}(n=1) \quad P_{cr} \sim n^2$$

the column will buckle when axial load P reaches its lowest P_{cr}

the only way to obtain higher P_{cr} is to provide lateral support at the inflection position

$$\therefore P_{cr} = \frac{\pi^2 E I}{L^2} \quad \begin{array}{l} P_{cr} \sim EI \\ P_{cr} \sim 1/L^2 \end{array}$$



i.e. $I \uparrow \Rightarrow P_{cr} \uparrow$

material away from NA may increase I

consider two cross sections A and B

$$A: I_{11} > I_{22}$$

$$B: I_{11} > I_{22}$$

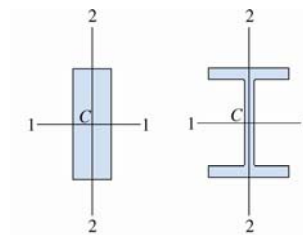


FIG. 11-9 Cross sections of columns showing principal centroidal axes with $I_1 > I_2$

so we must use I_{22} to calculate P_{cr}

but $(I_{22})_B > (I_{22})_A$ section B is better for buckling design

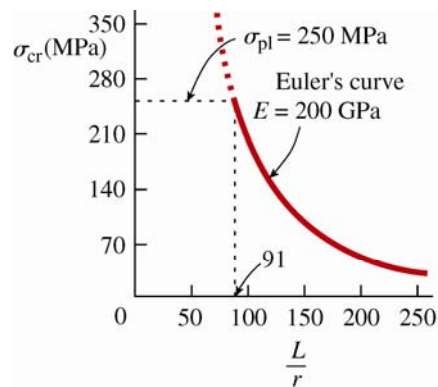
P_{cr} do not depends on σ_y or σ_u , but P_{cr} must be $< \sigma_y A$, and the critical stress can be calculated

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E I}{A L^2}$$

recalled $r = (I/A)^{1/2}$ is the radius of gyration, then

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

define the **slenderness ratio** $= L / r$ (depends only on the dimensions of the column), the critical stress is inverse proportional to the slenderness ratio of the column, it is obtained a plot of σ_{cr} vs. L/r as shown, this curve is called



Euler's curve, the curve is valid only when the critical stress is less than σ_y

FIG. 11-10 Graph of Euler's curve (from Eq. 11-16) for structural steel with $E = 200 \text{ GPa}$ and $\sigma_{pl} = 250 \text{ MPa}$

Effects of Large Deflections, Imperfections, and Inelastic Behavior

A : ideal elastic column (small deflection)

B : ideal elastic column (large deflection)

[exact expression for v'' are used]

C : elastic column with imperfections

D : inelastic column with imperfections

if column with imperfection, it will have a small initial curvature, thus it produce deflection from the onset of loading, the large the imperfections, curve C moves to the right, if column is constructed with great accuracy, curve C approaches more closely to curve A or B

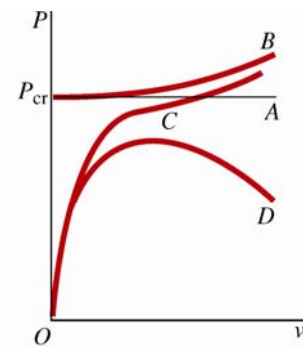


FIG. 11-11 Load-deflection diagram for columns: Line A, ideal elastic column with small deflections; Curve B, ideal elastic column with large deflections; Curve C, elastic column with imperfections; and Curve D, inelastic column with imperfections

if the stress exceed the proportional limit, the curve reaches a maximum and turns downward, only extremely slender columns remain elastic for the loading up to P_{cr}

stockier columns behave inelastically and follow curve D, P_{max}

may smaller than P_{cr} , the descending part of curve D represents catastrophic collapse

