

Vector Mechanics for Engineers : Statics

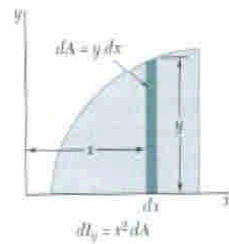
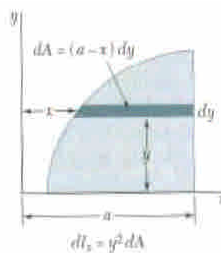
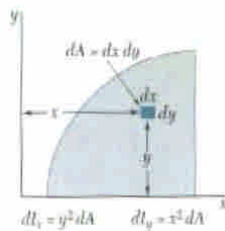
by Beer and Johnston Jr

Chapter 9 Distributed Forces : Moments of Inertia

9.3 Determination of the Moment of Inertia of an Area by Integration

$$I_x = \int y^2 dA \quad I_y = \int x^2 dA$$

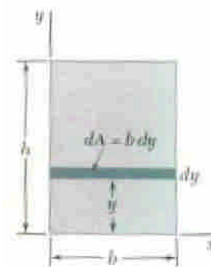
to compute the moments of inertia, three types of element can be chosen, small element $dx dy$, the strip parallel to x -axis and the strip parallel to y -axis



moment of inertia of a rectangular area

$$dI_x = y^2 dA = b y^2 dy$$

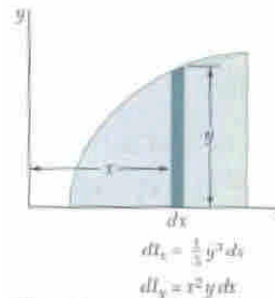
$$I_x = b \int_0^h y^2 dy = \frac{1}{3} b h^3$$



computing I_x and I_y using the same element strips

$$dI_x = \frac{1}{3} y^3 dx$$

$$dI_y = x^2 dA = x^2 y dx$$



9.4 Polar Moment of Inertia

the polar moment of inertia of an area with respect to a pole O is defined

$$J_0 = \int r^2 dA$$

note that $r^2 = x^2 + y^2$, then

$$\begin{aligned} J_0 &= \int r^2 dA = \int (x^2 + y^2) dA \\ &= I_x + I_y \end{aligned}$$

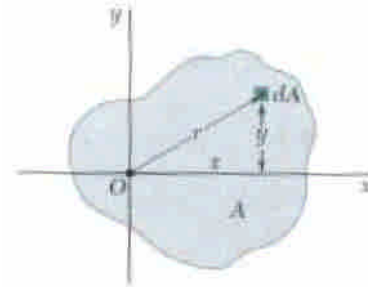


Fig. 9.6

9.5 Radius of Gyration of an Area

consider an area A which has I_x

let us imagine that the area concentrate into a strip parallel to x -axis with a distance k_x from the x -axis, if these two areas have same I_x , then

$$I_x = k_x^2 A$$

$$\text{or } k_x = (I_x / A)^{1/2}$$

the distance k_x is referred to as the radius of gyration of the area with respect to the x -axis, similarly k_y and k_0 can be defined

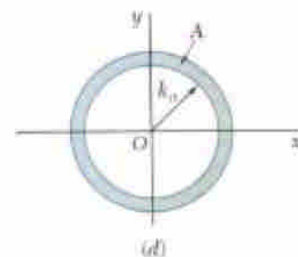
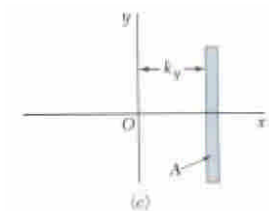
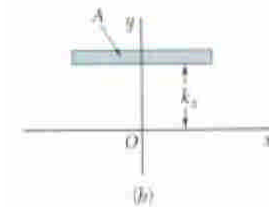
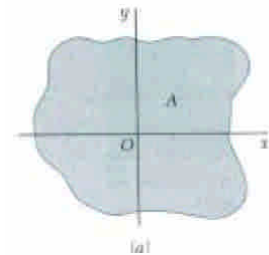
$$I_y = k_y^2 A \quad k_y = (I_y / A)^{1/2}$$

