Q.E.D.

Note: $\omega(f, x_0)$ 在衡量 f 在 x_0 有多不連續

Thm. (Lebesgue's Thm)
Let $f: A \subset \mathbb{R}^N \to \mathbb{R}$ be a bounded function on a bounded set A. Extend f to all $\mathbb{R}^N$ by letting it to be zero at points not contained in A. Then f is (Riemann) integrable on $A \iff$ The points at which the extended f which are discontinuous form a set of measure zero.

Remark. Lebesgue opened up new advances in integration theory by relating integrability to measure zero.

pf. It's equivalent to prove the thm for some rectangle R containing A s.t. the extended function $f(x)=0$ on $R \setminus A$.

$\Rightarrow$) 先不要一下子考慮所有 oscillation point, 可依多不連續分成几个 subset, 就其 countable union 來證此 union 為 measure zero

Let $D_{\frac{1}{m}} = \{x \in R \mid \omega(f,x) \geq \frac{1}{m}\}$. Then the set $\{x \in R \mid f \text{ is discontinuous at } x\}$

$= \bigcup_{m=1}^{\infty} D_{\frac{1}{m}} \triangleq D$

To show that $D_{\frac{1}{m}}$ is measure zero $\forall m \in \mathbb{N}$.
Then D is measure zero.

Let $\varepsilon > 0$ $\because f$ is integrable
$\therefore$ By Riemann's condition, $\exists$ a partition P of R st. $U(f,P) - L(f,P) < \varepsilon$

$$\Rightarrow \sum_{S \in P} [M_S(f) - m_S(f)] V(S) < \varepsilon$$

$\because D_{\frac{1}{m}} = \{x \in D_{\frac{1}{m}} \mid x \text{ lies on the boundary of } S \text{ for some } S \in P\} \cup \{x \in D_{\frac{1}{m}} \mid x \text{ lies in the interior of } S \text{ for some } S \in P\}$,
$\therefore$ Let $D_{\frac{1}{m}} = E_1 \cup E_2$, respectively. Then
$E_1 \subset \bigcup_{S \in P} bd(S)$ ↳ 這兩個都得是 measure zero

$\because S$ has volume
$\therefore bd(S)$ has volume zero
$\Rightarrow \bigcup_{S \in P} bd(S)$ is measure zero

$\Rightarrow E_1$ has measure zero

Let $x \in E_2$. Then $\exists S \in P$ s.t. $x \in int(S)$
$\Rightarrow \omega(f,x) \leq M_S(f) - m_S(f)$

$\because \omega(f,x) \geq \frac{1}{m}$
$\therefore \frac{1}{m} V(S) \leq (M_S(f) - m_S(f)) V(S)$

Let C be the collection of rectangles of P that contains an element of $D_{\frac{1}{m}}$
Then C is a finite rectangle cover of E_2

and $\sum_{s\in C} \frac{1}{m} V(S) \leq \sum_{s\in C} (M_S(f) - m_S(f)) V(S)$

$\leq \sum_{s\in P} (M_S(f) - m_S(f) V(S)$

$< \varepsilon$

$\Rightarrow \sum_{s\in C} V(S) \leq m\varepsilon$

$\Rightarrow E_2$ has measure zero

Hence $D_{\frac{1}{m}}$ has measure zero.

$\Leftarrow$) 最方便(不care積分值) $\Rightarrow$ 考虑 Riemann's anditim.

Let $\varepsilon > 0$. Let P be a partition of R

目標：找到 P s.t. $U(f,P) - L(f,P) < \varepsilon$

即 $\sum_{s\in P} [M_S(f) - m_S(f)] V(S) < \varepsilon$

$\geq \omega(f, \cdot)$

要此項要多小有多小，相当於要求S裡头那些矣要多小有多小的 oscillatim

但不是所有矣的oscillatim都要多小有多小，但反正 $M_S(f) - m_S(f)$ 有界，不好的矣改为控制 total volume 要多小有多小

$\Rightarrow$ 控制 oscillatim point 很大的矣，在 $\mathbb{R}$ 上的幅原能否要多小有多小、(指長方形 occupy 之体積)

$\Rightarrow$ 考虑 compact.

$< \varepsilon$ 所成集合：good

$> \varepsilon$ " (D_ε)：土不矣，控制其 volume $\Rightarrow$

D_ε compact?
⇒ 只要證 closed，因 D_ε 在 R 中，必 bounded

Let $D_\varepsilon = \{x \in R \mid \omega(f,x) \geq \varepsilon\}$ and $D = \{x \in R \mid$ the extended function f is not cont. at $x\}$
Then $D_\varepsilon \subset D$.

Claim: D_ε is closed, $\forall \varepsilon > 0$.
pf of claim: Let $y \in cl(D_\varepsilon)$. Let U be a nbd of y. Then $\exists x \in D_\varepsilon$ s.t. $x \in U$.
$\Rightarrow \omega(f,x) \leq \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in U\}$
$\because U$ is arbitrary nbd of y
$\therefore \omega(f,y) \geq \omega(f,x) \geq \varepsilon$
$\Rightarrow y \in D_\varepsilon$
$\Rightarrow D_\varepsilon$ is closed.

$\because D_\varepsilon$ is bounded in $\mathbb{R}^n$
$\therefore$ By Heine-Borel Thm, D_ε is compact.

$\because D$ has measure zero and $D_\varepsilon \subset D$
$\therefore D_\varepsilon$ has measure zero
For $\frac{\varepsilon}{2}$, $\exists$ a countable rectangles cover $\{R_1, R_2, \dots\}$ of D_ε s.t. $\sum_{i=1}^{\infty} v(R_i) < \frac{\varepsilon}{2}$

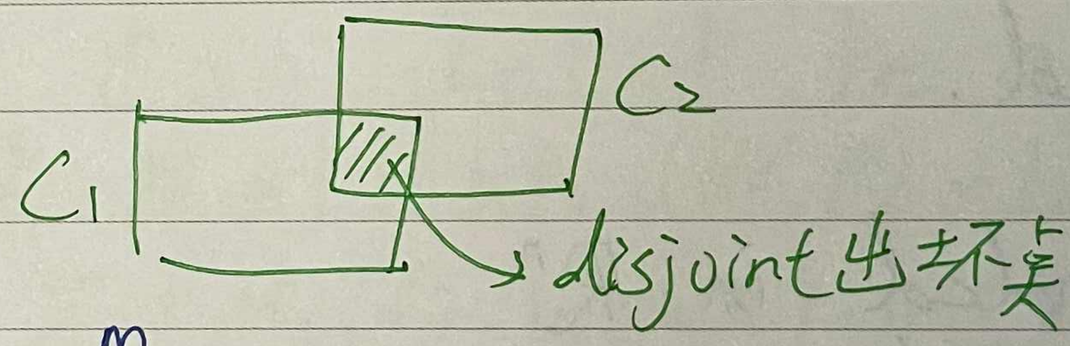
$\Rightarrow \exists \{\tilde{R}_1, \tilde{R}_2, \dots\}$ s.t. $\tilde{R}_i$ is a rectangle in $\mathbb{R}^n$ s.t. $R_i \subset \text{int}(\tilde{R}_i)$ and $\sum_{i=1}^{\infty} v(\tilde{R}_i) < \varepsilon$. $\forall i \in \mathbb{N}$.
↳ $= \text{int}(R_1)$
Then $\{\mathring{R}_1, \mathring{R}_2, \dots\}$ is an open cover of D_ε.
$\because D_\varepsilon$ is compact

$\therefore$ $\exists$ a finite subcover $\{\mathring{\tilde{R}}_1, \dots, \mathring{\tilde{R}}_m\}$ of $\{\mathring{\tilde{R}}_1, \mathring{\tilde{R}}_2, \dots\}$. Let's still denote $\tilde{R}_i$ by R_i.

Now pick up a partition P of R. By refining the partition, we may assume that each rectangle of P is either disjoint from D_ε or contained in one of the rectangles $R_1, \dots, R_m$.

Let $C_1 = \{S \in P \mid S \text{ is contained in one of } R_1, \dots, R_m\}$ and $C_2 = \{S \in P \mid S \cap D_\varepsilon = \emptyset\}$.

