

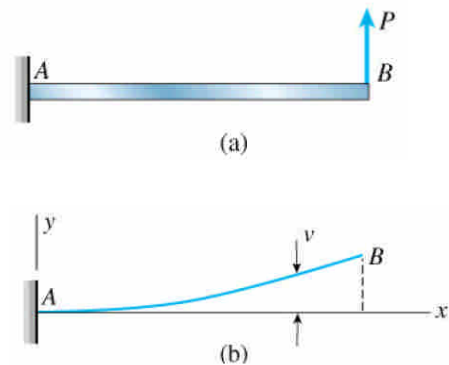
## Chapter 5 Stresses in Beam (Basic Topics)

### 5.1 Introduction

Beam : loads acting transversely to the longitudinal axis

the loads create shear forces and bending moments, stresses and strains due to  $V$  and  $M$  are discussed in this chapter

lateral loads acting on a beam cause the beam to bend, thereby deforming the axis of the beam into curve line, this is known as the **deflection curve** of the beam



the beams are assumed to be symmetric about  $x$ - $y$  plane, i.e.  $y$ -axis is an axis of symmetric of the cross section, all loads are assumed to act in the  $x$ - $y$  plane, then the bending deflection occurs in the same plane, it is known as the **plane of bending**

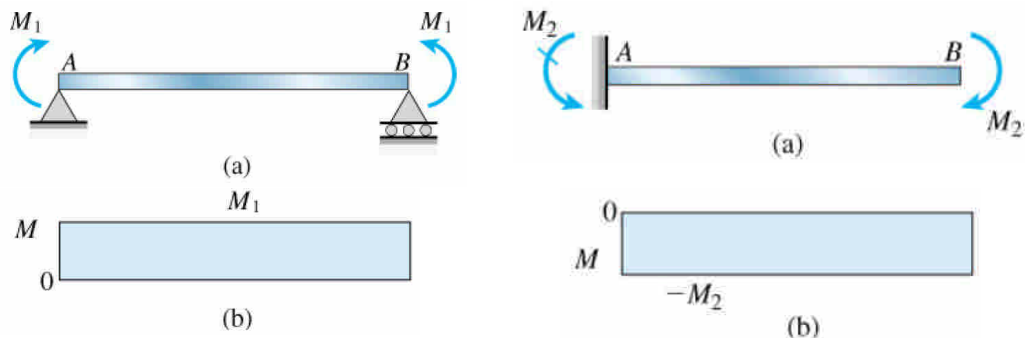
the **deflection** of the beam is the displacement of that point from its original position, measured in  $y$  direction

### 5.2 Pure Bending and Nonuniform Bending

pure bending :

$$M = \text{constant} \quad V = dM / dx = 0$$

pure bending in simple beam and cantilever beam are shown

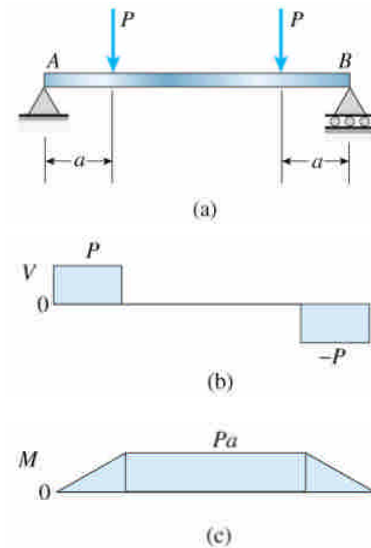


nonuniform bending :

$$M \neq \text{constant}$$

$$V = dM / dx \neq 0$$

simple beam with central region in pure bending and end regions in nonuniform bending is shown



### 5.3 Curvature of a Beam

consider a cantilever beam subjected to a load  $P$

choose 2 points  $m_1$  and  $m_2$  on the deflection curve, their normals intersect at point  $O'$ , is called the center of curvature, the distance  $m_1 O'$  is called radius of curvature  $\rho$ , and the curvature  $\kappa$  is defined as

$$\kappa = 1 / \rho$$

and we have  $\rho d\theta = ds$

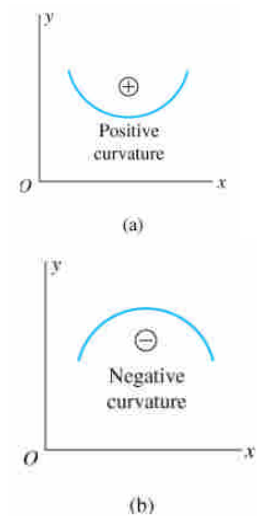
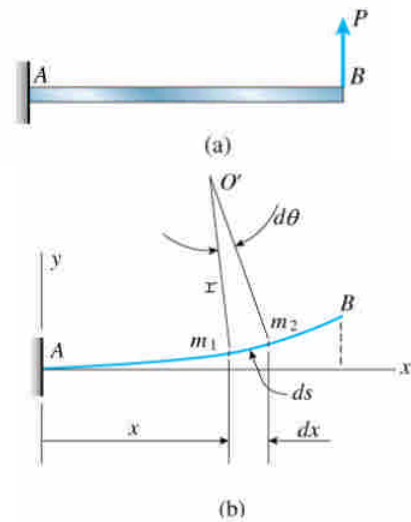
if the deflection is small  $ds \simeq dx$ , then

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx}$$

sign convention for curvature

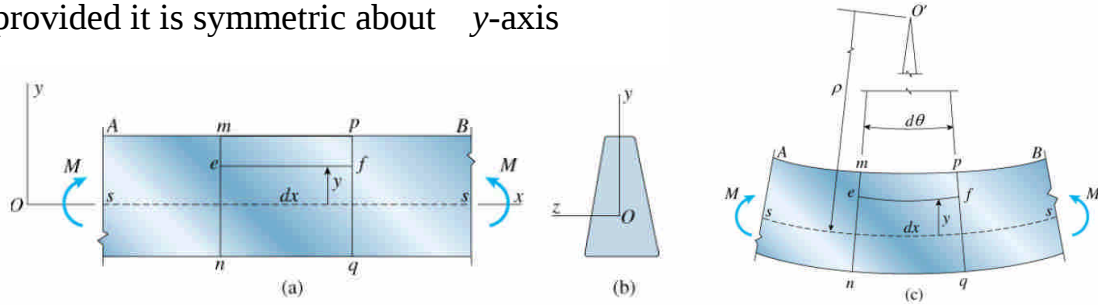
+ : beam is bent concave upward (convex downward)

- : beam is bent concave downward (convex upward)



## 5.4 Longitudinal Strains in Beams

consider a portion  $ab$  of a beam in pure bending produced by a positive bending moment  $M$ , the cross section may be of any shape provided it is symmetric about  $y$ -axis



under the moment  $M$ , its axis is bent into a circular curve, cross section  $mn$  and  $pq$  remain plane and normal to longitudinal lines (plane remains plane can be established by experimental result)

$\therefore$  the symmetry of the beam and loading, it requires that all elements of the beam deform in an identical manner ( $\therefore$  the curve is circular), this are valid for any material (elastic or inelastic)

due to bending deformation, cross sections  $mn$  and  $pq$  rotate w.r.t. each other about axes perpendicular to the  $xy$  plane

longitudinal lines on the convex (lower) side ( $nq$ ) are elongated, and on the concave (upper) side ( $mp$ ) are shortened

the surface  $ss$  in which longitudinal lines do not change in length is called the **neutral surface**, its intersection with the cross-sectional plane is called neutral axis, for instance, the  $z$  axis is the neutral axis of the cross section

in the deformed element, denote  $\rho$  the distance from  $O'$  to N.S. (or N.A.), thus

$$\rho d\theta = dx$$

consider the longitudinal line  $ef$ , the length  $L_1$  after bending is

$$L_1 = (\rho - y) d\theta = dx - \frac{y}{\rho} dx$$

$$\text{then } \Delta_{ef} = L_1 - dx = -\frac{y}{\rho} dx$$

and the strain of line  $ef$  is

$$\epsilon_x = \frac{\Delta_{ef}}{dx} = -\frac{y}{\rho} = -\kappa y$$

$\epsilon_x$  vary linear with  $y$  (the distance from N.S.)

$$y > 0 \text{ (above N. S.)} \quad \epsilon = -$$

$$y < 0 \text{ (below N. S.)} \quad \epsilon = +$$

the longitudinal strains in a beam are accompanied by transverse strains in the  $y$  and  $z$  directions because of the effects of Poisson's ratio

