

Chapter 10 Statically Indeterminate Beams

10.1 Introduction

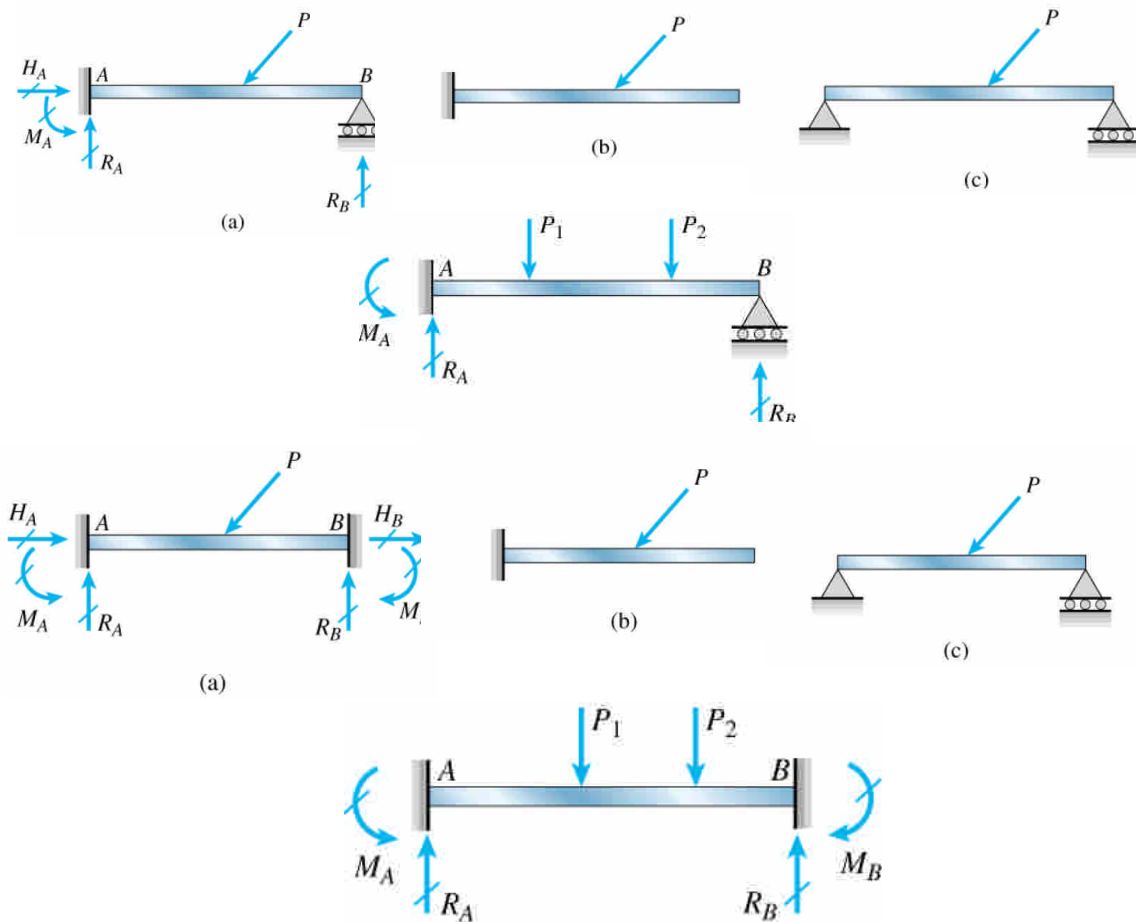
in this chapter we will analyze the beam in which the number of reactions exceed the number of independent equations of equilibrium

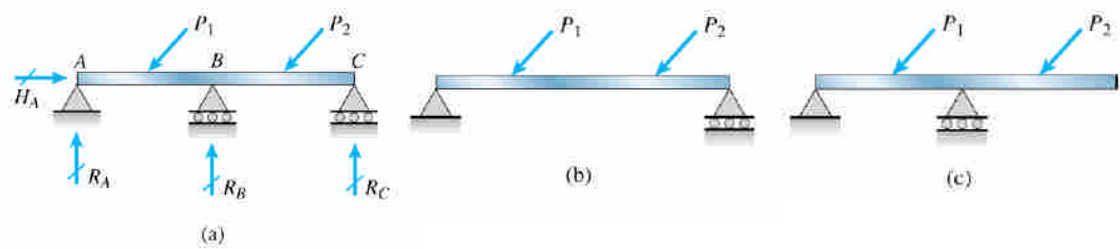
integration of the differential equation, method of superposition

compatibility equation (consistence of deformation)