$\because \{R_1, \dots, R_m\}$ covers D_ε



$\therefore \bigcup_{i=1}^{m} R_i \supset D_\varepsilon$

$\Rightarrow C_1 \cap C_2$ are not necessary disjoint

$$\Rightarrow U(f,P) - L(f,P) \le \sum_{S \in C_1} [M_S(f) - m_S(f)] v(S) + \sum_{S \in C_2} [M_S(f) - m_S(f)] v(S)$$

If $S \in C_2$, then $\forall x \in S$, $\omega(f,x) < \varepsilon$.
$\Rightarrow \exists$ a nbd U_x of x s.t. $M_{U_x}(f) - m_{U_x}(f) < \varepsilon$
$\because \{U_x \mid x \in S\}$ is an open cover of S and S is compact

$\therefore \exists$ a finite subcover $\{U_{x_1}, \dots, U_{x_k}\}$ of $\{U_x \mid x \in S\}$

Pick a refinement of P in S s.t. each rectangle of P in S is contained in U_{x_i} for some $i \in \{1, \dots, k\}$. We do this for each S in C_2, we get a new refined partition P of R s.t.

$$\sum_{S \in C_2} [M_S(f) - m_S(f)] V(S) \leq \sum_{S \in C_2} \varepsilon V(S)$$

$$\leq \sum_{S \in P} \varepsilon V(S)$$

$$= \varepsilon V(R)$$

On the other hand,

$$\sum_{S \in C_1} [M_S(f) - m_S(f)] V(S) \leq \sum_{S \in C_1} 2M V(S),$$

where $|f(x)| \leq M, \forall x \in A$.

$$= 2M \sum_{S \in C_1} V(S)$$

$$\leq 2M \sum_{i=1}^{m} V(R_i)$$

$$< 2M\varepsilon.$$

Therefore $U(f, P) - L(f, P) < \varepsilon V(R) + 2M\varepsilon$

$$= (V(R) + 2M)\varepsilon$$

By Riemann's condition, f is integrable

Q. E. D.

Cor1. A bounded set A in $\mathbb{R}^n$ has volume
$\iff$ $bd(A)$ is measure zero
(1_A 在 $bd(A)$ 不連續)

Cor2. Let $A \subset \mathbb{R}^n$ be bounded and have volume. A bounded function $f: A \subset \mathbb{R}^n \to \mathbb{R}$ with finite or countable number of discontinuities is integrable.

pf. Let D be the set of discontinuities of the extended function f and D_1 the set of discontinuities of f on A.
Then $D \subset D_1 \cup bd(A)$
$\because$ A has volume
$\therefore$ $bd(A)$ is measure zero.
$\because$ D_1 is countable
$\therefore$ D_1 is measure zero
$\Rightarrow$ $A \cup D_1$ is measure zero
$\Rightarrow$ D is measure zero.
By Lebesgue's thm, f is integrable.
Q.E.D.

Cor3. Any bounded function defined on a bounded set with volume is integrable

Thm. ① Let $A \subset \mathbb{R}^n$ be a bounded set and have measure zero. If $f: A \subset \mathbb{R}^n$ is bounded and integrable, then

f在 $\mathbb{R}^{n+1}$ 中与 A 所夹之广义体积

$\boxed{\int_A f = 0}$ A底面积几乎是0

② Let $f: A \overset{\text{bounded}}{\subset} \mathbb{R}^n \to \mathbb{R}$ be bounded and integrable and $f \geq 0$ on A. If $\int_A f = 0$, then the set $\{x \in A \mid f(x) \neq 0\}$ has measure zero.

pf. ① Let R be a rectangle on $\mathbb{R}^n$ containing A. Extend f to R by setting $f(x) = 0$ on $R \setminus A$. Let $P = \{R_1, \dots, R_N\}$ be a partition of R.

$\because v(R_i) > 0, \ \forall i = 1, \dots, N.$

$\therefore R_i \not\subset A, \ \forall i = 1, \dots, N.$

$\Rightarrow m(R_i) \leq 0, \ i = 1, \dots, N.$

$\Rightarrow L(f, P) \leq 0.$

$\Rightarrow \underline{\int}_A f \leq 0.$

By the same argument. $L(-f, P) \leq 0$

$\Rightarrow U(f, P) \geq 0$

$\Rightarrow \overline{\int}_A f \geq 0$

$\because f$ is integrable

$\therefore \int_A f = 0.$

② Let $A_m = \{x \in A \mid f(x) > \frac{1}{m}\}$. Then $\{x \in A \mid f(x) > 0\} = \bigcup_{m=1}^{\infty} A_m$

Let $\varepsilon > 0$. Let R be a rectangle containing A. Extend f to R by setting $f=0$ on $R \setminus A$.

$\because$ f is integrable

$\therefore$ By Riemann's condition, $\exists$ a partition P of R s.t. $U(f,P) - L(f,P) < \varepsilon$

希望包 A_m 之長方形可因此要多小有多小.

$\because f \geq 0$

$\therefore L(f,P) \geq 0$

$\because \underline{\int_A} f = \overline{\int_A} f = 0$ and $L(f,P) \leq \underline{\int_A} f$

$\therefore L(f,P) = 0$

$\Rightarrow U(f,P) < \varepsilon$. $\longrightarrow = \sum_{S \in P} \underbrace{M_S(f)}_{\geq 0} V(S)$, 但我只要管住 A_m 的那些地方

$\Rightarrow$ 此時項數較少, 故比 $U(f,P)$ 小 (控制一下 $M_S(f)$)

$\because$ 即 $\sum_{S \in P,\, S \cap A_m \neq \emptyset} \underbrace{\frac{1}{m}}_{\text{m}} V(S) \leq \sum_{S \in P} M_S(f) V(S) < \varepsilon$

$\therefore \sum_{S \in P,\, S \cap A_m \neq \emptyset} < m\varepsilon$

$\Rightarrow A_m$ has measure zero.

$\Rightarrow A$ has measure zero.

Q.E.D.

Note. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function on a bounded set A. The set of discontinuities of the extended $f = \{ x \in \text{int}(A) \mid f \text{ is discontinuous at } x \} \cup \{ x \in bd(A) \mid \text{the extended } f \text{ is discont.} \}$

at $x\}$. By Lebesgue's Thm, f is integrable $\iff \{x \in \text{int}(A) \mid f$ is discont. at $x\}$ has measure zero, and $\{x \in bd(A) \mid$ extented f is discont. at $x\}$ has measure zero

eg. Let $f(x) = \begin{cases} x & , \text{if } -1 \le x \le 0 \\ 3x+8 & , \text{if } 0 < x \le 1 \end{cases}$

Prove or disprove f is integrable.

-sol-. $\because [-1,1]$ is bounded and has volume and f has at most discontinuity at $x=0$
$\therefore$ By Lebesgue's thm, f is integrable. #

eg. $f(x) = \begin{cases} 0 & , \text{if } x \in \mathbb{Q}^c \cap (0,1] \\ 1 & , \text{if } x \in \mathbb{Q}^c \cap (0,1] \end{cases}$

Prove or disprove that f is integrable

-sol-. $\because f$ is discont. at every point $x \in (0,1)$, and $(0,1)$ is not measure zero
$\therefore$ By Lebesgue's thm, f is not integrable. #

eg. Let $f(x,y) = \begin{cases} x^2 + \sin\frac{1}{y} & , \text{if } y \neq 0 \\ x^2 & \text{if } y = 0 \end{cases}$

Prove or disprove that f is integrable on the set $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$

-sol-. f is bounded on A and f is discont. on $A \cap \{y=0\}$ which is

measure zero, and
$\because bd(A) = \{(x, \sqrt{1-x^2}) \mid -1 \le x \le 1\} \cup \{(x, -\sqrt{1-x^2}) \mid -1 \le x \le 1\}$
$=$ two graphs of two cont. functions on $[-1, 1]$
$\therefore bd(A)$ is measure zero
By Lebesgue thm, f is integrable. $\#$

Prop. If A and B have volume, then $A \cup B$, $A \cap B$ and $A \setminus B$ have volume.

pf. exercise

$*$結論：函数可不可積是看 extended 函数的連不連續性.

§8.4 Properties of the integral

Thm. Let A and B be bounded subsets of $\mathbb{R}^n$, $c \in \mathbb{R}$, and let the bounded functions $f, g: A \subset \mathbb{R}^n \to \mathbb{R}$ be integrable. Then ① $f+g$ is integrable and

$$\int_A (f+g) = \int_A f + \int_A g$$ → 用 Darboux 去證.

② cf is integrable and $\int_A (cf) = c\int_A f$, $\forall c \in \mathbb{R}$

③ If $f \le g$ on A, then $\int_A f \le \int_A g$ → by ① and ②

④ fg is integrable

pf. ④ Let S, T, W be sets of discontinuities of the extended f, the extended g, and the extended fg respectively.

$\because$ the extended (fg) = the extended f $\cup$ the extended g
$\therefore W \subset S \cup T$
$\because f, g$ is integrable
$\therefore$ By Lebesgue' Thm, S, T are measure zero
$\Rightarrow S \cup T$ is measure zero
$\Rightarrow W$ "
$\Rightarrow$ By Lebesgue Thm, fg is integrable.
Q.E.D.