### Example 5-1

a simply supported beam  $AB$ ,  $L = 8$  ft

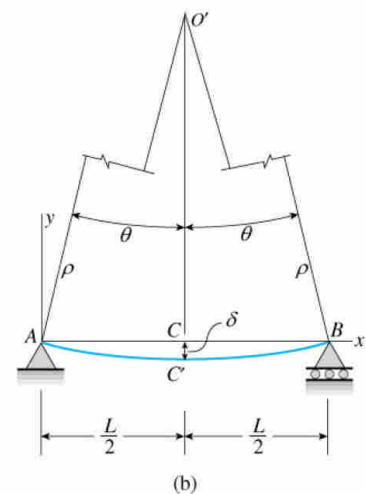
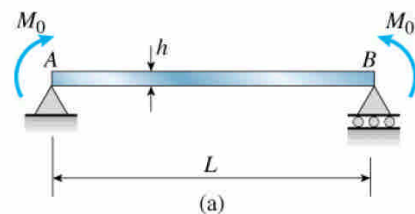
bent by  $M_0$  into a circular arc

$$\epsilon_{bottom} = \epsilon_x = 0.00125$$

$$y_{bottom} = -3 \text{ in}$$

determine  $\rho$ ,  $\kappa$ , and  $\delta$  (midpoint deflection)

$$\begin{aligned} \rho &= -\frac{y}{\epsilon_x} = -\frac{-3}{0.00125} \\ &= 2,400 \text{ in} = 200 \text{ ft} \end{aligned}$$



$$\kappa = \frac{1}{\rho} = 0.005 \text{ ft}^{-1}$$

$$\delta = \rho (1 - \cos \theta)$$

$\therefore \rho$  is large,  $\therefore$  the deflection curve is very flat

$$\text{then } \sin \theta = \frac{L/2}{\rho} = \frac{8 \times 12}{2 \times 2,400} = 0.020$$

$$\theta = 0.02 \text{ rad} = 1.146^\circ$$

$$\text{then } \delta = 2,400 (1 - \cos 1.146^\circ) = 0.48 \text{ in}$$

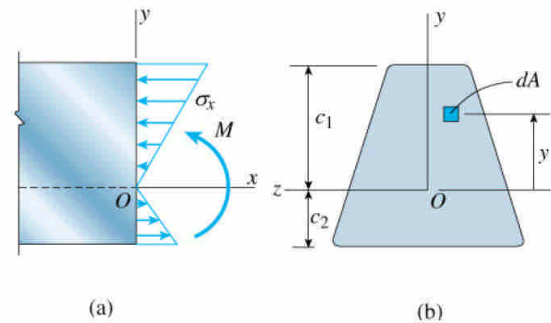
#### 5.4 Normal Stress in Beams (Linear Elastic Materials)

$\therefore \epsilon_x$  occurs due to bending,  $\therefore$  the longitudinal line of the beam is subjected only to tension or compression, if the material is linear elastic

$$\text{then } \sigma_x = E \epsilon_x = -E \kappa y$$

$\sigma$  vary linear with distance  $y$  from the neutral surface

consider a positive bending moment  $M$  applied, stresses are positive below N.S. and negative above N.S.



$\therefore$  no axial force acts on the cross section, the only resultant is  $M$ , thus two equations must satisfy for static equilibrium condition

$$\text{i.e. } \Sigma F_x = \int \sigma dA = - \int E \kappa y dA = 0$$

$\therefore E$  and  $\kappa$  are constants at the cross section, thus we have

$$\int y dA = 0$$

we conclude that the neutral axis passes through the centroid of the cross section, also for the symmetrical condition in  $y$  axis, the  $y$  axis must pass through the centroid, hence, the origin of coordinates  $O$  is located at the centroid of the cross section

the moment resultant of stress  $\sigma_x$  is

$$dM = -\sigma_x y dA$$

$$\text{then } M = - \int \sigma_x y dA = \int E \kappa y^2 dA = E \kappa \int y^2 dA$$

$$M = E \kappa I$$

where  $I = \int y^2 dA$  is the moment of inertia of the cross-sectional area w. r. t.  $z$  axis

$$\text{thus } \kappa = \frac{1}{\rho} = \frac{M}{EI}$$

this is the **moment-curvature equation**,

and  $EI$  is called flexural rigidity

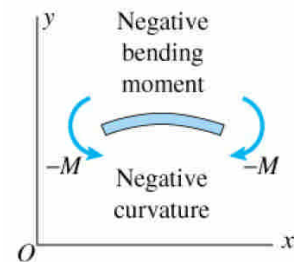
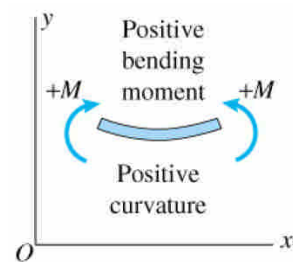
+ M => + curvature

- M => - curvature

the normal stress is

$$\sigma_x = -E \kappa y = -E y \left( \frac{M}{EI} \right) = -\frac{M y}{I}$$

this is called the **flexure formula**, the stress  $\sigma_x$  is called bending stresses or flexural stresses



$$\sigma_x \text{ vary linearly with } y$$

$$\sigma_x \simeq \frac{M}{I} y \quad \sigma_x \simeq \frac{1}{I} y$$

the maximum tensile and compressive stresses occur at the points located farthest from the N.A.

$$\sigma_1 = -\frac{M c_1}{I} = -\frac{M}{S_1}$$

$$\sigma_2 = \frac{M c_2}{I} = \frac{M}{S_2}$$

where  $S_1 = \frac{I}{c_1}$ ,  $S_2 = \frac{I}{c_2}$  are known as the section moduli

if the cross section is symmetric w.r.t.  $z$  axis (double symmetric cross section), then  $c_1 = c_2 = c$

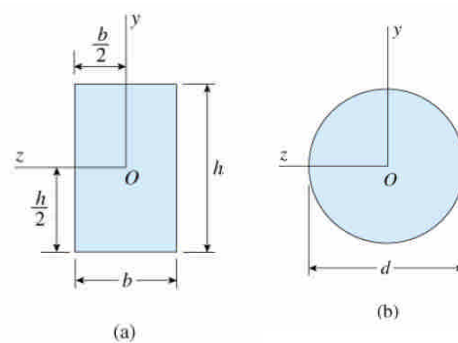
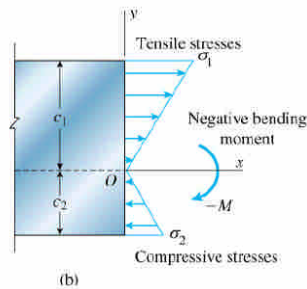
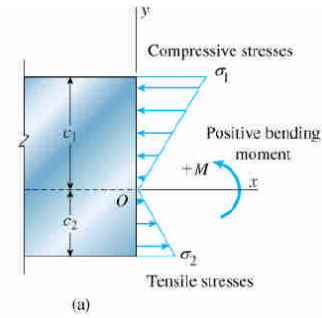
$$\text{thus } S_1 = S_2 \text{ and } \sigma_1 = -\sigma_2 = -\frac{M c}{I} = -\frac{M}{S}$$

for rectangular cross section

$$I = \frac{b h^3}{12} \quad S = \frac{b h^2}{6}$$

for circular cross section

$$I = \frac{\pi d^4}{64} \quad S = \frac{\pi d^3}{32}$$



the preceding analysis of normal stress in beams concerned pure bending, no shear force

in the case of nonuniform bending ( $V \neq 0$ ), shear force produces warping

(out of plane distortion), plane section no longer remain plane after bending, but the normal stress  $\sigma_x$  calculated from the flexure formula are not significantly altered by the presence of shear force and warping

we may justifiably use the theory of pure bending for calculating  $\sigma_x$  even when we have nonuniform bending

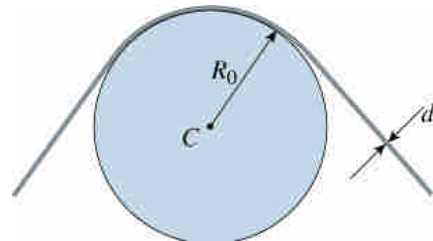
the flexure formula gives results in the beam where the stress distribution is not disrupted by irregularities in the shape, or by discontinuous in loading (otherwise, stress concentration occurs)

#### example 5-2

a steel wire of diameter  $d = 4 \text{ mm}$  is bent around a cylindrical drum of radius  $R_0 = 0.5 \text{ m}$