10.2 Types of Statically Indeterminate Beams

the number of reactions in excess of the number of equilibrium equations is called the degree of static indeterminacy





the excess reactions are called static redundants

the structure that remains when the redundants are released is called released structure or the primary structure

10.3 Analysis by the Differential Equations of the Deflection Curve

$$EIy'' = M \quad EIy''' = V \quad EIy^{iv} = -q$$

the procedure is essentially the same as that for a statically determinate beam and consists of writing the differential equation, integrating to obtain its general solution, and then applying boundary and other conditions to evaluate the unknown quantities, the unknowns consist of the redundant reactions as well as the constants of integration

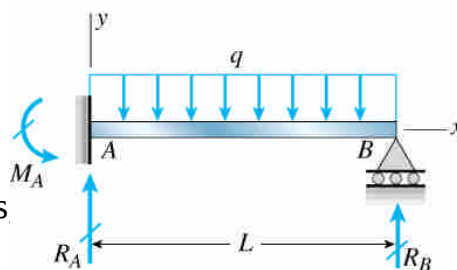
this method has the computational difficulties that arise when a large number of constants to be evaluated, it is practical only for relatively simple cases

Example 10-1

a propped cantilever beam AB supports a uniform load q

determine the reactions, shear forces, bending moments, slopes, and deflections

choose R_B as the redundant, then



$$R_A = qL - R_B \quad M_A = \frac{qL^2}{2} - R_B L$$

and the bending moment of the beam is

$$\begin{aligned} M &= R_A x - M_A - \frac{qx^2}{2} \\ &= qLx - R_B x - \frac{qL^2}{2} - R_B L - \frac{qx^2}{2} \end{aligned}$$

$$EIy'' = M = qLx - R_B x - \frac{qL^2}{2} - R_B L - \frac{qx^2}{2}$$

$$EIy' = \frac{qLx^2}{2} - \frac{R_B x^2}{2} - \frac{qL^2 x}{2} - R_B Lx - \frac{qx^3}{6} + C_1$$

$$EIy = \frac{qLx^3}{6} - \frac{R_B x^3}{6} - \frac{qL^2 x^2}{4} - \frac{R_B Lx^2}{2} - \frac{qx^4}{24} + C_1 x + C_2$$

boundary conditions

$$y(0) = 0 \quad y'(0) = 0 \quad y(L) = 0$$

it is obtained

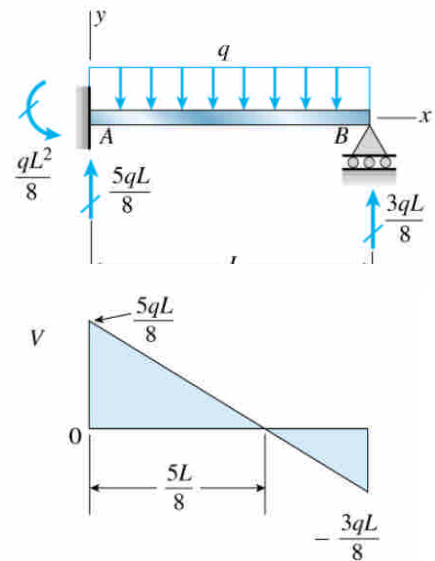
$$C_1 = C_2 = 0 \quad R_B = 3qL/8$$

and $R_A = 5qL/8$

$$M_A = qL^2/8$$

the shear force and bending moment are

$$V = R_A - qx = \frac{5qL}{8} - qx$$



$$\begin{aligned}
 M &= R_{Ax} - M_A - \frac{qx^2}{2} \\
 &= \frac{5qLx}{8} - \frac{qL^2}{8} - \frac{qx^2}{2}
 \end{aligned}$$