⑤ $|f|$ is integrable and $\left|\int_A f\right| \le \int_A |f|$

Notice that $S =$

pf ⑤ ∧ The set of discont. of the extended $|f|$ is contained in the set of discont. of the extended of f.

$\because f$ is integrable

$\therefore$ By Lebesgue's thm, T has measure zero

$\Rightarrow S$ has measure zero.

$\Rightarrow$ By Lebesgue's thm, $|f|$ is integrable.

$\because -|f| \le f \le |f|$

$\therefore$ By ③, $\int_A (-|f|) \le \int_A f \le \int_A |f|$.

$\Rightarrow \left|\int_A f\right| \le \int_A |f|$.

Q.E.D.

⑥. If A has volume, and $|f| \le M$ on A, then $\left|\int_A f\right| \le M v(A)$

pf ⑥ $\because -M \le f \le M$ on A

$\therefore -M 1_A \le f \cdot 1_A \le M 1_A$ on $\mathbb{R}^n$

$\because A$ has volume

$\therefore 1_A$ is integrable.

By ③, ②, $-M\int_A 1_A \le \int_A f \le M \int_A 1_A$.

$\Rightarrow -M v(A) \le \int_A f \le M v(A)$

$\Rightarrow \left|\int_A f\right| \le M v(A)$

Q.E.D.

⑦ (MVT for integral)
If f is cont. on A and A has volume and is compact and connected, then $\exists$ an $x_0 \in A$ s.t. $\int_A f = f(x_0)\, v(A)$.
The quantity $\frac{1}{V(A)} \int_A f$ is called the average of f over A.

pf ⑦: 同 $\mathbb{R}$ 之情形

⑧ Let $f: A \cup B\ (\subset \mathbb{R}^n) \to \mathbb{R}$. If $A \cap B$ has measure zero and $f|_A$, $f|_B$ and $f|_{A\cap B}$ are all integrable, then f is integrable on $A \cup B$ and $\int_{A\cup B} f = \int_A f + \int_B f$.

(若 domain 難積，可適當切割之！)

pf. Notice that $bd(A\cap B) \subset bd(A) \cup bd(B)$.
$\because f$ is integrable on A, B, and $A\cap B$
$\therefore f \cdot \underline{1}_A,\ f \cdot \underline{1}_B$ and $f \cdot \underline{1}_{A\cap B}$ are integrable
$\therefore f \cdot \underline{1}_A + f \cdot \underline{1}_B - f \cdot \underline{1}_{A\cap B}$ is the extended f on $A \cup B$
$\therefore f$ is integrable on $A \cup B$, and

$$\int_{A\cup B} f = \int_{A\cup B} (f \cdot \underline{1}_A + f \cdot \underline{1}_B - f \cdot \underline{1}_{A\cap B})$$

$$= \int_{A\cup B} f \cdot \underline{1}_A + \int_{A\cup B} f \cdot \underline{1}_B - \int_{A\cup B} f \cdot \underline{1}_{A\cap B}$$

$$= \int_A f \cdot \underline{1}_A + \int_B f \cdot \underline{1}_B - \int_{A\cap B} f \cdot \underline{1}_{A\cap B}$$

$$= \int_A f + \int_B f - \int_{A\cap B} f$$

$\because$ f is integrable on $A\cap B$ and $A\cap B$ has measure zero

$\therefore$ $\int_{A\cap B} f = 0$, the result holds.

Q.E.D.

⑨ Let $C \subset A$. If $bd(C)$ has measure zero. Then $f|_C$ and $f|_{A\setminus C}$ are integrable, and $\int_A f = \int_C f + \int_{A\setminus C} f$

pf. This follows directly from ⑧

Prop. If A and B have volume, then $A\cup B$, $A\cap B$ and $A\setminus B$ have volume.

hint. $1_{A\cap B} = 1_A \cdot 1_B$ is integrable
$1_{A\cup B} = 1_A + 1_B - 1_{A\cap B}$ "
$1_{A\setminus B} = 1_A - 1_{A\cap B}$ "

pf: exercise.

§8.5 Improper Integrals

在這個章節我們將会把積分的概念延伸到任意的函数上.

Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function 函数依其定義域(A是 bounded or unbounded set)与其取值的方式(f is bounded or unbounded)可分成四大類

① f is bounded and A is bounded
② f " bounded A " unbounded
③ f " unbounded " A " bounded
④ f " unbounded " A " unbounded

其中①已在§8.1節中定義与討論. 而②, ③, ④中的積分 $\int_A f$ 被称為 improper integral.
(也可通称這是所有積分).

這一節可看到其实一个定義可把所有种類積分涵蓋在一起, 並且在数学上將第一類定義推廣到②, ③, ④.

如何找到共用的積分定義, 使得後來①只是其中一种 special case?

⇒用①來處理②, ③, ④. ①和②只差在 domain. 我們知道 unbounded set 可用 bounded set 擴大來逼近, 由「極限」延伸出來的方式

又 unbounded function 可用 bounded function 來逼近, 如果可以做到這樣的事情, 就可把③, ④由②推廣出去(由極限). 故真正 crucial 的問題在如何由①推廣到②

透过極限

cube
Def. A n-dimensional $[-a,a]^n$ in $\mathbb{R}^n$ is the set $[-a,a]^n = \underbrace{[-a,a]\times\cdots\times[-a,a]}_{n次}$ for some $a>0$

① 我們希望極限和逼近的方式無关. 如果用 disk, ball 等逼近所得到的結果不一樣, 我們不会喜欢. 應證明這方式跟一開始延伸的方式無关 ↘ 是对的
先取 cube (∵容易描述), it turns out
② cube 太对称了, 造成一不太好的結果: 定積分描述的是一廣義体積。無論 bounded, unbounded 定積分在图形上表達的應是函数的图形跟定義域所夾的廣義体積。而对称之逼近方式, 在討論廣義体積的時候会產生一了正負对消的狀況. 如:

$$\int_{-\infty}^{\infty} x\,dx,\ \lim_{a\to\infty}\int_{-a}^{a} x\,dx = 0,\ \text{but}$$

$$\lim_{a\to-\infty\ b\to\infty}\int_{a}^{b} x\,dx \quad DNE$$

先處理非負函数避免此問題, 再設法推展到一般函数. 但一般函数可用2非負函数相減來表示. 故一般函数仍可做, 以避免此弊病

Def. Let $f: A\subset\mathbb{R}^n\to\mathbb{R}$ be a function define two functions $f^+, f^-: A\subset\mathbb{R}^n\to\mathbb{R}$ by

$$f^+(x) = \begin{cases} f(x), & \text{if } f(x)\geq 0 \\ 0, & \text{if } f(x)<0 \end{cases}$$

$$f^-(x) = \begin{cases} -f(x), & \text{if } f(x) \le 0 \\ 0, & \text{if } f(x) > 0 \end{cases}$$

Then $f^+ \ge 0$ and $f^- \ge 0$ on A and $f = f^+ - f^-$ on A and $f^+ \cdot f^- \equiv 0$ on A.

Prop. If $f: A \subset \mathbb{R}^n \to \mathbb{R}$ is cont. at x, then f^+ and f^- are cont. at x.

pf. $\because f$ is cont. at x

$\therefore |f|$ "

$\because f = f^+ - f^-$ and $|f| = f^+ + f^-$

$\therefore f^+ = \dfrac{|f| + f}{2}$ and $f^- = \dfrac{|f| - f}{2}$

are cont.

Q.E.D.

Def. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be bounded function s.t. $f \ge 0$ on A. Extend f to $\mathbb{R}^n$ by setting $f = 0$ outside A. Suppose that f is integrable on each n-dimensional cube $[-a, a]^n$. We say that the integral $\int_A f$ exists or $\int_A f$ converges if $\lim_{a \to \infty} \int_{[-a,a]^n} f$ exists and finite, and we define $\int_A f = \lim_{a \to \infty} \int_{[-a,a]} f$.
In this case we say that f is integrable.

Note: For A is bounded, $\exists a_0 > 0$ s.t.
$\forall a > a_0$, $\int_{[-a,a]^n} f = \int_{[-a_0,a_0]} f = \int_A f$.

$\Rightarrow \lim_{a\to\infty} \int_{[-a,a]^n} f = \int_A f$

故 A 不需管 bounded or unbounded, 此時①為②之特例

Thm. Let $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$ be bounded nonnegative functions. If f and g are two function. If f and g are integrable. Then

① $f+g$ is integrable on A and
$\int_A (f+g) = \int_A f + \int_A g$

② αf is integrable, $\forall \alpha \geq 0$, and $\int_A \alpha f = \alpha \int_A f$

③ If $0 \leq f \leq g$, then $\int_A f \leq \int_A g$

④ If $A = B \cup C$ and $B \cap C$ has measure zero, and $f|_B$, $f|_C$, $f|_{B\cap C}$ are all integrable, then f is integrable on A and $\int_A f = \int_B f + \int_C f$

⑤ If $B \subset A$ s.t. $bd(B)$ has measure zero, then f_B and $f_{A\setminus B}$ are integrable, and
$\int_A f = \int_B f + \int_{A\setminus B} f$

pf. ①: f is integrable on $[-a,a]^n$, and g ...