$E = 200 \text{ GPa}$        $\sigma_{pl} = 1200 \text{ MPa}$

determine  $M$  and  $\sigma_{max}$



the radius of curvature of the wire is

$$\rho = R_0 + \frac{d}{2}$$

$$\begin{aligned} M &= \frac{EI}{\rho} = \frac{2EI}{2R_0 + d} = \frac{\pi E d^4}{32(2R_0 + d)} \\ &= \frac{\pi (200 \times 10^3) 4^4}{32 (2 \times 500 + 4)} = 5007 \text{ N-mm} = 5.007 \text{ N-m} \end{aligned}$$

$$\begin{aligned} \sigma_{max} &= \frac{M}{S} = \frac{M}{I / (d/2)} = \frac{M d}{2 I} = \frac{2 E I d}{2 I (2 R_0 + d)} = \frac{E d}{2 R_0 + d} \\ &= \frac{200 \times 10^3 \times 4}{2 \times 500 + 4} = 796.8 \text{ MPa} < 1,200 \text{ MPa (OK)} \end{aligned}$$

### Example 5-3

a simple beam  $AB$  of length  $L = 22$  ft

$$q = 1.5 \text{ k/ft} \quad P = 12 \text{ k}$$

$$b = 8.75 \text{ in} \quad h = 27 \text{ in}$$

determine the maximum tensile and compressive stresses due to bending

firstly, construct the  $V$ -dia and  $M$ -dia

$\sigma_{max}$  occurs at the section of  $M_{max}$

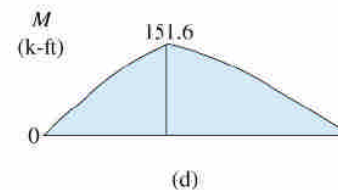
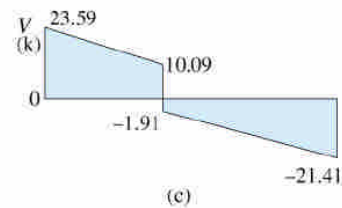
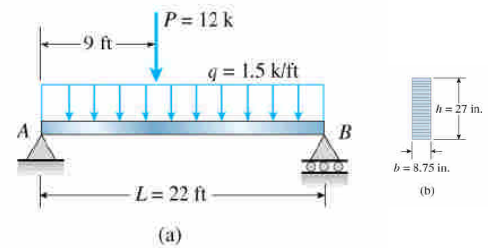
$$M_{max} = 151.6 \text{ k-ft}$$

the section modulus  $S$  of the section is

$$S = \frac{b h^2}{6} = \frac{8.75 \times 27^2}{6} = 1,063 \text{ in}^3$$

$$\sigma_t = \sigma_2 = \frac{M}{S} = \frac{151.6 \times 10^3 \times 12}{1,063} = 1,710 \text{ psi}$$

$$\sigma_c = \sigma_1 = -\frac{M}{S} = -1,710 \text{ psi}$$



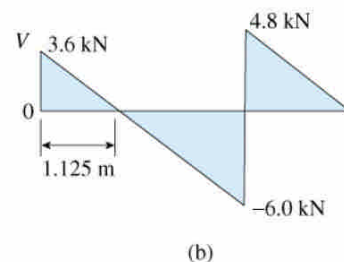
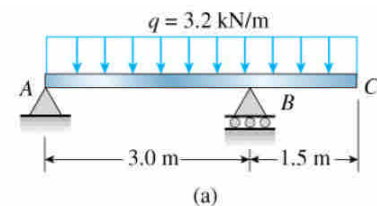
### Example 5-4

an overhanged beam  $ABC$  subjected to uniform load of intensity  $q = 3.2 \text{ kN/m}$

for the cross section (channel section)

$$t = 12 \text{ mm} \quad b = 300 \text{ mm} \quad h = 80 \text{ mm}$$

determine the maximum tensile and compressive stresses in the beam



construct the *V-dia.* and *M-dia.* first

we can find  $+M_{max} = 2.205 \text{ kN-m}$

$-M_{max} = -3.6 \text{ kN-m}$

next, we want to find the N. A. of the section

|       | $A(\text{mm}^2)$ | $y(\text{mm})$ | $Ay (\text{mm}^3)$ |
|-------|------------------|----------------|--------------------|
| $A_1$ | 3,312            | 6              | 19,872             |
| $A_2$ | 960              | 40             | 38,400             |
| $A_3$ | 960              | 40             | 38,400             |
| total | 5,232            |                | 96,672             |

$$c_1 = \frac{\sum A_i y_i}{\sum A_i} = \frac{96,672}{5,232} = 18.48 \text{ mm}$$

$$c_2 = h - c_1 = 61.52 \text{ mm}$$

moment of inertia of the section is

$$I_{z1} = I_{zc} + A_1 d_1^2$$

$$I_{zc} = \frac{1}{12} (b - 2t) t^3 = \frac{1}{12} 276 \times 12^3 = 39744 \text{ mm}^4$$

$$d_1 = c_1 - t/2 = 12.48 \text{ mm}$$

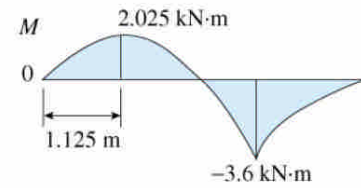
$$I_{z1} = 39,744 + 3,312 \times 12.48^2 = 555,600 \text{ mm}^4$$

similarly  $I_{z2} = I_{z3} = 956,000 \text{ mm}^4$

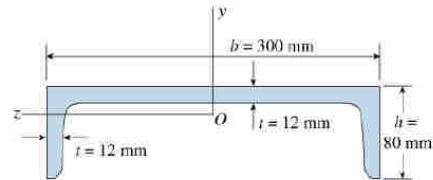
then the centroidal moment of inertia  $I_z$  is

$$I_z = I_{z1} + I_{z2} + I_{z3} = 2.469 \times 10^6 \text{ mm}^4$$

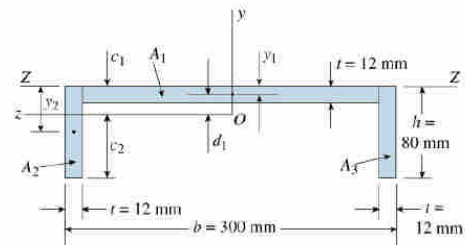
$$S_1 = \frac{I_z}{c_1} = 133,600 \text{ mm}^3 \quad S_2 = \frac{I_z}{c_2} = 40,100 \text{ mm}^3$$



(c)



(a)



(b)

at the section of maximum positive moment

$$\begin{aligned}\sigma_t = \sigma_2 &= \frac{M}{S_2} = \frac{2.025 \times 10^3 \times 10^3}{40,100} = 50.5 \text{ MPa} \\ \sigma_c = \sigma_1 &= -\frac{M}{S_1} = -\frac{2.025 \times 10^3 \times 10^3}{133,600} = -15.2 \text{ MPa}\end{aligned}$$

at the section of maximum negative moment

$$\begin{aligned}\sigma_t = \sigma_1 &= -\frac{M}{S_1} = -\frac{-3.6 \times 10^3 \times 10^3}{133,600} = 26.9 \text{ MPa} \\ \sigma_c = \sigma_2 &= \frac{M}{S_2} = \frac{-3.6 \times 10^3 \times 10^3}{40,100} = -89.8 \text{ MPa}\end{aligned}$$

thus  $(\sigma_t)_{max}$  occurs at the section of maximum positive moment

$$(\sigma_t)_{max} = 50.5 \text{ MPa}$$

and  $(\sigma_c)_{max}$  occurs at the section of maximum negative moment

$$(\sigma_c)_{max} = -89.8 \text{ MPa}$$

## 5.6 Design of Beams for Bending Stresses

design a beam : type of construction, materials, loads and environmental conditions

beam shape and size : actual stresses do not exceed the allowable stress

for the bending stress, the section modulus  $S$  must be larger than  $M / \sigma$

$$\text{i.e.} \quad S_{min} = M_{max} / \sigma_{allow}$$

$\sigma_{allow}$  is based upon the properties of the material and magnitude of the desired factor of safety

if  $\sigma_{allow}$  are the same for tension and compression, doubly symmetric section is logical to choose

if  $\sigma_{allow}$  are different for tension and compression, unsymmetric cross section such that the distance to the extreme fibers are in nearly the same ratio as the respective allowable stresses

select a beam not only the required  $S$ , but also the smallest cross-sectional area

### Beam of Standardized Shapes and Sizes

steel, aluminum and wood beams are manufactured in standard sizes

steel : American Institute of Steel Construction (AISC)

e.g. wide-flange section W 30 x 211 depth = 30 in, 211 lb/ft

other sections : S shape (I beam), C shape (channel section)

L shape (angle section)