the maximum shear force is

$$V_{max} = 5qL/8 \quad \text{at the fixed end}$$

the maximum positive and negative moments are

$$M_{pos} = 9qL^2/128 \quad M_{neg} = -qL^2/8$$

slope and deflection of the beam

$$y' = \frac{qx}{48EI} (-6L^2 + 15Lx - 8x^2)$$

$$y = -\frac{qx^2}{48EI} (3L^2 - 5Lx + 2x^2)$$

to determine the δ_{max} , set $v' = 0$

$$-6L^2 + 15Lx - 8x^2 = 0$$

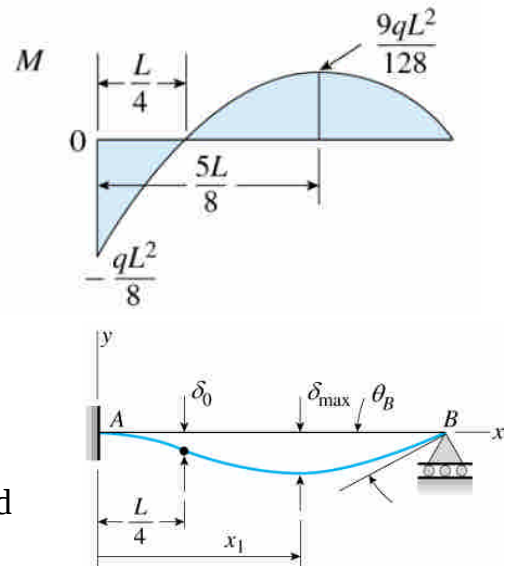
we have $x_1 = 0.5785L$

$$\delta_{max} = -(y)_{x=x_1} = 0.005416 \frac{qL^4}{EI}$$

the point of inflection is located at $M = 0$, i.e. $x = L/4$

$$\kappa < 0 \quad \text{and} \quad M < 0 \quad \text{for} \quad x < L/4$$

$$\kappa > 0 \quad \text{and} \quad M > 0 \quad \text{for} \quad x > L/4$$



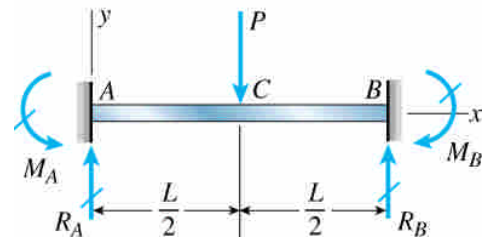
the slope at B is

$$\theta_B = (y')_{x=L} = \frac{qL^3}{48EI}$$

Example 10-2

a fixed-end beam ABC supports a concentrated load P at the midpoint

determine the reactions, shear forces, bending moments, slopes, and deflections



because the load P is in vertical direction and symmetric

$$H_A = H_B = 0 \quad R_A = R_B = P/2$$

$$M_A = M_B \quad (1 \text{ degree of indeterminacy})$$

$$M = \frac{Px}{2} - M_A \quad (0 \leq x \leq L/2)$$

$$EIy'' = M = \frac{Px}{2} - M_A \quad (0 \leq x \leq L/2)$$

after integration, it is obtained

$$EIy' = \frac{Px^2}{4} - M_A x + C_1 \quad (0 \leq x \leq L/2)$$

$$EIy = \frac{Px^3}{12} - \frac{M_A x^2}{2} + C_1 x + C_2 \quad (0 \leq x \leq L/2)$$

