

國立交通大學九十二學年度碩士班入學考試試題

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科目名稱：線性代數與機率(103,113,123,133) 考試日期：92年4月20日 第2節

系所班別：資訊科學系 組別：甲組 ~ 戊組 第1頁,共3頁

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Linear Algebra

- 1 For each statement below, indicate whether it is true or not. (5%)
 - a) If A is a symmetric matrix then $\text{rank}(A) = \text{rank}(A^2)$
 - b) If Ax and Ay are linearly independent then x and y are linearly independent. (A is a matrix and x and y are vectors.)
 - c) If $AB = 0$ then the null space of A contains the column space of B .
 - d) If A and B are 2×2 matrices such that $ABAB = 0$ then $BABA = 0$
 - e) If $A = BC$ where B is a 5×4 matrix and C is a 4×5 matrix then A is invertible.

- 2 Let F be the space of all functions from $\mathbb{R}$ to $\mathbb{R}$, E be the subspace of F that contains all functions satisfying $f(x) = f(-x)$, and O be the subspace of F that contains all functions satisfying $f(x) = -f(-x)$, show that $F = E + O$. (5%)

For problems 3~5, give your answers directly.

No explanation or computation is necessary.

3 Let $A = \begin{bmatrix} 0 & b & b+c \\ a & a+b & b \\ a & a & 0 \end{bmatrix}$, where $a \neq 0, b \neq 0, c \neq 0$.

Factor the matrix A into $PA = LDL^T$, where P is a permutation matrix, L is a lower triangular matrix, and D is a diagonal matrix. (5%)

4 Suppose the complete solution to the equation $Ax = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ is $x = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Find A . (5%)

- 5 Define the square matrices

$$A_n = \begin{bmatrix} a_1 & 1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & -1 & a_n \end{bmatrix}, \quad n \geq 1$$

and let $d_n = \det A_n$. Find a recurrence relation for d_n . (Don't solve it.) (5%)

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以下各計算題, 只需寫出答案.

6. (1) In $\mathbb{R}^3$, let T be the orthogonal projection onto the subspace S , where

$$S = \text{span} \{ (1, 1, 0), (1, 1, 1) \}.$$

Find the standard matrix representation A for this linear transformation.

(2) As in part (1), find the orthogonal projection of $(1, 0, 0)$ onto the subspace S . (5%)

7. Let $A = \begin{bmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{bmatrix}$ and let T be the linear transformation given by

$$T(x) = Ax. \text{ Find (1) kernel } (T), \text{ (2) nullity } (T), \text{ (3) range } (T), \text{ (4) rank } (T). \text{ (4\%)}$$

8. In $\mathbb{R}^3$, let B_1 and B_2 be two basis, where

$$B_1 = \{ (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) \} \text{ and}$$

$$B_2 = \{ (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) \}$$

Find the transition matrix from B_1 to B_2 . (2%)

9. In $\mathbb{R}^3$, find the rotation matrix that rotates a point 30° about the y -axis. (2%)

10. Are the following two matrices similar? If so, find a matrix P such that $B = P^{-1}AP$.

$$(1) \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(2) \quad A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(4%)

11. Find a basis B for the domain of T such that the matrix of T related to B is diagonal.

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad T(x, y) = (x+y, x+y). \text{ (4\%)}$$

12. (1) For an invertible matrix A , prove that A and A^{-1} have the same eigenvectors.

How are the eigenvalues of A^{-1} related to the eigenvalues of A ?

(2) Let A be a diagonalizable matrix with n real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

What is the determinant of A ? Prove your answer briefly. (4%)

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Problem 1 (15%)

A box contains N balls, labeled by numbers $1, 2, \dots, N$, respectively. We draw n balls from the box randomly and without replacement, where $n > 1$. Let Y and Z be the random variables for the minimum and maximum numbers labeled in the drawn balls, respectively.

- What is the probability mass function of Z ?
- What is the joint probability mass function of Y and Z ?
- Let $W = Y + Z$. What is the probability mass function of W ?
- For $N = 8$ and $n = 6$, what is $E(W)$?

Problem 2 (10%)

Let X_1 and X_2 be mutually independent exponential random variables with parameter λ . Find the cumulative distribution function for $Y = X_1 - X_2$.

Problem 3 (5%)

You ask your neighbor to water a sickly plant while you are on vacation. Without water it will die with probability 0.8; with water it will die with probability 0.15. You are 90 percent certain that your neighbor will remember to water the plant.

If the plant is alive, what is the probability your neighbor forgot to water it?

Problem 4 (5%)

The random variable X has probability density function

$$f(x) = \begin{cases} ax + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = 0.6$, find a and b .

Problem 5 (5%)

Suppose that the cumulative distribution function of the random variable X is given by

$$F(x) = 1 - e^{-x^2} \quad x > 0$$

Use the results $E[X^n] = \int_0^\infty nx^{n-1}P\{X > x\} dx$ and $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ to find $\text{Var}(X)$.

Problem 6 (5%)

If n balls are randomly selected from an urn containing N balls of which m are white, find the expected number of white balls selected.

Problem 7 (5%)

It is known that the moment generating function of a normal random variable N with mean μ and variance σ^2 is given by $M_N(t) = \exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\}$. Use this information to show that if X and Y are two independent normal random variables with means μ_X and μ_Y and variances σ_X^2 and σ_Y^2 , respectively, then $X + Y$ is also a normal random variable with mean $\mu_X + \mu_Y$ and variance $\sigma_X^2 + \sigma_Y^2$.