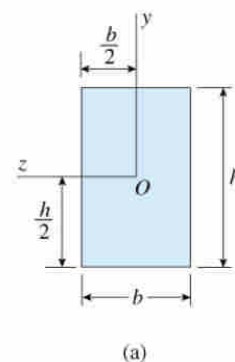
aluminum beams can be extruded in almost any desired shape since the dies are relatively easy to make

wood beam always made in rectangular cross section, such as 4" x 8", but its actual size is 3.5" x 7.25" after it is surfaced

consider a rectangular of width  $b$  and depth  $h$

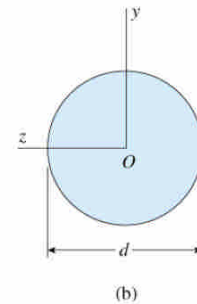
$$S = \frac{b h^2}{6} = \frac{A h}{6} = 0.167 A h$$

a rectangular cross section becomes more efficient as  $h$  increased, but very narrow section may fail because of lateral buckling



for a circular cross section of diameter  $d$

$$S = \frac{\pi d^3}{32} = \frac{A d}{8} = 0.125 A d$$



comparing the circular section to a square section of same area

$$h^2 = \pi d^2 / 4 \Rightarrow h = \sqrt{\pi d} / 2$$

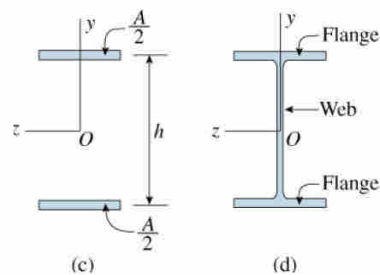
$$\frac{S_{\text{square}}}{S_{\text{circle}}} = \frac{0.167 A h}{0.125 A d} = \frac{0.167 \sqrt{\pi d} / 2}{0.125 d} = \frac{0.148}{0.125} = 1.184$$

$\therefore$  the square section is more efficient than circular section

the most favorable case for a given area  $A$  and depth  $h$  would have to distribute  $A/2$  at a distance  $h/2$  from the neutral axis, then

$$I = \frac{A}{2} \left(\frac{h}{2}\right)^2 \times 2 = \frac{A h^2}{4}$$

$$S = \frac{I}{h/2} = \frac{A h}{2} = 0.5 A h$$



the wide-flange section or an  $I$  - section with most material in the flanges would be the most efficient section

for standard wide-flange beams,  $S$  is approximately

$$S \simeq 0.35 A h$$

wide-flange section is more efficient than rectangular section of the same area and depth,  $\therefore$  much of the material in rectangular beam is located near the neutral axis where it is unstressed, wide-flange section have most of the material located in the flanges, in addition, wide-flange shape is wider and therefore more stable with respect to sideways bucking

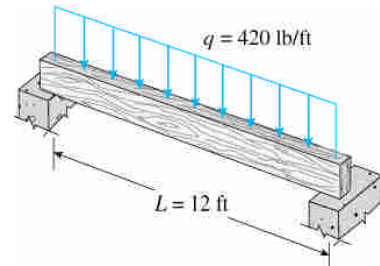
### Example 5-5

a simply supported wood beam carries uniform load

$$L = 12 \text{ ft} \quad q = 420 \text{ lb/ft}$$

$$\sigma_{allow} = 1800 \text{ psi} \quad \text{wood weights } 35 \text{ lb/ft}^3$$

select the size of the beam



(a) calculate the required  $S$

$$M_{max} = \frac{q L^2}{8} = \frac{(420/12) (12 \times 12)^2}{8} = 90,720 \text{ lb-in}$$

$$S = \frac{M_{max}}{\sigma_{allow}} = \frac{90,720}{1,800} = 50.40 \text{ in}^3$$

(b) select a trial size for the beam (with lightest weight)

$$\text{choose } 3 \times 12 \text{ beam, } S = 52.73 \text{ in}^3 \text{ and weight } 6.8 \text{ lb/ft}$$

(c) now the uniform load on the beam is increased to 426.8 lb/ft

$$S_{required} = (50.40) \frac{426.8}{420} = 51.22 \text{ in}^3$$

(d)  $S_{required} < S$  of  $3 \times 12$  beam ( $52.73 \text{ in}^3$ ) (O.K.)

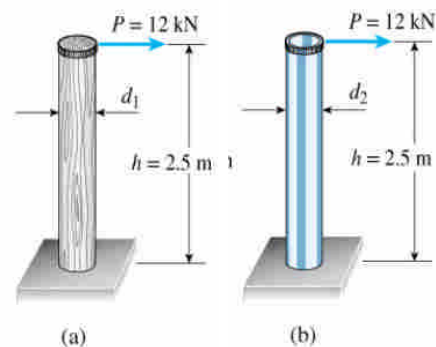
### Example 5-6

a vertical post 2.5 m high support a lateral load  $P = 12 \text{ kN}$  at its upper end

(a)  $\sigma_{allow}$  for wood = 15 MPa

determine the diameter  $d_1$

(b)  $\sigma_{allow}$  for aluminum tube = 50 MPa



determine the outer diameter  $d_2$  if  $t = d_2 / 8$

$$M_{max} = P h = 12 \times 2.5 = 30 \text{ kN-m}$$

(a) wood post

$$S_1 = \frac{\pi d_1^3}{32} = \frac{M_{max}}{\sigma_{allow}} = \frac{30 \times 10^3 \times 10^3}{15} = 2 \times 10^6 \text{ mm}^3$$

$$d_1 = 273 \text{ mm}$$

(b) aluminum tube

$$I_2 = \frac{\pi}{64} [d_2^4 - (d_2 - 2t)^4] = 0.03356 d_2^4$$

$$S_2 = \frac{I_2}{c} = \frac{0.03356 d_2^4}{d_2 / 2} = 0.06712 d_2^3$$

$$S_2 = \frac{M_{max}}{\sigma_{allow}} = \frac{30 \times 10^3 \times 10^3}{50} = 600 \times 10^3 \text{ mm}^3$$

$$\text{solve for } d_2 \Rightarrow d_2 = 208 \text{ mm}$$

### Example 5-7

a simple beam  $AB$  of length 21 ft

$$q = 2000 \text{ lb/ft} \quad \sigma_{allow} = 18,000 \text{ psi}$$

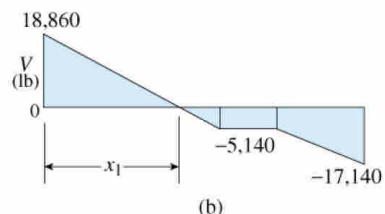
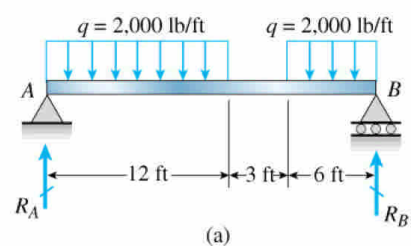
select a wide-flange shape

firstly, determine the support reactions

$$R_A = 18,860 \text{ lb} \quad R_B = 17,140 \text{ lb}$$

the shear force  $V$  for  $0 \leq x \leq 12 \text{ ft}$  is

$$V = R_A - q x$$



for  $V = 0$ , the distance  $x_1$  is

$$x_1 = \frac{R_A}{q} = \frac{18,860}{2,000} = 9.43 \text{ ft}$$

and the maximum moment at the section is

$$M_{max} = 18,860 \times 9.43 / 2 = 88,925 \text{ lb-ft}$$

the required section modulus is

$$S = \frac{M_{max}}{\sigma_{allow}} = \frac{88,925 \times 12}{18,000} = 59.3 \text{ in}^3$$

from table E-1, select the W 12 x 50 section with  $S = 64.7 \text{ in}^3$

the weight of the beam is 50 lb/ft, now recalculate the reactions,  $M_{max}$ ,

and  $S_{required}$ , we have

$$R_A = 19,380 \text{ lb} \quad R_B = 17,670 \text{ lb}$$

$$V = 0 \text{ at } x_1 = 9.455 \text{ ft} \Rightarrow M_{max} = 91,600 \text{ lb-ft}$$

$$S_{required} = \frac{M_{max}}{\sigma_{allow}} = 61.1 \text{ in}^3 < 64.7 \text{ in}^3 \quad (\text{O. K.})$$