boundary conditions

$$y(0) = 0 \quad y'(0) = 0$$

symmetric condition

$$y'(0) = 0$$

the constants C_1 , C_2 and the moment M_A are obtained

$$C_1 = C_2 = 0$$

$$M_A = \frac{PL}{8} = M_B$$

the shear force and bending moment diagrams can be plotted

thus the slope and deflection equations are

$$y' = -\frac{Px}{8EI}(L - 2x) \quad (0 \leq x \leq L/2)$$

$$y = -\frac{Px^2}{48EI}(3L - 4x) \quad (0 \leq x \leq L/2)$$

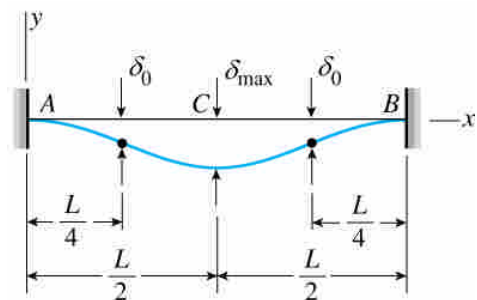
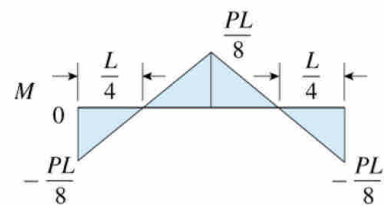
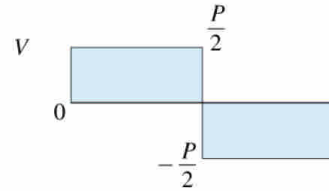
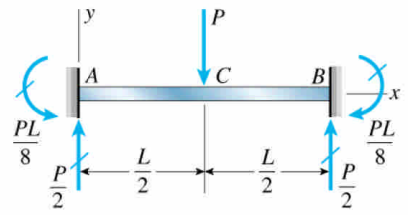
the maximum deflection occurs at the center

$$\delta_{max} = -y(L/2) = \frac{PL^3}{192EI}$$

the point of inflection occurs at the point where $M = 0$, i.e. $x = L/4$, the deflection at this point is

$$\delta = -y(L/4) = \frac{PL^3}{384EI}$$

which is equal $\delta_{max}/2$



10.4 Method of Superposition

1. selecting the reaction redundants
2. establish the force-displacement relations
3. consistence of deformation (compatibility equation)

consider a propped cantilever beam

(i) select R_B as the redundant, then

$$R_A = qL - R_B \quad M_A = \frac{qL^2}{2} - R_B L$$

force-displacement relation

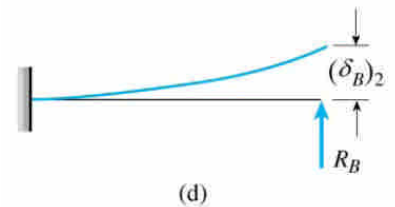
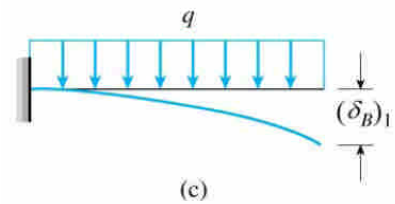
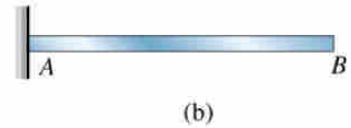
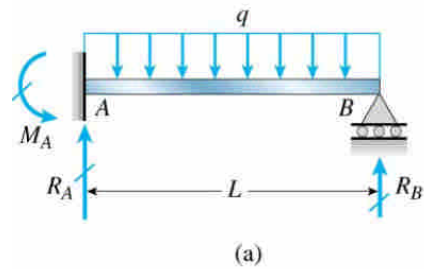
$$(\delta_B)_1 = \frac{qL^4}{8EI} \quad (\delta_B)_2 = \frac{R_B L^3}{3EI}$$

compatibility equation

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$$

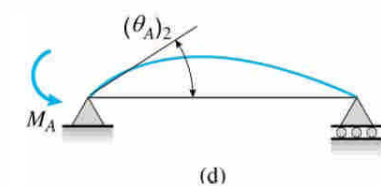
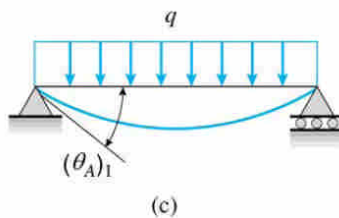
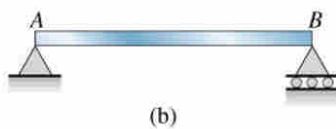
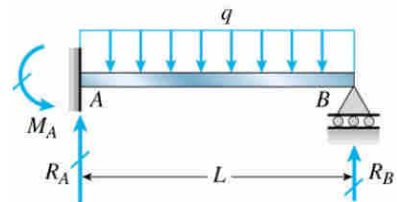
$$\frac{qL^4}{8EI} = \frac{R_B L^3}{3EI}$$

$$R_B = \frac{3qL}{8} \Rightarrow R_A = \frac{5qL}{8} \quad M_A = \frac{qL^2}{8}$$