$\therefore$ $f+g$ is integrable on $[-a,a]^n$, and

$$\int_{[-a,a]^n}(f+g)=\int_{[-a,a]^n}f+\int_{[-a,a]^n}g$$

$\because$ $\lim_{a\to\infty}\int_{[-a,a]^n}f$ and $\lim_{a\to\infty}\int_{[-a,a]^n}g$ exist and is finite

$\therefore$ $\lim_{a\to\infty}\int_{[-a,a]^n}(f+g)=\lim_{a\to\infty}\int_{[-a,a]^n}f+\lim_{a\to\infty}\int_{[-a,a]}g$ exist and is finite.

③ $\because$ $0\le g\le f$ on A (and on $[-a,a]^n$

$\therefore$ by Thm, $\int_{[-a,a]^n}g\le\int_{[-a,a]^n}f$

$\because$ "$\le$" 两边对 A 之极限存在且 finite

$\therefore \lim_{a\to\infty}\int_{[-a,a]^n}g\le\lim_{a\to\infty}\int_{[-a,a]^n}f$

$\Rightarrow \int_A f\le\int_A g$.

其余与8.4节同理

Q.E.D.

Prop. (The comparison test A) for bounded nonnegative functions.

Let $f,g:A\subset\mathbb{R}^n\to\mathbb{R}$ are bounded non-negative functions s.t. $0\le g\le f$ on A. If f is integrable and g is integrable on each cube, then g is integrable

pf. $\because f \geq 0$ and f is integrable

$\therefore \int_{[-a,a]^n} f \nearrow \int_A f$ as $a \to \infty$

$\because 0 \leq g \leq f$ on A and g is integrable on each cube $[-a,a]^n$, and f, g are bdd

$\therefore 0 \leq \int_{[-a,a]^n} g \leq \int_{[-a,a]^n} f \leq \int_A f$

$\because \int_{[-a,a]^n} g$ increases as a increases and is bound by $\int_A f$

$\therefore \lim_{a\to\infty} \int_{[-a,a]^n} g$ exists and it is not bigger than $\int_A f$ (and finite)

$\Rightarrow \int_A g \leq \int_A f$

Q.E.D.

Thm. Suppose that $f: A \subset \mathbb{R}^n \to \mathbb{R}$ is a bounded nonnegative function s.t. f is integrable on each cube $[-a,a]^n$. Then f is integrable $\Leftrightarrow$ for each sequence B_k of bounded sets with volumes s.t. $B_k \subset B_{k+1}$, $\forall k \in \mathbb{N}$ and for each cube C we have $C \subset B_k$ for some $k \in \mathbb{N}$,

$\lim_{k\to\infty} \int_{B_k} f$ exists.

In this case, $\lim_{k\to\infty} \int_{B_k} f = \int_A f$

Note: 此定理告訴我們 $\int_A f$ 中我們將 bounded set 延拓到 unbounded set 的方法無关.

pf. $\Rightarrow$) $\because$ B_k is bounded, $\forall k \in \mathbb{N}$

$\therefore$ For each $k \in \mathbb{N}$, $\exists a_n > 0$ s.t.

$B_k \subset [-a_k, a_k]^n$

$\because$ f is integrable

$\therefore$ f " on $[-a_k, a_k]^n$

$\because$ B_k has volume

$\therefore$ $bd(B_k)$ is measure zero.

$\Rightarrow$ f is integrable on B_k and $[-a_k, a_k]^n \setminus B_k$, and

$$\int_{[-a_k, a_k]^n} f = \int_{B_k} f + \int_{[-a_k, a_k]^n \setminus B_k} f$$

$\because$ $f \geq 0$

$\therefore$ $\int_{[-a_k, a_k]^n \setminus B_k} f \geq 0$

$\Rightarrow$ $\int_{B_k} f \leq \int_{[-a_k, a_k]^n} f \nearrow \int_A f$

$\because$ $f \geq 0$

$\therefore$ $\int_{B_k} f$ increases as k increases

$\Rightarrow$ $\lim_{k \to \infty} \int_{B_k} f$ exists and is not bigger than $\int_A f$.

$\because \forall N \in \mathbb{N}, \exists k \in \mathbb{N}$ s.t. $[-N, N]^n \subset B_k$

$\therefore \int_{[-N,N]^n} f \leq \int_{B_k} f \leq \lim_{k\to\infty} \int_{B_k} f$

$\because \lim_{N\to\infty} \int_{[-N,N]^n} f = \int_A f$

$\therefore \int_A f \leq \lim_{k\to\infty} \int_{B_k} f$

$\Rightarrow \lim_{k\to\infty} \int_{B_k} f = \int_A f.$

$\Leftarrow$ Similarly.

Q.E.D.

下一步要把 bounded function 逼到 unbounded function，方法是用砍的，但越砍越高（上人一起 M（砍之））

Def. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function.
Let $M \in \mathbb{R}$. Define a function
$f_M: A \to \mathbb{R}$ by $f_M(x) = \begin{cases} f(x), & \text{if } f(x) \leq M \\ M, & \text{if } f(x) > M \end{cases}$
(i.e. $f_M = \min(f, M)$)

Prop. If f is cont. at x, then f_M is cont. at x, $\forall M \in \mathbb{R}$

pf. Notice that

從半開值往上/下走一半.

$$\begin{cases} \max(f,g) = \dfrac{f+g}{2} + \dfrac{|f-g|}{2} \\ \min(f,g) = \dfrac{f+g}{2} - \dfrac{|f-g|}{2} \end{cases}$$

The result holds.

Q.E.D.

Note: $f^+ = \min(f, 0)$, $f^- = \min(-f, 0)$.

Def. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a nonnegative function. Suppose that f_M is integrable for each $M > 0$. Then $\int_A f_M$ increases as M increases.

($\because f \geq 0$)

We say that the integral $\int_A f$ exists or converges if $\lim_{M\to\infty} \int_A f_M$ exists and finite. and we thus define $\int_A f = \lim_{M\to\infty} \int_A f_M$. In this case, we say that f is integrable.

Note. ① f is bounded and $f \geq 0$, then $\exists$ $M_0 > 0$ s.t. $\forall M > M_0$, $f_M = f$

$\Rightarrow \lim_{M\to\infty} \int_A f_M = \int_A f$

② Case ② 為此定義之 special case!

Prop. (The comparison test B) for nonnegative functions.

Let $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$ be nonnegative functions s.t. $0 \le g \le f$ on A. If f is integ. and g_M is integrable, $\forall M > 0$, then g is integrable. and $\int_A g \le \int_A f$

pf. $\because 0 \le g \le f$ on A s.t. $0 \le g_M \le f_M$ on A, $\forall M > 0$, and integrable

$\therefore g_M$ and f_M are bounded nonnegative and integrable functions, $\forall M > 0$

$\Rightarrow \int_A g_M \le \int_A f_M \longrightarrow \int_A f$, $\forall M > 0$.

$\because g \ge 0$

$\therefore \int_A g_M$ increases as $M \to \infty$

$\Rightarrow \lim_{M \to \infty} \int_A g_M$ is finite and is not larger than $\int_A f$

$\Rightarrow g$ is integrable and $\int_A g \le \int_M f$.

Q.E.D.

Cor. If $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$ are nonnegative integrable s.t. $0 \le g \le f$ on A, then $\int_A g \le \int_A f$.

Def. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function.
We say that f is integrable, if f^+ and f^- are integrable, and we thus define $\int_A f = \int_A f^+ - \int_A f^-$

Def. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function and $M > 0$. We define a function $f_{[-M,M]}: A \subset \mathbb{R}^n \to \mathbb{R}$ by $f_{[-M,M]}(x) = \begin{cases} -M, & \text{if } f(x) < -M \\ f(x), & \text{if } -M \le f(x) \le M \\ M, & \text{if } f(x) > M \end{cases}$

Then $f_{[-M,M]}$ is a bounded function by M.

Prop. If f is cont. at x, then $f_{[-M,M]}$ is cont. at x.

pf. In fact, $f_{[-M,M]} = \max(\min(f, M), -M)$

Q.E.D.