### Example 5-8

the vertical posts  $B$  are supported  
planks  $A$  of the dam

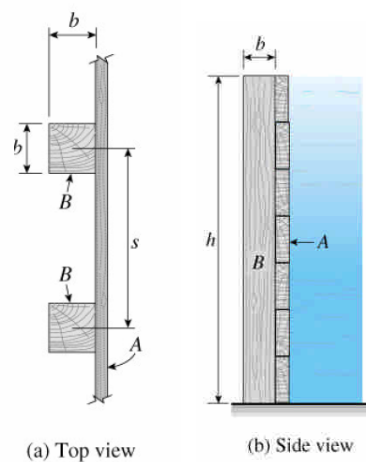
post  $B$  are of square section  $b \times b$

the spacing of the posts  $s = 0.8 \text{ m}$

water level  $h = 2.0 \text{ m}$

$$\sigma_{allow} = 8.0 \text{ MPa}$$

determine  $b$



the post  $B$  is subjected to the water pressure (triangularly distributed)

the maximum intensity  $q_0$  is

$$q_0 = \gamma h s$$

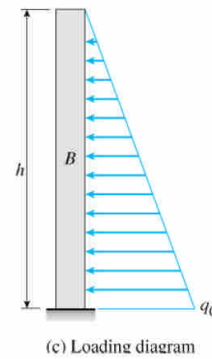
the maximum bending moment occurs at the base is

$$M_{max} = \frac{q_0 h}{2} \left( \frac{h}{3} \right) = \frac{\gamma h^3 s}{6}$$

$$\text{and } S = \frac{M_{max}}{\sigma_{allow}} = \frac{\gamma h^3 s}{6 \sigma_{allow}} = \frac{b^3}{6}$$

$$b^3 = \frac{\gamma h^3 s}{\sigma_{allow}} = \frac{9.81 \times 2^3 \times 0.8}{8 \times 10^6} = 0.007848 \text{ m}^3 = 7.848 \times 10^6 \text{ mm}^3$$

$$b = 199 \text{ mm} \quad \text{use } b = 200 \text{ mm}$$



## 5.7 Nonprismatic Beams

nonprismatic beams are commonly used to reduce weight and improve appearance, such beams are found in automobiles, airplanes, machinery, bridges, building etc.

$\sigma = M / S$ ,  $S$  varying with  $x$ , so we cannot assume that the maximum stress occur at the section with  $M_{max}$

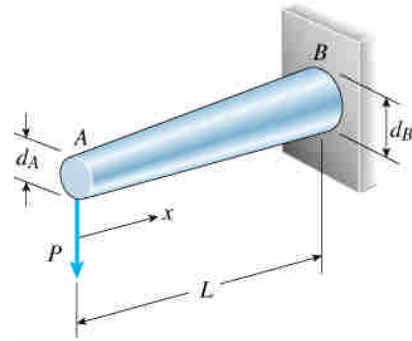
### Example 5-9

a tapered cantilever beam  $AB$  of solid circular cross section supports a load  $P$  at the free end with  $d_B / d_A = 2$

determine  $\sigma_B$  and  $\sigma_{max}$

$$d_x = d_A + (d_B - d_A) \frac{x}{L}$$

$$S_x = \frac{\pi d_x^3}{32} = \frac{\pi}{32} \left[ d_A + (d_B - d_A) \frac{x}{L} \right]^3$$



$\therefore M_x = P x$ , then the maximum bending stress at any cross section is

$$\sigma_1 = \frac{M_x}{S_x} = \frac{32 P x}{\pi [d_A + (d_B - d_A) (x/L)]^3}$$

at support B,  $d_B = 2 d_A$ ,  $x = L$ , then

$$\sigma_B = \frac{4 P L}{\pi d_A^3}$$

to find the maximum stress in the beam, take  $d\sigma_1 / dx = 0$

$$\Rightarrow x = L / 2$$

at that section ( $x = L/2$ ), the maximum is

$$\sigma_{max} = \frac{128 P L}{27 \pi d_A^3} = 4.741 \frac{P L}{\pi d_A^3}$$

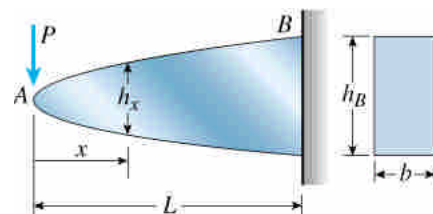
it is 19% greater than the stress at the built-in end

### Example 5-10

a cantilever beam of length  $L$  support a load  $P$  at the free end

cross section is rectangular with constant width  $b$ , the height may vary such that

$\sigma_{max} = \sigma_{allow}$  for every cross section



(fully stressed beam)

determine the height of the beam

$$M = P x \quad S = \frac{b h_x^2}{6}$$

$$\sigma_{allow} = \frac{M}{S} = \frac{P x}{b h_x^2 / 6} = \frac{6 P x}{b h_x^2}$$

solving the height for the beam, we have

$$h_x = \left( \frac{6 P x}{b \sigma_{allow}} \right)^{1/2}$$

at the fixed end ( $x = L$ )

$$h_B = \left( \frac{6 P L}{b \sigma_{allow}} \right)^{1/2}$$

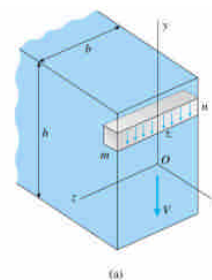
then  $h_x = h_B \left( \frac{x}{L} \right)^{1/2}$  the idealized beam has the parabolic shape

## 5-8 Shear Stress in Beam of Rectangular Cross Section

for a beam subjected to  $M$  and  $V$  with rectangular cross section having width  $b$  and height  $h$ , the shear stress  $\tau$  acts parallel to the shear force  $V$

assume that  $\tau$  is uniform across the width of the beam

consider a beam section subjected the a shear force  $V$ , we isolate a small element  $mn$ , the shear stresses  $\tau$  act vertically and accompanied horizontally as shown



$\therefore$  the top and bottom surfaces are free,  
then the shear stress must be vanish, i.e.  
 $\tau = 0$  at  $y = \pm h/2$

for two equal rectangular beams of height  $h$  subjected to a concentrated load  $P$ , if no friction between the beams, each beam will be in compression above its N.A., the lower longitudinal line of the upper beam will slide w.r.t. the upper line of the lower beam

for a solid beam of height  $2h$ , shear stress must exist along N.A. to prevent sliding, thus single beam of depth  $2h$  will much stiffer and stronger than two separate beams each of depth  $h$

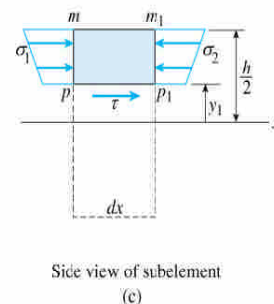
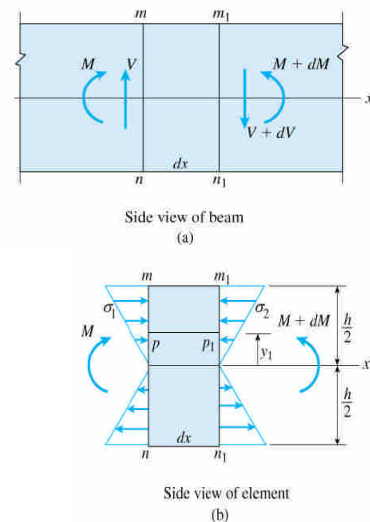
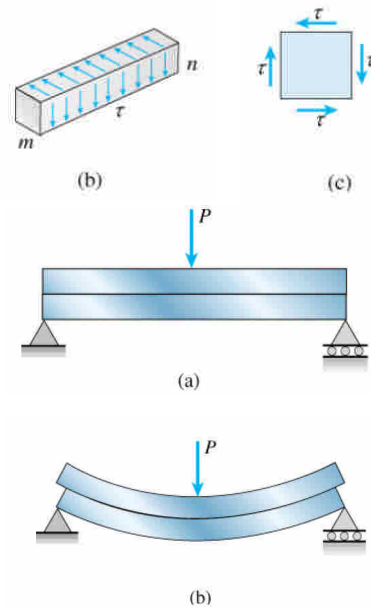
consider a small section of the beam subjected  $M$  and  $V$  in left face and  $M + dM$  and  $V + dV$  in right face

for the element  $mm_1p_1p$ ,  $\tau$  acts on  $p_1p$  and no stress on  $mm_1$

if the beam is subjected to pure bending ( $M = \text{constant}$ ),  $\sigma_x$  acting on  $mp$  and  $m_1p_1$  must be equal, then  $\tau = 0$  on  $pp_1$

for nonuniform bending,  $M$  acts on  $mn$  and  $M + dM$  acts on  $m_1n_1$ , consider  $dA$  at the distance  $y$  from N.A., then on  $mn$