Optimum Shapes of Columns

a column shaped shown in figure will have a large critical load that a prismatic column made from the same volume of materials

for a prismatic column with pinned ends that is free to buckle in any lateral direction, thus P_{cr} is calculated by using the smallest I for the cross section

is the circular shape is the most efficient section for column? the answer is "no", for same cross-sectional area $I(\triangle) > I(\circ)$ by 21%, i.e. P_{cr} is 21% higher

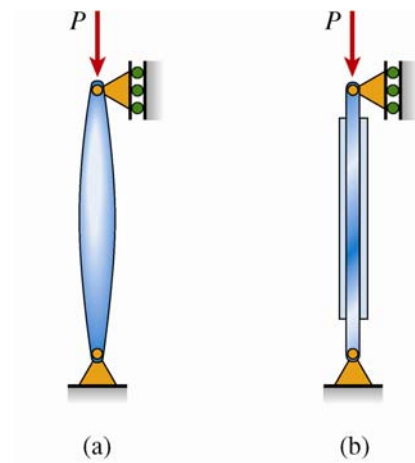


FIG. 11-12 Nonprismatic columns

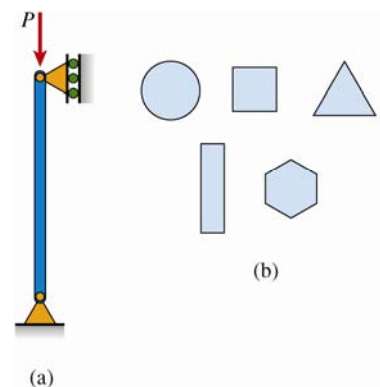


FIG. 11-13 Which cross-sectional shape is the optimum shape for a prismatic column?

Example 11-1

a pinned column ABC of IPN 220 wide-flange section have lateral support at midpoint B

$$E = 220 \text{ GPa}$$

$$\sigma_{pl} = 300 \text{ MPa}$$

$$L = 8 \text{ m} \quad n = 2.5$$

$$P_{allow} = ?$$

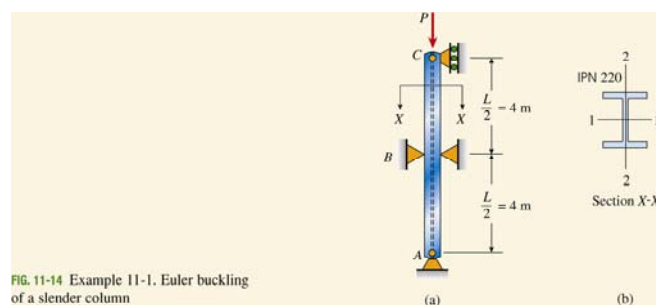


FIG. 11-14 Example 11-1. Euler buckling of a slender column

for the IPN 220 section

$$I_1 = 3,060 \text{ cm}^4 \quad I_2 = 162 \text{ cm}^4 \quad A = 39.5 \text{ cm}^2$$

case 1. if the column buckles in the plane $\perp$ 2-2 direction

$$P_{cr} = \frac{\pi^2 E I_2}{(L/2)^2} = \frac{4\pi^2 E I_2}{L^2} = \frac{4\pi^2 \times 200 \text{ GPa} \times 162 \text{ cm}^4}{(8 \text{ m})^2} = 200 \text{ kN}$$

case 2. if the column buckles in the plane $\perp$ 1-1 direction

$$P_{cr} = \frac{\pi^2 E I_1}{L^2} = \frac{\pi^2 \times 200 \text{ GPa} \times 3,060 \text{ cm}^4}{(8 \text{ m})^2} = 943.8 \text{ kN}$$