$$J_0 = k_0^2 A \quad k_0 = (J_0 / A)^{1/2}$$

and the relation can be found

$$k_0^2 = k_x^2 + k_y^2$$

for a rectangular area



$$k_x^2 = I_x / A = (1/3 b h^3) / b h = h^2 / 3$$

or $k_x = h / \sqrt{3}$

Sample Problem 9.1

determine I_x of the triangle

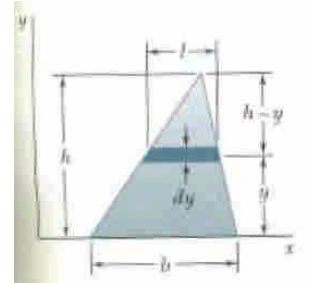
$$dI_x = y^2 dA = y^2 l dy$$

$$l / b = (h - y) / h \quad l = (h - y) b / h$$

$$I_x = \int_0^h y^2 dA = \int_0^h y^2 [(h - y) b / h] dy$$

$$= b/h \int_0^h (y^2 h - y^3) dy = b/h [h y^3 / 3 - y^4 / 4]_0^h$$

$$= b h^3 / 12$$



Sample Problem 9.2

Determine J_0 and I_{diameter}

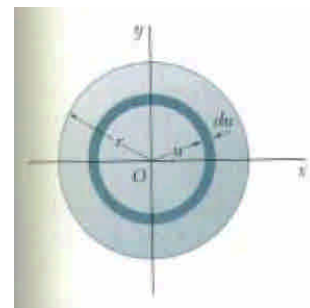
$$dJ_0 = u^2 dA = u^2 2\pi u du$$

$$J_0 = \int dJ_0 = \int_0^r 2\pi u^3 du = 1/2 \pi r^4$$

because of symmetry, $I_x = I_y$

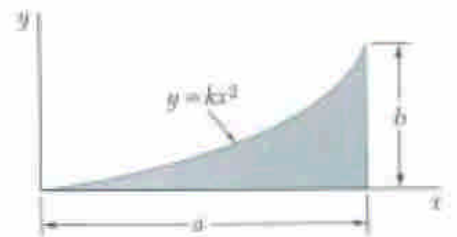
$$J_0 = I_x + I_y = 2 I_x = 2 I_{\text{diameter}}$$

$$I_{\text{diameter}} = J_0 / 2 = \pi r^4 / 4$$



Sample Problem 9.3

determine I_x , I_y , k_x , and k_y



$$y = kx^2$$

$$y = a \text{ at } x = b, \quad k = b/a^2$$

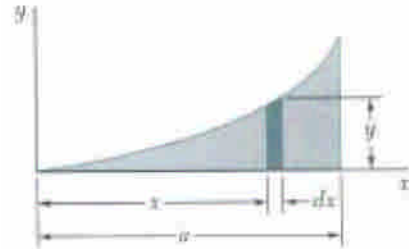
$$y = bx^2/a^2$$

$$A = \frac{1}{3}ab$$

moment of inertia I_x

$$dI_x = \frac{1}{3}y^3 dx = \frac{1}{3}(b^3 x^6 / a^6) dx$$

$$I_x = \int dI_x = b^3 / 3a^6 \int_0^a x^6 dx = ab^3 / 21$$



moment of inertia I_y

$$dI_y = x^2 dA = x^2 y dx = (bx^4 / a^2) dx$$

$$I_y = \int dI_y = b/a^2 \int_0^a x^4 dx = a^3 b / 5$$

radii of gyration k_x and k_y

$$k_x = (I_x / A)^{1/2} = [(ab^3/21) / (ab/3)]^{1/2} = b/\sqrt{7}$$

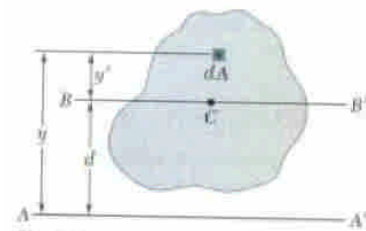
$$k_y = (I_y / A)^{1/2} = [(a^3b/5) / (ab/3)]^{1/2} = a/\sqrt{3/5}$$