Prop. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function and $M > 0$. Then $f_{[-M,M]}$ is integrable $\iff (f^+)_M$ and $(f^-)_M$ are integrable and $\int_A f_{[-M,M]} = \int_A (f^+)_M - \int_A (f^-)_M$

pf. ($\because$ $f_{[-M,M]}$ is bounded
$\therefore$ 由證 $f_{[-M,M]}$ 是可積的所求想起，
要證 $(f_{[-M,M]})^+$ and $(f_{[-M,M]})^-$ 是可積)

$\because (f_{[-M,M]})^+ = (f^+)_M$

$\therefore f_{[-M,M]}$ is integrable and

$\int_A f_{[-M,M]} = \int_A (f^+)_M - \int_A (f^-)_M$

Q.E.D.

Thm. ① If $f: A \subset \mathbb{R}^n \to \mathbb{R}$ is an integrable function. Then $f_{[-M,M]}$ is integrable, $\forall M > 0$ and $\lim_{M\to\infty} \int_A f_{[-M,M]} = \int_A f$

② Furthermore, if f is bounded, then f is integrable on each cube $[-a,a]^n$ and $\lim_{a\to\infty} \int_{[-a,a]^n} f = \int_A f$

pf. ① $\because f$ is integrable

$\therefore f^+$ and f^- are integrable and

$\int_A f = \int_A f^+ - \int_A f^-$

$\because f^+$ and f^- are integrable

$\therefore (f^+)_M$ and $(f^-)_M$ are integrable,

$\forall M > 0$ and $\lim_{M\to\infty} \int_A (f^{+(-)})_M = \int_A f^{+(-)}$

$\Rightarrow f_{[-M,M]}$ is integrable and

$\int_A f_{[-M,M]} = \int_A (f^+)_M - \int_A (f^-)_M$, $\forall M > 0$

$\Rightarrow \lim_{M\to\infty} \int_A f_{[-M,M]} = \int_A f^+ - \int_A f^- = \int_A f$

② ∵ f is bounded and integrable
∴ f^+ and f^- are integrable and
$$\int_A f = \int_A f^+ - \int_A f^-$$

∵ f^+ and f^- are integrable
∴ f^+ and f^- are integrable on each cube $[-a,a]^n$, and
$$\lim_{a\to\infty} \int_{[-a,a]^n} f^+ = \int_A f^+ \text{ and}$$
$$\lim_{a\to\infty} \int_{[-a,a]^n} f^- = \int_A f^-$$
∵ f^+ and f^- are bounded and $[-a,a]^n$ is bounded, $\forall a>0$
∴ By §8.4, $f^+ - f^-$ are integrable on each cube $[-a,a]^n$ and
$$\int_{[-a,a]^n} (f^+ - f^-) = \int_{[-a,a]^n} f^+ - \int_{[-a,a]^n} f^-$$
$\Rightarrow f$ is integrable on each cube $[-a,a]^n$ and $\int_{[-a,a]^n} f = \int_{[-a,a]^n} f^+ - \int_{[-a,a]^n} f^-$

$$\Rightarrow \lim_{a\to\infty} \int_{[-a,a]^n} f = \int_A f^+ - \int_A f^-$$

Q.E.D.

Thm. Let $f: A\subset \mathbb{R}^n \to \mathbb{R}$ be a function
① If f is integrable, then $|f|$ is integrable and $\left|\int_A f\right| \le \int_A |f| = \int_A f^+ + \int_A f^-$
② If $|f|$ is integrable and $f_{[-M,M]}$ is integrable $\forall M>0$, then f is integrable.

Note: 在②中, f需要額外的條件才能保証②的成立. Why? 如反例

$$f(x)=\begin{cases}1, & \text{if } x\in\mathbb{Q}\cap[0,1]\\ -1, & \text{if } x\in\mathbb{Q}^c\cap[0,1]\end{cases}$$

②幾乎處處

pf. ① $\because |f|\geq 0$, $\therefore$ 要去証 $|f|_M$ 可積, $\forall M>0$ and $\lim_{M\to\infty}\int_A |f|_M$ is finite.

Notice that $|f|_M=(f^+)_M+(f^-)_M$

$\because f$ is integrable

$\therefore f^+$ and f^- are integrable and

$$\int_A f=\int_A f^+-\int_A f^-$$

$\because f^+$ and f^- are integrable

$\therefore (f^+)_M$ and $(f^-)_M$ are integrable, $\forall M>0$, and $\lim_{M\to\infty}\int_A (f^+)_M=\int_A f^+$ and

$\lim_{M\to\infty}\int_A (f^-)_M=\int_A f^-$

$\because (f^+)_M$ and $(f^-)_M$ are bounded nonnegative, $\forall M>0$.

$\therefore |f|_M$ is integrable and

$$\int_A |f|_M=\int_A (f^+)_M+\int_A (f^-)_M,\quad \forall M>0.$$

$$\Rightarrow \lim_{M\to\infty}\int_A |f|_M=\int_A f^+ +\int_A f^-$$

Double A

$\Rightarrow f$ is integrable, and

$$\int_A |f| = \int_A f^+ + \int_A f^-$$

$\because$ $\int_A f^+ + \int_A f^- \geq \left| \int_A f^+ - \int_A f^- \right| = \left| \int_A f \right|$

$\therefore$ $\int_A |f| \geq \left| \int_A f \right|$.

② 由所求想起要去証 f^+ and f^- are integrable

$\Rightarrow$ 再由 f^+, f^- 可积的所求想起, 要去証 $(f^{\pm})_M$ 可積, $\forall M>0$, and $\lim_{M\to\infty} \int_A (f^{\pm})_M$ are finite

$\because f_{[-M,M]}$ is integrable, $\forall M>0$

$\therefore (f^+)_M$ and $(f^-)_M$ are integrable, $\forall M>0$

Notice that $0 \leq f^+ \leq |f|$,

$\because |f|$ is integrable

$\therefore$ By comparison test B, f^+ is integrable.

Similarly f^- is integrable

$\Rightarrow f$ is integrable.

Q.E.D.

if f, g 可积, 但 fg 未必可积!

e.g. Give an example that $f: A \subset \mathbb{R}^n \to \mathbb{R}$ is an integrable function as an improper integral but f^2 is not integrable.

-sol-. Let $f(x) = \frac{1}{\sqrt{x}}$ on $(0,1]$. Then

$\int_0^1 \frac{1}{\sqrt{x}}\,dx < \infty$, but $\int_0^1 \frac{1}{x}\,dx = \infty$ ⁂

☆ Thm. If $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be an integrable function, then the set of discontinuities of the extended function $f \cdot 1_A$ is measure zero.

pf. Let $S = \{x \in \mathbb{R}^n \mid f \cdot 1_A$ is not cont. at $x\}$, and $S_N = \{x \in \mathbb{R}^n \mid f_{[-N,N]} \cdot 1_A$ is not cont. at $x\}$, $\forall N \in \mathbb{N}$. Then $S = \bigcup_{N=1}^{\infty} S_N$

$\because f$ is integrable
$\therefore f_{[-N,N]}$ is integrable, $\forall N \in \mathbb{N}$.
$\because f_{[-N,N]}$ is bounded, $\forall N \in \mathbb{N}$.
$\therefore f_{[-N,N]}$ is integrable on each cube $[-a,a]^n$
Let $B_{N,m} = \{x \in \mathbb{R}^n \mid f_{[-N,N]} \cdot 1_{[-m,m]^n}$ is not cont. at $x\}$, $\forall m \in \mathbb{N}$. Then $S_N = \bigcup_{m=1}^{\infty} B_{N,m}$

$\because f_{[-N,N]}$ is bounded and $[-m,m]^n$ is bounded
$\therefore$ By Lebesgue's thm, $B_{N,m}$ is measure zero, $\forall m \in \mathbb{N}$
$\Rightarrow S_N$ is measure zero
$\Rightarrow S$ is measure zero

Q.E.D.