$$\text{thus } P_{cr} = \min [200, 943.8] = 200 \text{ kN}$$

$$\text{and } \sigma_{cr} = P_{cr} / A$$

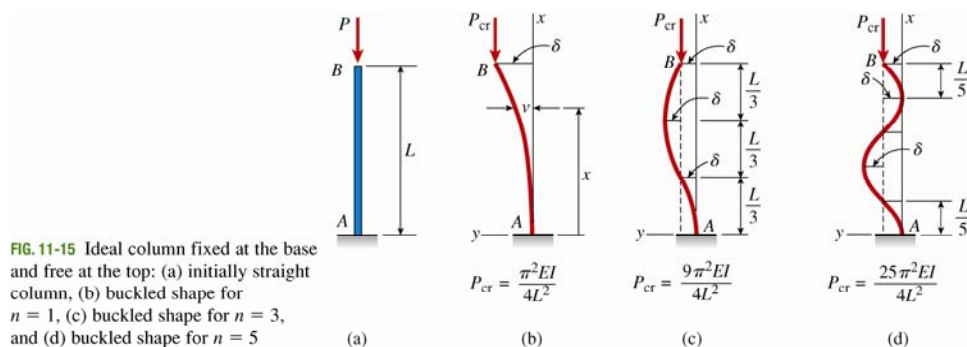
$$\text{case 1. } \sigma_{cr} = 50.63 \text{ MPa} < \sigma_{pl} \quad (\text{OK})$$

$$\text{case 2. } \sigma_{cr} = 238.9 \text{ MPa} < \sigma_{pl} \quad (\text{OK})$$

$$P_{allow} = P_{cr} / n = 200 / 2.5 = 80 \text{ kN}$$

11.4 Columns with other Support Conditions

for a column with pinned end, it is known as fundamental case



bending moment at distance x is

$$M = P(\delta - v)$$

the differential equation of deflection becomes

$$EI v'' = M = P(\delta - v)$$

$$\text{or } v'' + k^2 v = k^2 \delta$$

where $k^2 = P / EI$, this is a second order differential equation with constant coefficients, the general solution can be obtained

$$v = C_1 \sin kx + C_2 \cos kx + \delta$$

$$\text{boundary conditions } v(0) = v'(0) = 0$$

$$v(0) = 0 \Rightarrow C_2 = -\delta$$

$$v' = C_1 \cos kx - C_2 \sin kx$$

$$v'(0) = 0 \Rightarrow C_1 = 0$$

thus the deflection curve for the buckled column is

$$v = \delta(1 - \cos kx)$$

$$\therefore v(L) = \delta \Rightarrow \delta \cos kL = 0$$

$$\therefore \delta = 0 \quad \text{trivial solution}$$

$$\text{or } \cos kL = 0 \Rightarrow kL = n\pi / 2 \quad n = 1, 3, 5, \dots$$

therefore the critical load for the column are

$$P_{cr} = \frac{n^2 \pi^2 E I}{4 L^2} \quad n = 1, 3, 5, \dots$$

and the buckled mode shapes are

$$v = \delta \left(1 - \cos \frac{n\pi x}{2L} \right) \quad n = 1, 3, 5, \dots$$

$$\text{for } n = 1 \quad P_{cr} = \pi^2 EI / 4L^2 \quad v = \delta (1 - \cos \pi x / 2L)$$

$$n = 3 \quad P_{cr}(n = 3) = 9 P_{cr}(n = 1)$$

Effective Lengths of Columns

the critical load for columns with various support conditions can be related to P_{cr} of a pinned-end column (fundamental mode)

the effective length of a column is the distance between point of inflection (zero moment) in its deflection curve

for fixed-free column, the effective length is

$$L_e = 2L$$

define the effective length factor K as

$$L_e = K L$$

then the general formula for critical load can be written

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{(KL)^2}$$

$$K = 2 \quad \text{for fixed-free column}$$

$$K = 1/2 \quad \text{for fixed-fixed column}$$

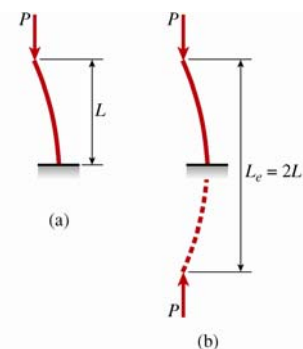


FIG. 11-16 Deflection curves showing the effective length L_e for a column fixed at the base and free at the top