$$\sigma_x dA = \frac{My}{I} dA$$



hence the total horizontal force on  $mp$  is

$$F_1 = \int \frac{M y}{I} dA$$

similarly

$$F_2 = \int \frac{(M + dM) y}{I} dA$$

and the horizontal force on  $pp_1$  is

$$F_3 = \tau b dx$$

equation of equilibrium

$$F_3 = F_2 - F_1$$

$$\tau b dx = \int \frac{(M + dM) y}{I} dA - \int \frac{M y}{I} dA$$

$$\tau = \frac{dM}{dx} \frac{1}{Ib} \int y dA = \frac{V}{Ib} \int y dA$$

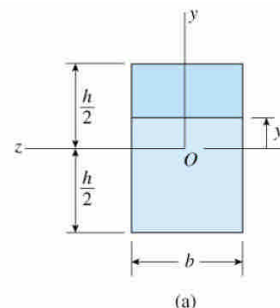
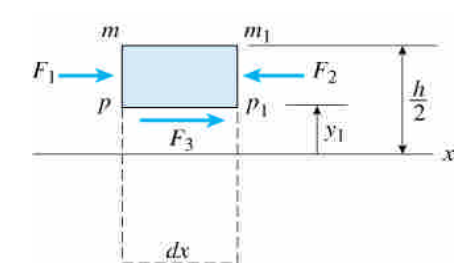
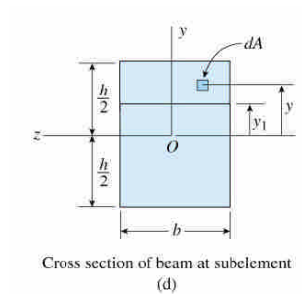
denote  $Q = \int y dA$  is the first moment of the cross section area above the level  $y$  (area  $mm_1p_1p$ ) at which the shear stress  $\tau$  acts, then

$$\tau = \frac{VQ}{Ib} \quad \text{shear stress formula}$$

for  $V, I, b$  are constants,  $\tau \sim Q$

for a rectangular cross section

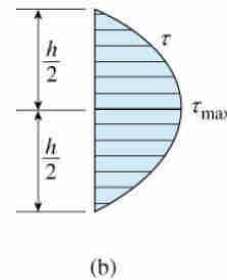
$$Q = b \left( \frac{h}{2} - y_1 \right) \left( y_1 + \frac{h/2 - y_1}{2} \right) = \frac{b}{2} \left( \frac{h^2}{4} - y_1^2 \right)$$



$$\text{then } \tau = \frac{V}{2I} \left( \frac{h^2}{4} - y_1^2 \right)$$

$\tau = 0$  at  $y_1 = \pm h/2$ ,  $\tau_{max}$  occurs

at  $y_1 = 0$  (N.A.)



$$\tau_{max} = \frac{V h^2}{8I} = \frac{V h^2}{8 b h^3/12} = \frac{3V}{2A} = \frac{3}{2} \tau_{ave}$$

$\tau_{max}$  is 50% larger than  $\tau_{ave}$

$\therefore V =$  resultant of shear stress,  $\therefore V$  and  $\tau$  in the same direction

### Limitations

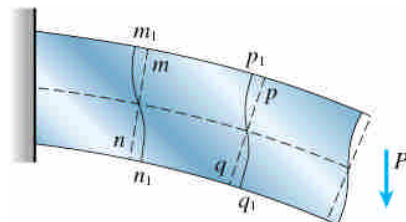
the shear formula are valid only for beams of linear elastic material with small deflection

the shear formula may be consider to be exact for narrow beam ( $\therefore \tau$  is assumed constant across  $b$ ), when  $b = h$ , true  $\tau_{max}$  is about 13% larger than the value given by the shear formula

### Effects of Shear Strains

$\therefore \tau$  vary parabolically from top to bottom, and  $\gamma = \tau / G$  must vary in the same manner

thus the cross sections were plane surfaces become warped, no shear strains occur on the surfaces, and maximum shear strain occurs on N.A.



$\therefore \gamma_{max} = \tau_{max} / G$ , if  $V$  remains constant along the beam, the

warping of all sections is the same, i.e.  $mm_1 = pp_1 = \dots$ , the stretching or shortening of the longitudinal lines produced by the bending moment is unaffected by the shear strain, and the distribution of the normal stress  $\sigma$  is the same as it is in pure bending

for shear force varies continuously along the beam, the warping of cross sections due to shear strains does not substantially affect the longitudinal strains by more experimental investigation

thus, it is quite justifiable to use the flexure formula in the case of nonuniform bending, except the region near the concentrate load acts of irregularly change of the cross section (stress concentration)

#### Example 5-11

a metal beam with span  $L = 3$  ft

$q = 160$  lb/in  $b = 1$  in  $h = 4$  in

determine  $\sigma_C$  and  $\tau_C$  at point  $C$

the shear force  $V_C$  and bending moment  $M_C$  at the section through  $C$  are found

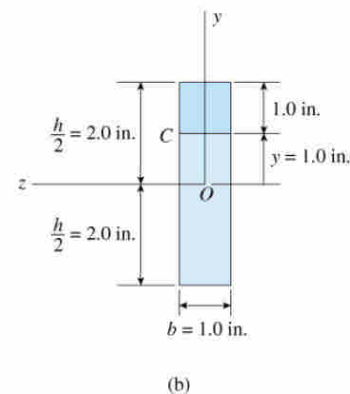
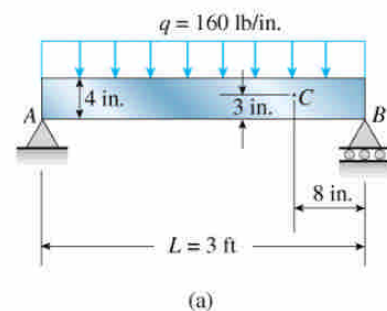
$$M_C = 17,920 \text{ lb-in}$$

$$V_C = -1,600 \text{ lb}$$

the moment of inertia of the section is

$$I = \frac{b h^3}{12} = \frac{1}{12} \times 1 \times 4^3 = 5.333 \text{ in}^4$$

normal stress at  $C$  is



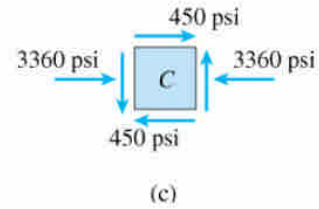
$$\sigma_C = -\frac{M y}{I} = -\frac{17,920 \times 1}{5.333} = -3,360 \text{ psi}$$

shear stress at  $C$ , calculate  $Q_C$  first

$$A_C = 1 \times 1 = 1 \text{ in}^2 \quad y_C = 1.5 \text{ in}$$

$$Q_C = A_C y_C = 1.5 \text{ in}^3$$

$$\tau_C = \frac{V_C Q_C}{I b} = \frac{1,600 \times 1.5}{5.333 \times 1} = 450 \text{ psi}$$



the stress element at point  $C$  is shown

### Example 5-12

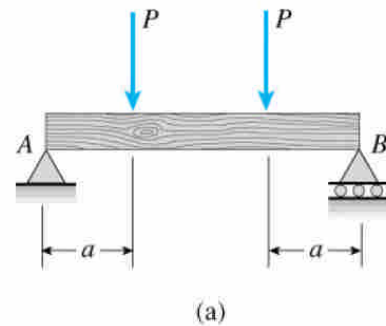
a wood beam  $AB$  supporting two concentrated loads  $P$

$$b = 100 \text{ mm} \quad h = 150 \text{ mm}$$

$$a = 0.5 \text{ m} \quad \sigma_{allow} = 11 \text{ MPa}$$

$$\tau_{allow} = 1.2 \text{ MPa}$$

determine  $P_{max}$

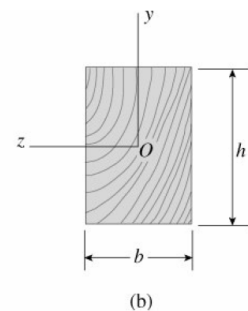


the maximum shear force and bending moment are

$$V_{max} = P \quad M_{max} = P a$$

the section modulus and area are

$$S = \frac{b h^2}{6} \quad A = b h$$



maximum normal and shear stresses are

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{6 P a}{b h^2} \quad \tau_{max} = \frac{3 V_{max}}{2 A} = \frac{3 P}{2 b h}$$

$$P_{bending} = \frac{\sigma_{allow} b h^2}{6 a} = \frac{11 \times 100 \times 150^2}{6 \times 500} = 8,250 \text{ N} = 8.25 \text{ kN}$$

$$P_{shear} = \frac{2 \tau_{allow} b h}{3} = \frac{2 \times 1.2 \times 100 \times 150}{3} = 12,000 \text{ N} = 12 \text{ kN}$$

$$\therefore P_{max} = 8.25 \text{ kN}$$

## 8-9 Shear Stresses in Beam of Circular Cross Section

$$\tau = \frac{V Q}{I b} \quad I = \frac{\pi r^4}{4} \quad \text{for solid section}$$

the shear stress at the neutral axis

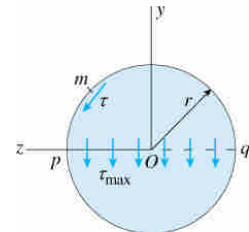
$$Q = A y = \left( \frac{\pi r^2}{2} \right) \left( \frac{4 r}{3 \pi} \right) = \frac{2 r^3}{3} \quad b = 2 r$$

$$\tau_{max} = \frac{V (2 r^3 / 3)}{(\pi r^4 / 4) (2 r)} = \frac{4 V}{3 \pi r^2} = \frac{4 V}{3 A} = \frac{4}{3} \tau_{ave}$$

for a hollow circular cross section

$$I = \frac{\pi}{4} (r_2^4 - r_1^4) \quad Q = \frac{2}{3} (r_2^3 - r_1^3)$$

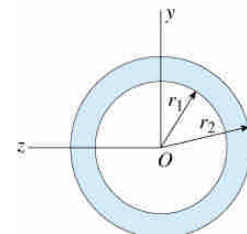
$$b = 2 (r_2 - r_1)$$



then the maximum shear stress at N.A. is

$$\tau_{max} = \frac{V Q}{I b} = \frac{4 V}{3 A} \left( \frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^2 + r_1^2} \right)$$

where  $A = \pi (r_2^2 - r_1^2)$



### Example 5-13

a vertical pole of a circular tube

$$d_2 = 4.0 \text{ in} \quad d_1 = 3.2 \text{ in} \quad P = 1,500 \text{ lb}$$

(a) determine the  $\tau_{\max}$  in the pole

(b) for same  $P$  and same  $\tau_{\max}$ , calculate  $d_0$  of a solid circular pole