(ii) select the moment M_A as the redundant

$$R_A = \frac{qL}{2} + \frac{M_A}{L} \quad R_B = \frac{qL}{2} - \frac{M_A}{L}$$



force-displacement relation

$$(\theta_A)_1 = \frac{qL^3}{24EI} \quad (\theta_A)_2 = \frac{M_AL}{3EI}$$

compatibility equation

$$\theta_A = (\theta_A)_1 - (\theta_A)_2 = \frac{qL^3}{24EI} - \frac{M_AL}{3EI} = 0$$

thus $M_A = qL^2/8$

and $R_A = 5qL/8 \quad R_B = 3qL/8$

Example 10-3

a continuous beam ABC supports a uniform load q

determine the reactions

select R_B as the redundant, then

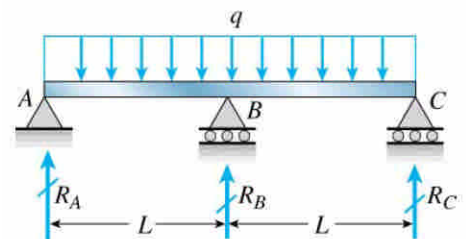
$$R_A = R_C = qL - \frac{qL}{2}$$

force-displacement relation

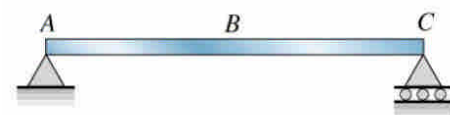
$$(\delta_B)_1 = \frac{5qL(2L)^4}{384EI} = \frac{5qL^4}{24EI}$$

$$(\delta_B)_2 = \frac{R_B(2L)^3}{48EI} = \frac{R_B L^3}{6EI}$$

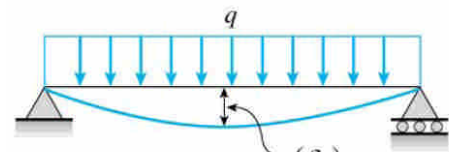
compatibility equation



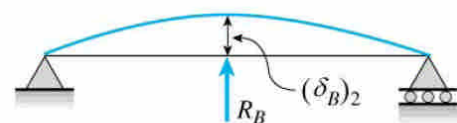
(a)



(b)



(c)



(d)

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = \frac{5qL^4}{24EI} - \frac{R_B L^3}{6EI} = 0$$

thus $R_B = 5qL/4$

and $R_A = R_C = 3qL/8$

Example 10-4

a fixed-end beam AB is loaded by a force P acting at point D

determine reactions at the ends

also determine δ_D

this is a 2-degree of indeterminacy problem

select M_A and M_B as the redundants

$$R_A = \frac{Pb}{L} + \frac{M_A}{L} - \frac{M_B}{L}$$

$$R_B = \frac{Pa}{L} - \frac{M_A}{L} + \frac{M_B}{L}$$

force-displacement relations

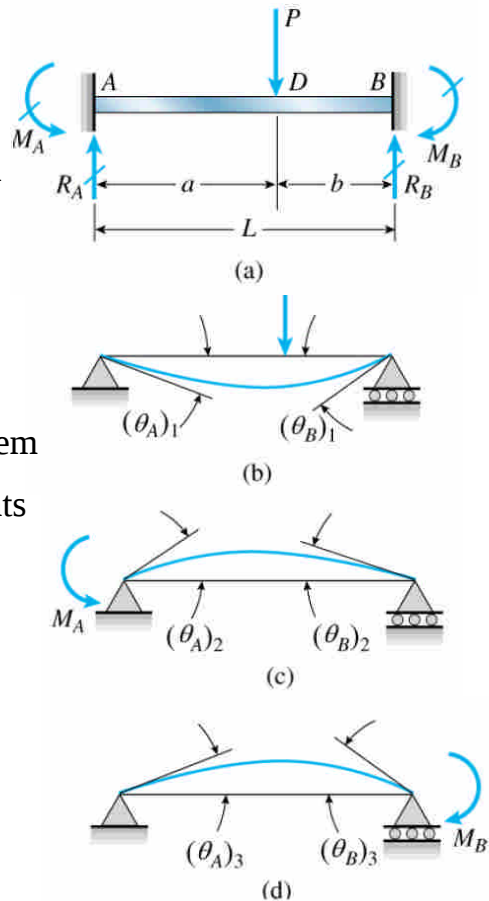
$$(\theta_A)_1 = \frac{Pab(L+b)}{6LEI}$$