9.6 Parallel-Axis Theorem

consider I of the area about AA' is

$$I = \int y^2 dA$$

draw an axis BB' , for which $BB' \parallel AA'$
 BB' through the centroid of the area



let $y = y' + d$

$$I_{AA'} = \int y^2 dA$$

$$= \int (y' + d)^2 dA = \int (y'^2 + 2d y' + d^2) dA$$

$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

$$I_{AA'} = \underline{I} + A d^2$$

$\underline{I}$: the moment of inertia of the area with respect to the centroidal axis BB' , note that $\underline{I}$ is the minimum I of the area with respect to a set of parallel axes

$$I = k^2 A \quad \text{and define} \quad \underline{I} = \underline{k}^2 A$$

$$\text{then} \quad k^2 A = \underline{k}^2 A + A d^2$$

$$\text{thus} \quad k^2 = \underline{k}^2 + d^2$$

similarly for the polar moment of inertia

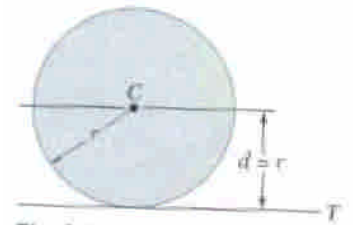
$$J_0 = \underline{J}_C + A d^2 \quad k_0^2 = \underline{k}_0^2 + d^2$$

where $\underline{J}_C$ is the polar moment of inertia about its centroid C , and d is the distance between O and C

example 1 for a circular area

$$I_T = \underline{I} + A d^2$$

$$= \pi r^4 / 4 + \pi r^2 r^2 = 5 \pi r^4 / 4$$



example 2 for a triangular area

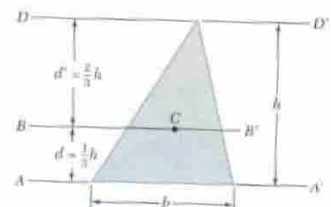
$$I_{AA'} = \underline{I}_{BB'} + A d^2$$

$$\underline{I}_{BB'} = I_{AA'} - A d^2$$

$$= bh^3 / 12 - \frac{1}{2} bh (h/3)^2 = bh^3 / 36$$

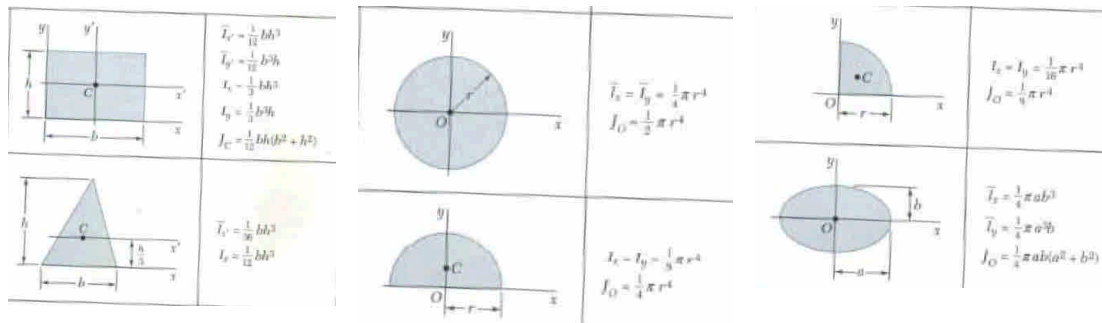
$$I_{DD'} = \underline{I}_{BB'} + A d^2$$

$$= bh^3 / 36 + \frac{1}{2} bh (2h/3)^2 = bh^3 / 4$$



note that $I_{DD'} \neq I_{AA'} + A h^2$

because AA' is not the axis pass through the centroid C



9.7 Moment of Inertia of Composite Areas

consider a composite area A made of several component areas A_1 , A_2 , ... etc.

$$\text{then} \quad (I_x)_A = (I_x)_{A_1} + (I_x)_{A_2} + (I_x)_{A_3}$$

Sample Problem 9.4

determine I , k about axis through C and // the plate
first, we need to determine the location of C