(a) The maximum shear stress of a circular tube is

$$\tau_{\max} = \frac{4P}{3\pi} \left( \frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^4 - r_1^4} \right)$$

$$\text{for } P = 1,500 \text{ lb} \quad r_2 = 2 \text{ in} \quad r_1 = 1.6 \text{ in}$$

$$\tau_{\max} = 658 \text{ psi}$$

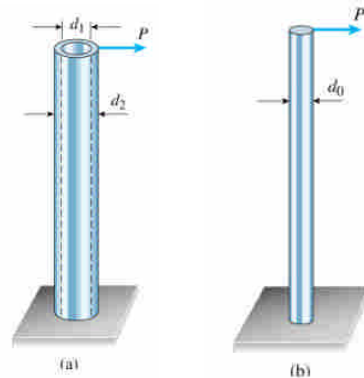
(b) for a solid circular pole,  $\tau_{\max}$  is

$$\tau_{\max} = \frac{4P}{3\pi (d_0/2)^2}$$

$$d_0^2 = \frac{16P}{3\pi \tau_{\max}} = \frac{16 \times 1,500}{3\pi \times 658} = 3.87 \text{ in}^2$$

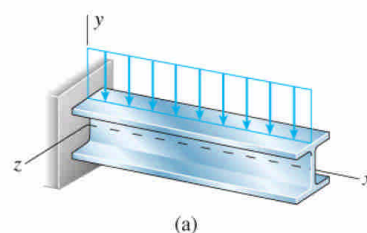
$$\text{then } d_0 = 1.97 \text{ in}$$

the solid circular pole has a diameter approximately one-half that of the tubular pole



### 5-10 Shear Stress in the Webs of Beams with Flanges

for a beam of wide-flange shape subjected to shear force  $V$ , shear stress

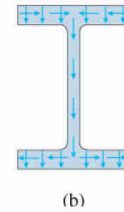


is much more complicated

most of the shear force is carried by shear stresses in the web

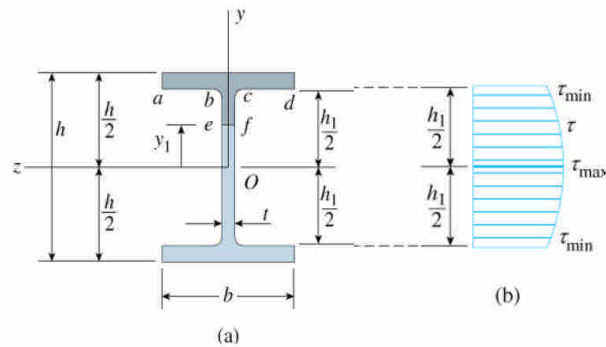
consider the shear stress at  $ef$ , the same assumption as in the case in rectangular beam, i.e.

$\tau \parallel y$  axis and uniformly distributed across  $t$



$$\tau = \frac{VQ}{Ib} \text{ is still valid with } b = t$$

the first moment  $Q$  of the shaded area is divided into two parts, i.e. the upper flange and the area between  $bc$  and  $ef$  in the web



$$A_1 = b \left( \frac{h}{2} - \frac{h_1}{2} \right) \quad A_2 = t \left( \frac{h_1}{2} - y_1 \right)$$

then the first moment of  $A_1$  and  $A_2$  w.r.t. N.A. is

$$\begin{aligned} Q &= A_1 \left( \frac{h_1}{2} + \frac{h/2 - h_1/2}{2} \right) + A_2 \left( y_1 + \frac{h_1/2 - y_1}{2} \right) \\ &= \frac{b}{8} (h^2 - h_1^2) + \frac{t}{8} (h_1^2 - 4y_1^2) \end{aligned}$$

$$\tau = \frac{VQ}{Ib} = \frac{V}{8It} \left[ \frac{b}{8} (h^2 - h_1^2) + \frac{t}{8} (h_1^2 - 4y_1^2) \right]$$

$$\text{where } I = \frac{b h^3}{12} - \frac{(b-t) h_1^3}{12} = \frac{1}{12} (b h^3 - b h_1^3 + t h_1^3)$$

maximum shear stress in the web occurs at N.A.,  $y_1 = 0$

$$\tau_{max} = \frac{V}{8 I t} (b h^2 - b h_1^2 + t h_1^2)$$

minimum shear stress occurs where the web meets the flange,  $y_1 = \pm h_1/2$

$$\tau_{min} = \frac{V b}{8 I t} (h^2 - h_1^2)$$

the maximum stress in the web is from 10% to 60% greater than the minimum stress

the shear force carried by the web consists two parts, a rectangle of area  $h_1 \tau_{min}$  and a parabolic segment of area  $\frac{2}{3} h_1 (\tau_{max} - \tau_{min})$

$$\begin{aligned} V_{web} &= h_1 \tau_{min} + \frac{2}{3} h_1 (\tau_{max} - \tau_{min}) \\ &= \frac{t h_1}{3} (2 \tau_{max} + \tau_{min}) \end{aligned}$$

$V_{web} = 90\% \sim 98\%$  of total  $V$

for design work, the approximation to calculate  $\tau_{max}$  is

$$\tau_{max} = \frac{V}{t h_1} \quad \begin{array}{l} \leq \text{ total shear force} \\ \leq \text{ web area} \end{array}$$

for typical wide-flange beam, error is within  $\pm 10\%$

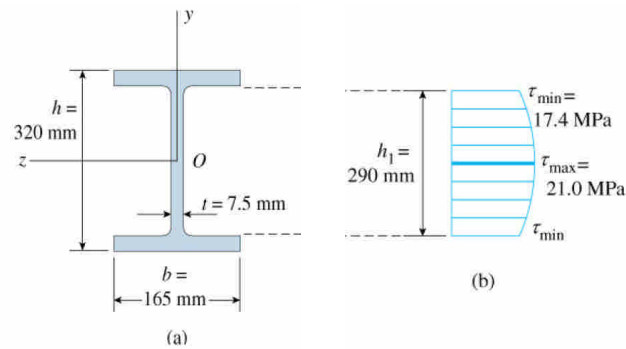
when considering  $y$  in the flange, constant  $\tau$  across  $b$  cannot be made, e.g. at  $y_1 = h_1/2$ ,  $t$  at  $ab$  and  $cd$  must be zero, but on  $bc$ ,  $\tau = \tau_{min}$

actually the stress is very complicated here, the stresses would become very large at the junction if the internal corners were square

### Example 5-14

a beam of wide-flange shape with  $b = 165 \text{ mm}$ ,  $t = 7.5 \text{ mm}$ ,  $h = 320 \text{ mm}$ , and  $h_1 = 290 \text{ mm}$ , vertical shear force  $V = 45 \text{ kN}$

determine  $\tau_{max}$ ,  $\tau_{min}$  and total shear force in the web



the moment of inertia of the cross section is

$$I = \frac{1}{12} (b h^3 - b h_1^3 + t h_1^3) = 130.45 \times 10^6 \text{ mm}^4$$

the maximum and minimum shear stresses are

$$\tau_{max} = \frac{V}{8 I t} (b h^2 - b h_1^2 + t h_1^2) = 21.0 \text{ MPa}$$

$$\tau_{min} = \frac{V b}{8 I t} (h^2 - h_1^2) = 17.4 \text{ MPa}$$

the total shear force is

$$V_{web} = \frac{t h_1}{3} (2 \tau_{max} + \tau_{min}) = 43.0 \text{ kN}$$

and the average shear stress in the web is

$$\tau_{ave} = \frac{V}{t h_1} = 20.7 \text{ MPa}$$

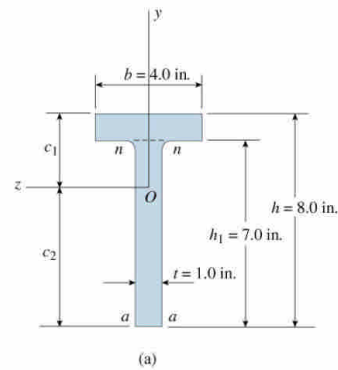
### Example 5-15

a beam having a *T*-shaped cross section

$$b = 4 \text{ in} \quad t = 1 \text{ in} \quad h = 8 \text{ in}$$

$$V = 10,000 \text{ lb}$$

determine  $\tau_{nn}$  (top of the web) and  $\tau_{max}$



$$c_1 = \frac{3 \times 1 \times 0.5 + 8 \times 1 \times 4}{3 \times 1 + 8 \times 1} = 3.045 \text{ in}$$