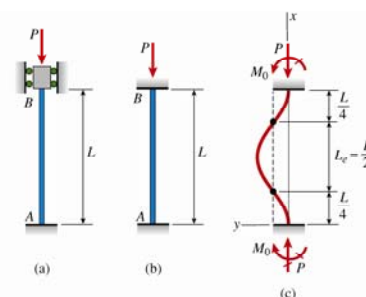


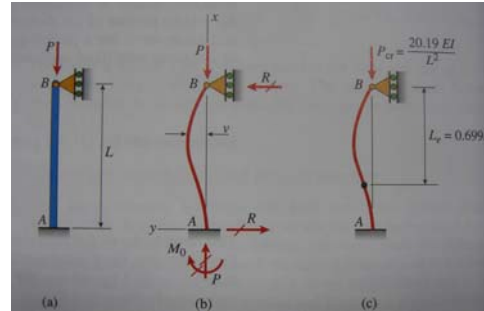
FIG. 11-17 Buckling of a column with both ends fixed against rotation

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{4 \pi^2 EI}{L^2}$$

for a fixed-pinned column

$$M_0 = R L$$

$$\begin{aligned} M &= M_0 - P v - R x \\ &= -P v + R(L - x) \end{aligned}$$



and $EI v'' = M = -P v + R(L - x)$

then $v'' + k^2 v = \frac{R}{EI}(L - x) \quad k^2 = \frac{P}{EI}$

the general solution for this equation is

$$v = C_1 \sin kx + C_2 \cos kx + R(L - x) / P$$

with boundary conditions $v(0) = v'(0) = 0, v(L) = 0$

we have

$$C_2 + \frac{RL}{P} = 0 \quad C_1 k - \frac{R}{P} = 0 \quad C_1 \tan kL + C_2 = 0$$

for $C_1 = C_2 = R = 0 \Rightarrow$ trivial solution

another solution can be obtained by eliminate R from the first two equations, which yields

$$C_1 kL + C_2 = 0 \quad C_2 = -C_1 kL$$

the 3rd equation can be written

$$kL = \tan kL \Rightarrow kL = 4.4934$$

the corresponding critical load is

$$P_{cr} = \frac{20.19 EI}{L^2} = \frac{2.046 \pi^2 EI}{L^2} = \frac{\pi^2 EI}{L_e^2}$$

where $L_e = 0.699 L \simeq 0.7 L$

and the mode shape is obtained

$$v = C_1 [\sin kx - kL \cos kx + k(L - x)]$$

in which $k = 4.4934 / L$

the lowest critical loads and L_e for the four columns are listed below

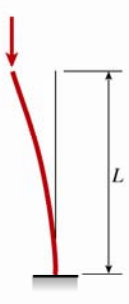
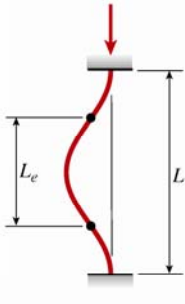
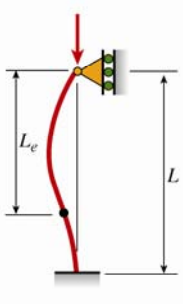
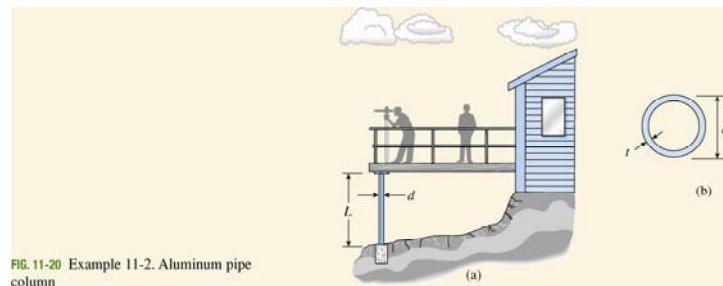
(a) Pinned-pinned column	(b) Fixed-free column	(c) Fixed-fixed column	(d) Fixed-pinned column
$P_{cr} = \frac{\pi^2 EI}{L^2}$	$P_{cr} = \frac{\pi^2 EI}{4L^2}$	$P_{cr} = \frac{4\pi^2 EI}{L^2}$	$P_{cr} = \frac{2.046 \pi^2 EI}{L^2}$
			