Cor. ① If $f: A \subset \mathbb{R}^n \to \mathbb{R}$ is integrable, then the sets of discontinuities of $f^+ \cdot 1_A$ and $f^- \cdot 1_A$ have measure zero. Hence so are $(f^+)_M \cdot 1_A$ and $(f^-)_M \cdot 1_A$

② If $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$ are integrable, then $[(\alpha f + \beta g)^{\pm}]$, $[(f \cdot g)^{\pm}]$, $[(\alpha f + \beta g)^{\pm}]_M$, $[-(f \cdot g)^{\pm}]_M$ are integrable on each cube, $\forall \alpha, \beta \in \mathbb{R}$

Thm. If $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$ are integrable function
Then ① $f+g$ is integrable and
$\int_A (f+g) = \int_A f + \int_A g$
② αf is integrable and $\int_A (\alpha f) = \alpha \int_A f$, $\forall \alpha \in \mathbb{R}$
③ If $g \leq f$ on A, then $\int_A g \leq \int_A f$.
④ Suppose that $A = B \cup C$ s.t. $B \cap C$ has measure zero. If f is integrable on B and C, then f is integrable on A s.t. $\int_A f = \int_B f + \int_C f$
⑤ If $B \subset A$ s.t. $bd(B)$ is measure zero, then f is integrable on B and $A \setminus B$ s.t. $\int_A f = \int_B f + \int_{A \setminus B} f$.

pf. ① case I: If $f \geq 0$ and $g \geq 0$ on A. Then
$f+g \geq 0$ on A
$\therefore$ f and g are integrable
$\therefore$ $(f+g)_M$ is integrable on each cube, $\forall M > 0$.
$\because f \geq 0$ and $g \geq 0$ on A
$\therefore 0 \leq (f+g)_M \leq f_M + g_M$, $\forall M > 0$
$\because f$ and g are integrable
$\therefore f_M$ and g_M are integrable, $\forall M > 0$

$\because f_M$ and g_M are bounded and nonnegative,

$\therefore f_M + g_M$ is integrable, and $\int_A (f_M + g_M) = \int_A f_M + \int_A g_M$

By comparison test A, $(f+g)_M$ is integrable & and $\int_A (f+g)_M \le \int_A (f_M + g_M)$, $\forall M > 0$. $\forall M > 0$

$\because f \ge 0$ and $g \ge 0$

$\therefore 0 \le (f_{\frac{M}{2}} + g_{\frac{M}{2}})(x) = \begin{cases} f(x)+g(x), & \text{if } f(x) \le \frac{M}{2} \\ f(x)+\frac{M}{2}, & \text{if } f(x) \le \frac{M}{2} \\ \frac{M}{2}+g(x), & \text{if } f(x) > \frac{M}{2} \\ M, & \text{if } f(x) > \frac{M}{2} \end{cases}$

$\Rightarrow (f_{\frac{M}{2}} + g_{\frac{M}{2}})(x) \le (f+g)_M(x)$, $\forall x \in A$.

$\Rightarrow 0 \le f_{\frac{M}{2}} + g_{\frac{M}{2}} \le (f+g)_M$

$\because f_{\frac{M}{2}} + g_{\frac{M}{2}}$ and $(f+g)_M$ are bounded nonnegative integrable, $\forall M > 0$

$\therefore \int_A f_{\frac{M}{2}} + \int_A g_{\frac{M}{2}} \le \int_A (f+g)_M$, $\forall M > 0$

$\Rightarrow \int_A f_{\frac{M}{2}} + \int_A g_{\frac{M}{2}} \le \int_A (f+g)_M \le \int_A f_M + \int_A g_M$

$\because \lim_{M\to\infty} \int_A f_{\frac{M}{2}} = \lim_{M\to\infty} \int_A f_M = \int_A f$, and similarly for g

$\therefore$ By Squeez Thm,

$\lim_{M\to\infty} \int_A (f+g)_M = \int_A f + \int_A g$

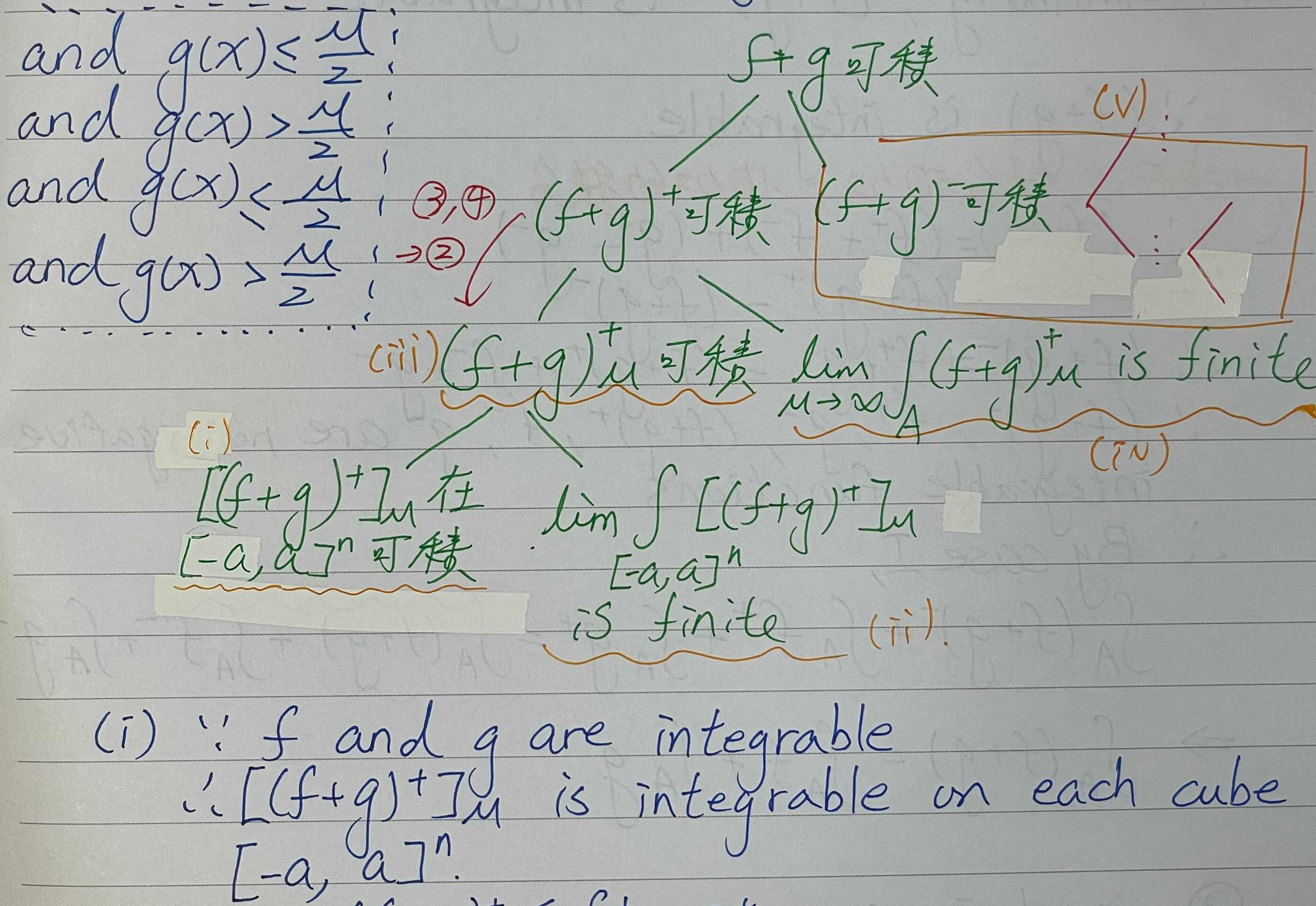
$\Rightarrow f+g$ is integrable, and

$$\int_A (f+g) = \int_A f + \int_A g$$

Case II. If f, g are integrable, to show that $f+g$ is integrable and

$$\int_A (f+g) = \int_A f + \int_A g.$$

and $g(x) \leq \frac{M}{2}$;
and $g(x) > \frac{M}{2}$;
and $g(x) < \frac{M}{2}$; ③,④
and $g(x) > \frac{M}{2}$; →②



(i) ∵ f and g are integrable
∴ $[(f+g)^+]_M$ is integrable on each cube $[-a,a]^n$.
∵ $0 \leq (f+g)^+ \leq f^+ + g^+$,
∴ $0 \leq [(f+g)^+_M] \leq (f^+)_M + (g^+)_M$
∵ f and g are integrable
∴ f^+ and g^+ are integrable
⇒ $(f^+)_M$ and $(g^+)_M$ are integrable, $\forall M > 0$.
⇒ By case I, $(f^+)_M + (g^+)_M$ are integrable
(ii) By comparison test A, $(f+g)^+_M$ is integrable, $\forall M > 0$. ⇒ 得 (iii)

(iv) $\because f^+$ and g^+ are integrable,
$\therefore$ By case I, $f^+ + g^+$ are integrable
$\because 0 \le (f+g)^+ \le f^+ + g^+$
$\therefore$ By comparison test B, $(f+g)^+$ is integrable

(v). Similarly, $(f+g)^-$ is integrable

$\Rightarrow (f+g)$ is integrable

(現在證積分相加 = 相加的積分)

$\because (f+g) = (f^+ + f^-) + (g^+ - g^-)$
$= (f+g)^+ - (f+g)^-$
$\therefore (f+g)^- + f^+ + g^+ = (f+g)^+ + f^- + g^-$
$\because (f+g)^-, f^+, g^+, (f+g)^+, f^-, g^-$ are nonnegative integrable functions
$\therefore$ By case I,
$\int_A (f+g)^- + \int_A f^+ + \int_A g^+ = \int_A (f+g)^+ + \int_A f^- + \int_A g^-$

$\Rightarrow \int_A (f+g) = \int_A f + \int_A g$

② is obvious

③ follows by ① + ② ($\because 0 \le g \le f \Rightarrow 0 \le f - g$, 用①, ②)

④, ⑤ ⇒ 與前面的定理類似

Q.E.D.