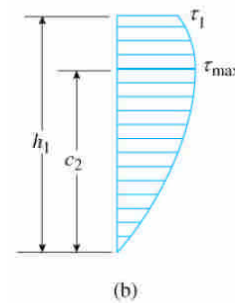
$$c_2 = 8 - c_1 = 4.955 \text{ in}$$

$$I = I_{aa} - A c_2^2$$

$$I_{aa} = \frac{b h^3}{3} - \frac{(b - t) h_1^3}{3} = 339.67 \text{ in}^3$$

$$A c_2^2 = 270.02 \text{ in}^3$$

$$I = 69.65 \text{ in}^3$$



to find the shear stress  $\tau_1$  ( $\tau_{nn}$ ), calculate  $Q_1$  first

$$Q_1 = 4 \times 1 \times (3.045 - 0.5) = 10.18 \text{ in}^3$$

$$\tau_1 = \frac{V Q_1}{I t} = \frac{10,000 \times 10.18}{69.65 \times 1} = 1460 \text{ psi}$$

to find  $\tau_{max}$ , we want to find  $Q_{max}$  at N.A.

$$Q_{max} = t c_2 (c_2/2) = 1 \times 4.955 \times (4.955/2) = 12.28 \text{ in}^3$$

$$\tau_{max} = \frac{V Q_{max}}{I t} = \frac{10,000 \times 12.28}{69.65 \times 1} = 1760 \text{ psi}$$

## 5.11 Built-up Beams and Shear Flow

## 5.12 Beams with Axial Loads

beams may be subjected to the simultaneous action of bending loads and axial forces,

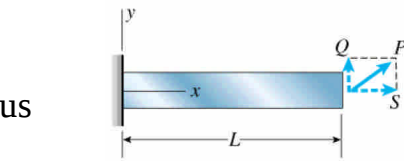
e.g. cantilever beam subjected to an inclined force  $P$ , it may be resolved into two components  $Q$  and  $S$ , then

$$M = Q(L - x) \quad V = -Q \quad N = S$$

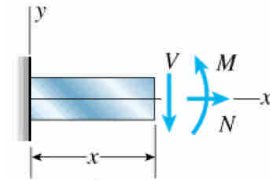
and the stresses in beams are

$$\sigma = -\frac{My}{I} \quad \tau = \frac{VQ}{Ib}$$

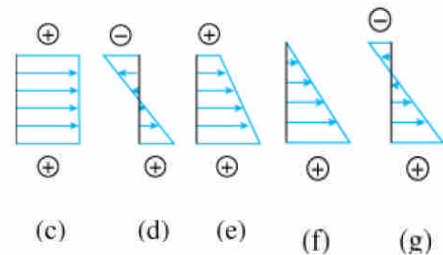
$$\sigma = \frac{N}{A}$$



(a)



(b)



(c)

(d)

(e)

(f)

(g)

the final stress distribution can be obtained by combining the stresses associated with each stress resultant

$$\sigma = -\frac{My}{I} + \frac{N}{A}$$

whenever bending and axial loads act simultaneously, the neutral axis no longer passes through the centroid of the cross section

### Eccentric Axial Loads

a load  $P$  acting at distance  $e$  from the  $x$  axis,  $e$  is called eccentricity

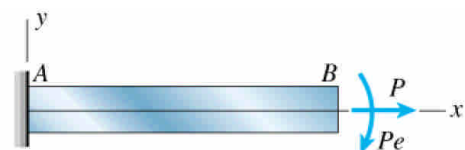
$$N = P \quad M = -Pe$$

then the normal stress at any point is

$$\sigma = \frac{Pe y}{I} + \frac{P}{A}$$



(a)

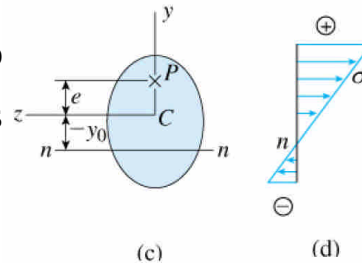


(b)

the position of the N.A.  $nn$  can be obtained by setting  $\sigma = 0$

$$y_0 = -\frac{I}{Ae} \quad \text{minus sign shows the N.A. lies below } z\text{-axis}$$

if  $e$  increased, N.A. moves closer to the centroid, if  $e$  reduced, N.A. moves away from the centroid

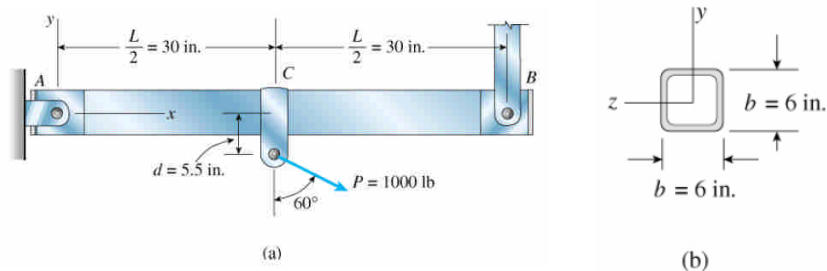


### Example 5-15

a tubular beam  $ACB$  of length  $L = 60$  in loaded by a inclined force  $P$  at mid length

$$P = 1000 \text{ lb} \quad d = 5.5 \text{ in} \quad b = 6 \text{ in} \quad A = 20 \text{ in}^2$$

$$I = 86.67 \text{ in}^4$$

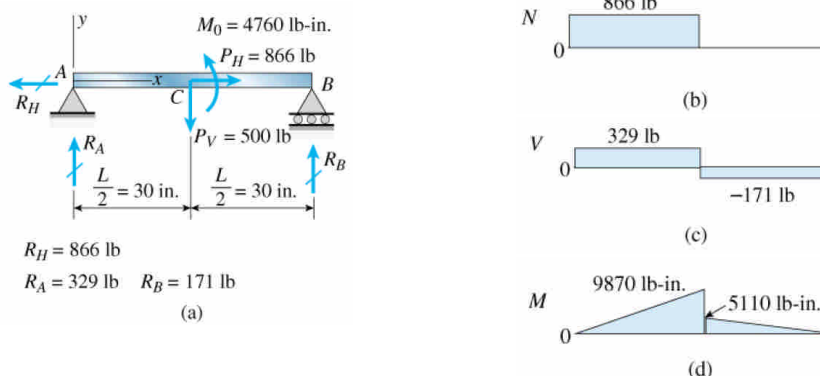


$$P_h = P \sin 60^\circ = 866 \text{ lb}$$

$$P_v = P \cos 60^\circ = 500 \text{ lb}$$

$$M_0 = P_h d = 866 \times 5.5 = 4760 \text{ in-lb}$$

the axial force, shear force and bending moment diagrams are sketched first



the maximum tensile stress occurs at the bottom of the beam,  $y = -3$  in

$$\begin{aligned}(\sigma_t)_{max} &= \frac{N}{A} - \frac{M y}{I} = \frac{866}{20} - \frac{9870 (-3)}{86.67} \\&= 43 + 342 = 385 \text{ psi}\end{aligned}$$

the maximum compressive stress occurs at the top of the beam,  $y = 3$  in

$$\begin{aligned}(\sigma_c)_{left} &= \frac{N}{A} - \frac{M y}{I} = \frac{866}{20} - \frac{9870 \times 3}{86.67} \\&= 43 - 342 = -299 \text{ psi} \\(\sigma_c)_{right} &= \frac{N}{A} - \frac{M y}{I} = \frac{0}{20} - \frac{5110 \times 3}{86.67} \\&= 0 - 177 = -177 \text{ psi}\end{aligned}$$

thus  $(\sigma_c)_{max} = -299$  psi occurs at the top of the beam to the left of point C

### 5.13 Stress Concentration in Beams