$L_e = L$	$L_e = 2L$	$L_e = 0.5L$	$L_e = 0.699L$
$K = 1$	$K = 2$	$K = 0.5$	$K = 0.699$

FIG. 11-19 Critical loads, effective lengths, and effective-length factors for ideal columns

Example 11-2

Aluminum pipe fixed-pinned column,



$$L = 3.25 \text{ m} \quad d = 100 \text{ mm} \quad P = 100 \text{ kN}$$

$$n = 3 \quad E = 72 \text{ GPa} \quad \sigma_{pl} = 480 \text{ MPa}$$

Critical load

$$P_{cr} = \frac{2.046 \pi^2 EI}{L^2} = nP = 300 \text{ kN}$$

Where $I = \pi [d^4 - (d - 2t)^4] / 64$

$$300,000 \text{ N} = \frac{2.046 \pi^2 (72 \times 10^9 \text{ Pa})}{(3.25 \text{ m})^2} \frac{\pi [(0.1 \text{ m})^4 - (0.1 \text{ m} - 2t)^4]}{64}$$

$$t = 0.008625 \text{ m} = 8.63 \text{ mm}$$

and $I = \pi [d^4 - (d - 2t)^4] / 64 = 2.18 \times 10^6 \text{ mm}^4$

$$A = \pi [d^2 - (d - 2t)^2] / 4 = 1,999 \text{ mm}^2$$

then $r = (I / A)^{1/2} = 33.0 \text{ mm}$

the slenderness ratio is

$$L / r \simeq 98$$

and diameter-to-thickness ratio is

$$d / t \simeq 15$$

it is OK for prevent local buckling

and the critical stress is

$$\sigma_{cr} = P_{cr} / A = 150 \text{ MPa} < \sigma_{pl} = 480 \text{ MPa} \quad (\text{OK})$$

11.5 Columns with Eccentric Axial Loads

assume that a column is compressed by loads P that are applied with a small eccentricity e , it is equivalent to a centric load P and a couple of $M_0 = Pe$

the bending moment in the column at distance x is

$$M = M_0 + P(-v) = Pe - Pv$$

the differential equation of deflection curve is

$$EI v'' = M = Pe - Pv$$

or
$$v'' + k^2 v = k^2 e$$

where $k^2 = P / EI$ as before

the general solution is

$$v = C_1 \sin kx + C_2 \cos kx + e$$

boundary conditions

$$v(0) = 0 \quad v(L) = 0$$

then we have

$$C_1 = -e \quad C_2 = -e (1 - \cos kL) / \sin kL = -e \tan (kL/2)$$

then
$$v = -e [\tan (kL/2) \sin kx + \cos kx - 1]$$

each value of e produces a definite value of the deflection, the maximum deflection δ at the midpoint is

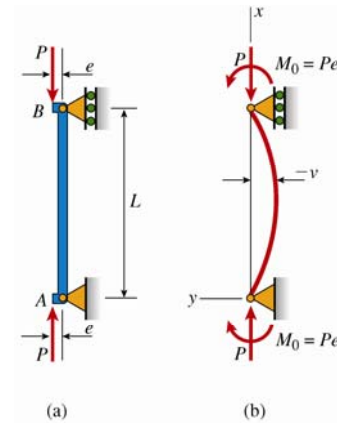


FIG. 11-21 Column with eccentric axial loads

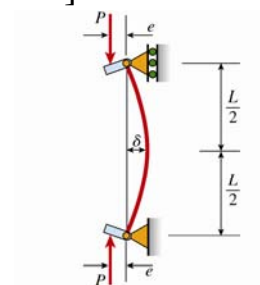


FIG. 11-22 Maximum deflection δ of a column with eccentric axial loads

$$\begin{aligned}\delta &= -v(L/2) = e [\tan(kL/2) \sin(kL/2) + \cos(kL/2) - 1] \\ &= e [\sec(kL/2) - 1]\end{aligned}$$