(續)

MON TUE WED THU FRI SAT SUN | DATE: / / | NO.: | SUBJECT:

§8.5 Improper Integrals (Cont.)

Remark. ① $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ 可積 but $\int_0^1 \frac{1}{x}\,dx$ 不可積

⇒ 乘法法則不会成立

→ $\frac{1}{x} = \sqrt{\frac{1}{x^2}}$ on $[1,\infty)$

② $\int_1^\infty \frac{1}{x}\,dx$ 不可積但 $\int_1^\infty \frac{1}{x^2}\,dx$ 可積

⇒ 合成法則不会成立 (即使 boundary 是 measure zero 也不成立)

* 在 n=1 時, improper integral 可以有更簡單的計算方式

Thm ① Suppose $f:[0,\infty)\to\mathbb{R}$ is a bounded cont. function s.t. $f\geq 0$, Let F be an antiderivative of f. Then f is integrable $\Leftrightarrow \lim_{x\to\infty} F(x)$ is finite

In this case, $\int_{[a,\infty)} f = \left(\lim_{x\to\infty} F(x)\right) - F(a)$

$\left(= \lim_{x\to\infty}\left(F(x)-F(a)\right) = \lim_{b\to\infty} \int_a^b f(x)\,dx\right)$

② Suppose that $f:(a,b]\to\mathbb{R}$ is cont. and $f\geq 0$. Then f is integrable $\Leftrightarrow \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$ exists. In this case, $\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$

pf. ① ∵ f is cont. on $[-b,b]$ and $[-b,b]$ has volume $\forall b>0$

∴ f is integrable on $[-b,b]$, $\forall b>0$

Double A

$\Rightarrow \int_{[-b,b]} f = \int_{[-b,b]} f \cdot 1_{[a,\infty)} = \int_{[-b,b]} f \cdot 1_{[a,b]}$,
$\forall b > |a|$

$\Rightarrow \int_{[-b,b]} f = \int_a^b f$.

$\because$ f is cont. on $[a,b]$

$\therefore$ By 2nd fundamental Thm of calculus, 柯西值

$$\int_a^b f = F(b) - F(a).$$

$\Rightarrow$ f is integrable $\Leftrightarrow \lim_{b\to\infty} F(b)$ is finite

In this case, $\int_a^b f(x)\,dx = \left(\lim_{b\to\infty} F(b)\right) - F(a)$.

② $\Leftarrow$) Let $\varepsilon > 0$. Define a function f^ε to be zero at $[a, a+\varepsilon]$ and and f on $(a+\varepsilon, b]$. Then f^ε is integrable on $[a,b]$ and

$$\int_a^b f^\varepsilon = \int_a^{a+\varepsilon} f^\varepsilon + \int_{a+\varepsilon}^b f^\varepsilon$$

$$= \int_{a+\varepsilon}^b f.$$

$$\int_a^b f_M - \int_a^b f^\varepsilon = \left(\int_a^{a+\varepsilon} f + \int_{a+\varepsilon}^b f_M\right) - \int_{a+\varepsilon}^b f \qquad f \geq f_M$$

$$\leq \int_a^{a+\varepsilon} f_M$$

$$\leq \int_a^{a+\varepsilon} M\,dx$$

未定式! $\varepsilon \to 0$
$M \to \infty$ 取决于
谁先取, 谁后取 $= \varepsilon M$

Let $I = \int_a^b f(x)\,dx$

$\because$ f is integrable and $f \geq 0$

$\therefore \int_a^b f_M \nearrow I$ as $M \nearrow \infty$ and $I \geq 0$

$\because f \geq 0$

$\therefore f^\varepsilon \nearrow$ as $\varepsilon \searrow 0^+$

極值定理 $\because$ f is cont. on $[a+\varepsilon, b]$

↳ $\therefore \exists \tilde{M} > 0$ s.t. $f(x) \leq \tilde{M}$, $\forall x \in [a+\varepsilon, b]$

$\Rightarrow \int_a^b f^\varepsilon = \int_{a+\varepsilon}^b f \leq \int_{a+\varepsilon}^b f_{\tilde{M}} \leq \int_a^b f_{\tilde{M}} \leq I$,

$\forall \varepsilon > 0^+$.

$\Rightarrow \lim_{\varepsilon \to 0^+} \int_a^b f^\varepsilon$ is finite and $\lim_{\varepsilon \to 0^+} \int_a^b f^\varepsilon \leq I$.

想證 $=$,

Let $\eta > 0$, 由 $\left| \int_a^b f^\varepsilon - I \right| < \eta$ 去解 ε.

$$\left| \int_a^b f^\varepsilon - I \right| = I - \int_a^b f^\varepsilon$$

$$= \left(I - \int_a^b f_M \right) + \left(\int_a^b f_M - \int_a^b f^\varepsilon \right)$$

$$\leq \left(I \overset{M \to \infty}{\longleftarrow} \int_a^b f_M \right) + \boxed{\varepsilon M} \text{ (by } \Leftarrow\text{)}$$

M 夠大, 此項 $< \frac{\eta}{2}$

$\because \int_a^b f_M \nearrow I$ as $M \nearrow \infty$

$\therefore$ For $\frac{\eta}{2}$, $\exists M \in \mathbb{N}$ s.t. $I - \int_a^b f_M < \frac{\eta}{2}$

$\Rightarrow$ Take $\delta = \frac{\eta}{2M}$, Then $\forall 0 < \varepsilon < \delta$, $\varepsilon M < \frac{\eta}{2}$

$$\left|\int_a^b f^\varepsilon - I\right| < \frac{\eta}{2} + \frac{\eta}{2M}\cdot M = \eta$$

$$\Rightarrow \lim_{\varepsilon\to 0^+}\int_a^b f^\varepsilon = I.$$

$$\Rightarrow \lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^b f = \int_a^b f.$$

$\Leftarrow$). Similarly

Q.E.D.

Def. Let $f:[a,\infty)\to\mathbb{R}$ be a function.
We say that $\int_a^\infty f(x)dx$ converges conditionally if $\lim_{b\to\infty}\int_a^b f(x)dx$ is finite

We thus write $\int_a^\infty f(x)dx$ (conditional)

Remark. Conditional convergence 不能 imply f is integrable (見下13.)

e.g. Show that $\int_1^\infty \frac{\sin x}{x}dx$ (conditional),

but $\int_1^\infty \frac{\sin x}{x}dx$ doesn't converge

(i.e. $\frac{\sin x}{x}$ is not integrable on $[1,\infty)$.)

-sol-. Let $b>1$. $\int_1^b \frac{\sin x}{x}dx = \frac{-\cos x}{x}\Big|_1^b - \int_1^b \frac{\cos x}{x^2}dx$

本32P(e存在)

Double A

$$= \cos 1 - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2}\,dx$$

$\because 0 \le \int_1^b \frac{|\cos x|}{x^2}\,dx \le \int_1^b \frac{1}{x^2}\,dx = 1 - \frac{1}{b}$,

$\forall b > 1$, and $\int_1^b \frac{|\cos x|}{x}\,dx \nearrow$ as $b \nearrow \infty$

$\therefore \lim_{b\to\infty} \int_1^b \frac{|\cos x|}{x}\,dx$ is finite

後面会再以一个 Thm 去證多何 $\lim_{b\to\infty} \int_1^b \frac{|\cos x|}{x}\,dx$ 存在会 imply $\int_1^b \frac{\cos x}{x}\,dx$ 極限存在

$\because \int_1^\infty \frac{|\sin x|}{x}\,dx \ge \int_\pi^{n\pi} \frac{|\sin x|}{x}\,dx$

$$= \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx$$

$$\ge \sum_{k=2}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} \sin x\,dx$$

$$= \sum_{k=2}^{n} \frac{-\cos x}{k\pi} \Big|_{(k-1)\pi}^{k\pi}$$

$$= \frac{2}{\pi} \sum_{k=2}^{n} \frac{1}{k}$$

and $\sum \frac{1}{k}$ diverges

$\therefore \int_1^\infty \frac{|\sin x|}{x}\,dx$ diverges

$\Rightarrow \frac{\sin x}{x}$ is not integrable on $[1, \infty)$ #

Thm. ① $\int_1^\infty x^p dx$ $\begin{cases} \text{converges, if } p<-1 \\ \text{diverges, if } p\geq -1 \end{cases}$

② $\int_0^a x^p dx$ $\begin{cases} \text{converges, if } p>-1 \\ \text{diverges, if } p\leq -1 \end{cases}$

③ $\int_1^\infty e^{-x}x^p dx$ converges for all p.