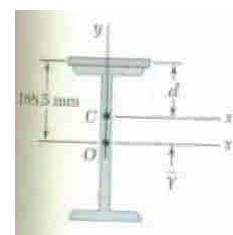
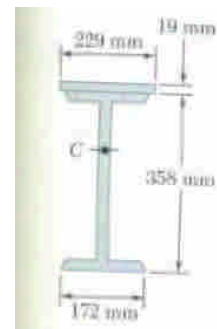
	$A \text{ (mm}^2\text{)}$	$\bar{y} \text{ (mm)}$	$\bar{y} A \text{ (mm}^3\text{)}$
plate	4,351	188.5	0.82×10^6
wide-flange	7,230	0	0
Σ	11,581		0.82×10^6

$$\bar{Y} = \Sigma \bar{y} A / \Sigma A = 70.8 \text{ mm}$$

for the wide-flange section

$$\begin{aligned} I_{x'} &= I_x + A \bar{Y}^2 = 160.2 \times 10^6 + 7,230 \times 70.8^2 \\ &= 196.4 \times 10^6 \text{ mm}^4 \end{aligned}$$

for the plate



$$\begin{aligned}
 I_{x'} &= I_x + A d^2 = 229 \times 19^3 / 12 + 4,351 \times (188.5 - 70.8)^2 \\
 &= 60.4 \times 10^6 \text{ mm}^4
 \end{aligned}$$

for the composite area

$$I_{x'} = 196.4 \times 10^6 + 60.4 \times 10^6 = 256.8 \times 10^6 \text{ mm}^4$$

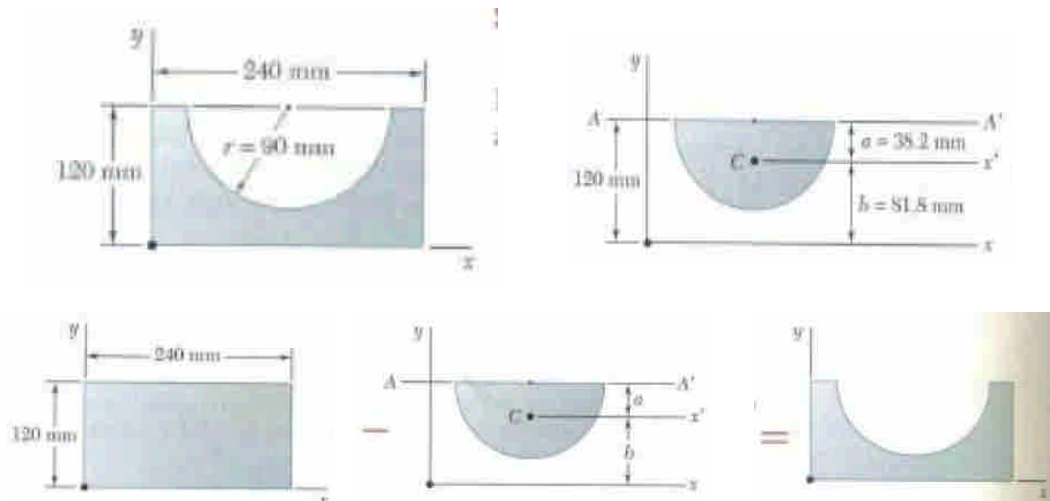
radius of gyration

$$k_{x'}^2 = I_{x'} / A = 256.8 \times 10^6 / 11,581$$

$$k_{x'} = 149 \text{ mm}$$

Sample Problem 8.5

determine I_x of the area



moment of inertia of rectangle

$$I_x = bh^3 / 3 = 240 \times 120^3 / 3 = 138.2 \times 10^6 \text{ mm}^4$$

moment of inertia of the half circle

$$a = 4r / 3\pi = 4 \times 90 / 3\pi = 38.2 \text{ mm}$$

$$b = 120 - a = 81.8 \text{ mm}$$

$$I_{AA'} = \pi r^4 / 8 = \pi 90^4 / 8 = 25.76 \times 10^6 \text{ mm}^4$$

$$A = \pi r^2 / 2 = \pi 90^2 / 2 = 12.72 \times 10^3 \text{ mm}^2$$

$$\begin{aligned} \underline{I}_{x'} &= I_{AA'} - A a^2 = 25.76 \times 10^6 - 12.72 \times 10^3 \times 38.2^2 \\ &= 7.20 \times 10^6 \text{ mm}^4 \end{aligned}$$

$$\begin{aligned} I_x &= \underline{I}_{x'} + A b^2 = 7.20 \times 10^6 + 12.72 \times 10^3 \times 81.8^2 \\ &= 92.3 \times 10^6 \text{ mm}^4 \end{aligned}$$

moment of inertia of given area

$$I_x = 138.2 \times 10^6 - 92.3 \times 10^6 = 45.9 \times 10^6 \text{ mm}^4$$