$$(\theta_B)_1 = \frac{Pab(L+a)}{6LEI}$$

$$(\theta_A)_2 = \frac{M_A L}{3EI} \quad (\theta_B)_2 = \frac{M_A L}{6EI}$$

$$(\theta_A)_3 = \frac{M_B L}{6EI} \quad (\theta_B)_3 = \frac{M_B L}{3EI}$$

compatibility equations



$$\theta_A = (\theta_A)_1 - (\theta_A)_2 - (\theta_A)_3 = 0$$

$$\theta_B = (\theta_B)_1 - (\theta_B)_2 - (\theta_B)_3 = 0$$

$$\text{i.e. } \frac{M_A L}{3EI} + \frac{M_B L}{6EI} = \frac{Pab(L+b)}{6LEI}$$

$$\frac{M_A L}{6EI} + \frac{M_B L}{3EI} = \frac{Pab(L+a)}{6LEI}$$

solving these equations, we obtain

$$M_A = \frac{Pab^2}{L^2} \quad M_B = \frac{Pa^2b}{L^2}$$

and the reactions are

$$R_A = \frac{Pb^2}{L^3} (L+2a) \quad R_B = \frac{Pa^2}{L^3} (L+2b)$$

the deflection δ_D can be expressed as

$$\delta_D = (\delta_D)_1 - (\delta_D)_2 - (\delta_D)_3$$

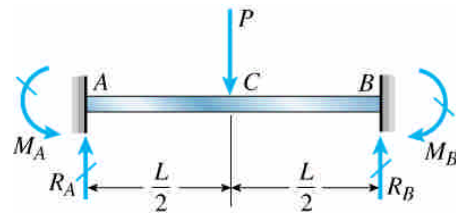
$$(\delta_D)_1 = \frac{Pa^2b^2}{3LEI}$$

$$(\delta_D)_2 = \frac{M_A ab}{6LEI} (L+b) = \frac{Pa^2b^3}{6L^3EI} (L+b)$$

$$(\delta_D)_3 = \frac{M_B ab}{6LEI} (L+a) = \frac{Pa^3b^2}{6L^3EI} (L+a)$$

$$\text{thus } \delta_D = \frac{Pa^3b^3}{3L^3EI}$$

$$\text{if } a = b = L/2$$



$$\text{then } M_A = M_B = \frac{PL}{8} \quad R_A = R_B = \frac{P}{2}$$

$$\text{and } \delta_C = \frac{PL^3}{192EI}$$

Example 10-5

a fixed-end beam AB supports a uniform load q acting over part of the span

determine the reactions of the beam

to obtain the moments caused by qdx ,
replace P to qdx , a to x , and b
to $L - x$

$$dM_A = \frac{qx(L - x)^2 dx}{L^2}$$

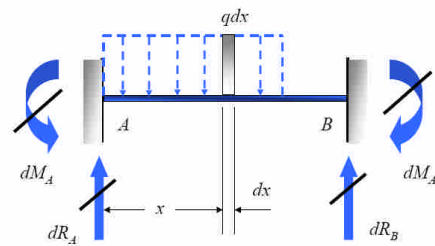
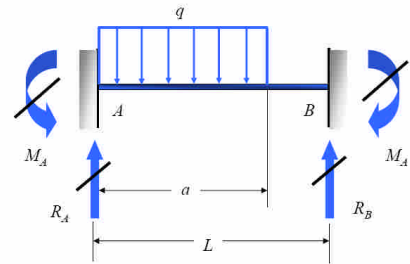
$$dM_B = \frac{qx^2(L - x) dx}{L^2}$$

integrating over the loaded part

$$M_A = \int dM_A = \frac{q}{L^2} \int_0^a x(L - x)^2 dx = \frac{qa^2}{12L^2} (6L^2 - 8aL + 3a^2)$$

$$M_B = \int dM_B = \frac{q}{L^2} \int_0^a x^2(L - x) dx = \frac{qa^3}{12L^2} (4L^2 - 3a)$$