$$\therefore P_{cr} = \pi^2 EI / L^2$$

$$\therefore k = (P / EI)^{1/2} = (P \pi^2 / P_{cr} L^2)^{1/2} = \pi / L (P / P_{cr})^{1/2}$$

$$\text{then } \delta = e [\sec \{ \pi / L (P / P_{cr})^{1/2} \} - 1]$$

1. δ is equal zero for e or P equal zero

2. δ becomes ∞ for $P = P_{cr}$

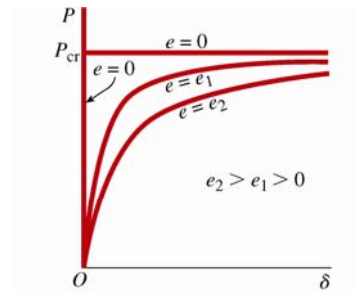


FIG. 11-23 Load-deflection diagram for a column with eccentric axial loads (see Fig. 11-22 and Eq. 11-54)

Maximum bending moment

$$\begin{aligned}M_{max} &= P(e + \delta) \\ &= P e \sec [\pi / L (P / P_{cr})^{1/2}]\end{aligned}$$

The manner of M_{max} vs P as shown

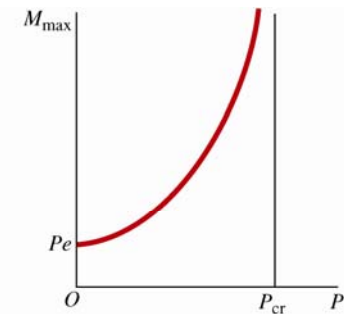


FIG. 11-24 Maximum bending moment in a column with eccentric axial loads (see Fig. 11-22 and Eq. 11-56)

ex. 11.3

$$\begin{aligned}P &= 7 \text{ kN} & e &= 11 \text{ mm} \\ h &= 30 \text{ mm} & b &= 15 \text{ mm} \\ E &= 110 \text{ GPa} & \delta_{max} &= 3 \text{ mm} \\ L_{max} &= ?\end{aligned}$$

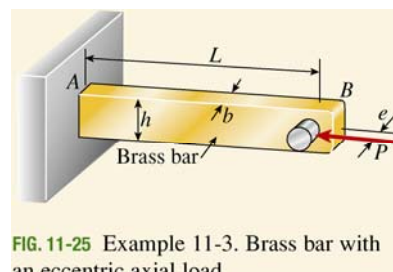


FIG. 11-25 Example 11-3. Brass bar with an eccentric axial load

the critical load for fixed-free column is

$$P_{cr} = \pi^2 EI / 4L^2$$

$$I = h b^3 / 12 = 0.844 \text{ cm}^2$$

$$\therefore P_{\text{cr}} = \frac{\pi^2 (110 \text{ GPa})(0.844 \text{ cm}^2)}{4L^2} = \frac{2.29 \text{ kN-m}^2}{L^2}$$

$$\delta = e [\sec \{ \pi/L (P / P_{\text{cr}})^{1/2} \} - 1]$$

$$3 \text{ mm} = (11 \text{ mm}) [\sec \{ \pi/L (7 \text{ kN } L^2 / 2.29)^{1/2} \} - 1]$$

$$0.2727 = \sec (2.746 L) - 1$$

$$\text{then } L = L_{\text{max}} = 0.243 \text{ m}$$

11.6 The Secant Formula for Columns

11.7 Elastic and Inelastic Column Behavior

11.8 Inelastic Buckling