Similarly



$$dR_A = \frac{q(L-x)^2(L+2x)dx}{L^3}$$

$$dR_B = \frac{qx^2(3L-2x)dx}{L^3}$$

integrating over the loaded part

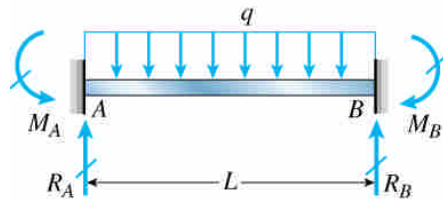
$$R_A = \int dR_A = \frac{q}{L^3} \int_0^a (L-x)^2(L+2x)dx = \frac{qa}{2L^3} (2L^3 - 2a^2L + a^3)$$

$$R_B = \int dR_B = \frac{q}{L^3} \int_0^a x^2(3L-2x)dx = \frac{qa^3}{2L^3} (2L-a)$$

for the uniform acting over the entire length, i.e. $a = L$

$$M_A = M_B = \frac{qL^2}{12}$$

$$R_A = R_B = \frac{qL}{2}$$



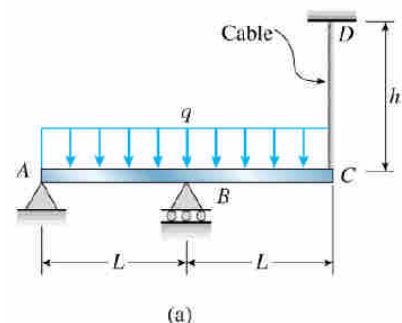
the center point deflections due to uniform load and the end moments are

$$(\delta_C)_1 = \frac{5qL^4}{384EI} \quad (\delta_C)_2 = \frac{M_AL}{8EI} = \frac{(qL^2/12)L^2}{8EI} = \frac{qL^4}{96EI}$$

$$\delta_C = (\delta_C)_1 - (\delta_C)_2 = \frac{qL^4}{384EI}$$

Example 10-6

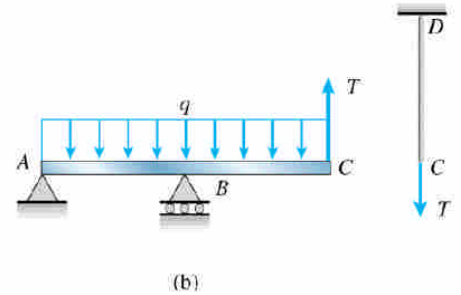
a beam ABC rests on supports A and B and is supported by a cable at C



find the force T of the cable

take the cable force T as redundant

the deflection $(\delta_C)_1$ due the uniform load can be found from example 9.9 with $a = L$



$$(\delta_C)_1 = \frac{qL^4}{4E_b I_b}$$

the deflection $(\delta_C)_2$ due to a force T acting on C is obtained

use conjugate beam method

$$\begin{aligned} (\delta_C)_2 &= M = \frac{TL^2}{3E_b I_b} L + \frac{TL}{E_b I_b} \frac{L}{2} \frac{2L}{3} \\ &= \frac{2TL^3}{3E_b I_b} \end{aligned}$$

the elongation of the cable is

$$(\delta_C)_3 = \frac{Th}{E_c A_c}$$

compatibility equation

$$\begin{aligned} (\delta_C)_1 - (\delta_C)_2 &= (\delta_C)_3 \\ \frac{qL^4}{4E_b I_b} - \frac{2TL^3}{3E_b I_b} &= \frac{Th}{E_c A_c} \\ T &= \frac{3qL^4 E_c A_c}{8L^3 E_c A_c + 12hE_b I_b} \end{aligned}$$

10.5 Temperature Effects

10.6 Longitudinal Displacements at the Ends of the Beams