④ $\int_0^1 e^{-x}x^p dx$ diverges for all p

⑤ $\int_0^a \ln x\, dx$ converges

⑥ $\int_1^\infty \ln x\, dx$ diverges

⑦ $\int_1^\infty \frac{1}{\ln x} dx$ diverges.

Thm. Let $f: A\subset \mathbb{R}^n \to \mathbb{R}$ be a function.

① If f is integrable, then $|f|$ is integrable

② If $|f|$ is integrable and $f_{[-M,M]}$ is integrable, $\forall M>0$, then f is integrable

③ Suppose that f is bounded. If $|f|$ is integrable and f is intergrable on each cube $[-a,a]^n$, then f is integrable.

e.g. $\int_1^\infty \frac{|\cos x|}{x^2} dx$ converges

-sol- $\because \frac{\cos x}{x^2}$ is cont. on $[1,\infty)$

$\therefore \frac{\cos x}{x^2}$ is integrable on each cube $[-a, a]$

$\Rightarrow \int_1^\infty \frac{\cos x}{x^2} dx$ converges

$\Rightarrow \int_1^\infty \frac{\sin x}{x} dx$ (conditional) ※

要确定若可积才可再定义取积分

e.g. $0 \le \frac{1}{\sqrt{x^3+1}} \le \frac{1}{x^{\frac{3}{2}}}$

-sol-. $\because \frac{1}{\sqrt{x^3+1}}$ is cont. on $[1, \infty)$

$\therefore \frac{1}{\sqrt{x^3+1}}$ is integrable on each cube $[-a, a]$.

$\Rightarrow$ By comparison test A, $\int \frac{1}{\sqrt{x^3+1}} dx$ converges #

§8.6 Some Convergence Theorem

Q: 当 $f, f_n : A \subset \mathbb{R}^n \to \mathbb{R}$ s.t. $f_n \to f$. 什麼時候 $\int_A f_n \to \int_A f$ 呢?

lemma. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function on the set A with $v(A) > 0$ s.t. $|f| \le M$. If f is integrable s.t. $\int_A f \ge \alpha > 0$. Then the set

$E = \{ x \in A \mid f(x) \ge \frac{\alpha}{2v(A)} \}$ contains a finite union of interior nonoverlapping rectangles of total volume $l \ge \frac{\alpha}{4M}$.

$2v(A) \to$ 可以 $v(R)$

pf. $\because f$ is bounded and integrable on the brunded set A

$\therefore$ For $\frac{\alpha}{4}$, $\exists$ a partition P of R

s.t. $0 \le \int_A f - L(f, P) \le \frac{\alpha}{4}$

$\because \int_A f \ge \alpha$

$\therefore \alpha - L(f, P) \le \frac{\alpha}{4}$

$\Rightarrow L(f, P) \ge \frac{3}{4}\alpha$

$\because L(f, P) = \sum_{S \in P \text{ s.t. } S \subset E} \underbrace{m_S(f)}_{\le M} v(S) + \sum_{S \in P \text{ s.t. } S \not\subset E} \underbrace{m_S(f)}_{\le \frac{\alpha}{2v(A)}} v(S)$

Let l be the total volume of those

rectangles $S \in P$ s.t. $S \subset E$.

$$\therefore L(f,P) \le \mu l + \frac{\alpha}{2V(A)}(V(R)-l)$$

$$= \mu l + \frac{\alpha V(R)}{2V(A)}$$

$$\Rightarrow \mu l \ge \frac{3}{4}\alpha - \frac{\alpha V(R)}{2V(A)}$$

$$\Rightarrow l \ge \frac{\alpha}{4\mu}\left(3 - \frac{2V(R)}{V(A)}\right)$$

定数. 可找一足够形状R使这一项 <1.

Note: 当 $A=R$, $l \ge \frac{\alpha}{4\mu}$

Q.E.D.

Thm. (Lebesgue's Monotone convergence Thm)
Let $g_n : A \subset \mathbb{R}^n \to \mathbb{R}$ be a sequence of nonnegative integrable functions on the set A with $V(A) > 0$. Suppose that $0 \le g_{n+1} \le g_n$, $\forall n \in \mathbb{N}$ and $g_n \to 0$ on A (i.e. $g_n \downarrow 0$ on A). Then $\int_A g_n \to 0$.

先来看上方lemma的(更精细)的 pf.

$$L(f,P) = \sum_{S\in P \text{ s.t. } S\subset E} \underbrace{m_S(f)}_{\le M} V(S) + \sum_{\substack{S\in P \text{ s.t.} \\ S\subset A \text{ but } S\not\subset E}} \underbrace{m_S(f)}_{\le \frac{\alpha}{2V(A)}} V(S) + \sum_{S\in P \text{ s.t. } S\not\subset A} \underbrace{m_S(f)}_{\le 0} V(S)$$

$$= \sum_{S\in P, S\subset E} m_S(f)V(S) + \sum_{S\in P \text{ s.t. } S\not\subset A, S\not\subset E} m_S(f)V(S)$$

$$\le \mu l + \frac{\alpha}{2V(A)}(V(A)-l)$$

$$\le \mu l + \frac{\alpha}{2}$$

$\Rightarrow \frac{3}{4}\alpha \leq M\ell + \frac{\alpha}{2}$

$\Rightarrow \ell \geq \frac{\alpha}{4M}$ ※

pf of Thm. $\because g_{n+1} \leq g_n \quad \forall n \in \mathbb{N}$

$\therefore g_n \leq g_1, \quad \forall n \in \mathbb{N}$.

$\Rightarrow \int_A g_n \searrow$ as $n \nearrow \infty$ and $\int_A g_n \geq 0$.

$\forall n \in \mathbb{N}$.

$\Rightarrow \lim_{n\to\infty} \int_A g_n$ exists, say λ, and $\lambda \geq 0$.

Assume $\lambda > 0$. Then $\int_A g_n \geq \lambda > 0$. (想要矛盾) Let $M > 0$. Notice that $0 \leq g_n - (g_n)_M \leq g_1 - (g_1)_M$

$\because g_n$ is integrable

$\therefore (g_n)_M$ is integrable

$\Rightarrow g_n - (g_n)_M$ is integrable

$\Rightarrow 0 \leq \int_A (g_n - (g_n)_M)$

$\leq \int_A (g_1 - (g_1)_M)$

$= \int_A g_1 - \int_A (g_1)_M$

$\Rightarrow \lim_{M\to\infty} \int_A (g_1 - g_{1M}) = 0$

$\Rightarrow \exists K \in \mathbb{N}$ s.t. $\forall M > K$, $\int_A (g_n - (g_n)_M) < \frac{\lambda}{5}$

$\Rightarrow \forall m > K$; $\int_A g_n - \int_A (g_n)_M < \frac{\lambda}{5}$ ($\int_A g_n \geq \lambda$)

Double A

$\Rightarrow \forall M > K, \int_A (g_n)_M \geq \frac{4}{5}\lambda$

Let $M > \max(K, \frac{2\lambda}{5V(A)})$ and $E_n = \{x \in A \mid (g_n)_M(x) \geq \frac{2\lambda}{5V(A)}\}$

$\forall n \in \mathbb{N}$.

$\because g_n \downarrow$

$\therefore E_{n+1} \subset E_n, \forall n \in \mathbb{N}$.

$\because (g_n)_M$ is bounded by M and

$\therefore$ By lemma, E_n contains a finite union of interior nonoverlapping rectangles in $\mathbb{R}^n$ of total volume $\geq \frac{\lambda}{5M}, \forall n \in \mathbb{N}$.

Let $D = \bigcup_{n=1}^{\infty} \{x \in A \mid g_n \text{ is not cont. at } x\}$

$\because g_n$ is integrable, $\forall n \in \mathbb{N}$

$\therefore U_D$ is measure zero.

$\Rightarrow D$ is contained in the union U of countable open rectangles in $\mathbb{R}^n$ with total volume $< \frac{\lambda}{5M}$.

$\Rightarrow E_n \not\subset U, \forall n \in \mathbb{N}$

Let $x \in cl(E_n) \setminus E_n$, then $(g_n)_M(x) < \frac{2\lambda}{5V(A)}$

$\Rightarrow (g_n)_M$ is not cont. at $x, \forall n \in \mathbb{N}$

$\Rightarrow g_n$ is not cont. at x

$\Rightarrow x \in D \subset U$.

$\Rightarrow cl(E_n) \subset E_n \cup U, \forall n \in \mathbb{N}$

不在E_n裡就會在U裡

Let $F_n = cl(E_n) \setminus U = E_n \setminus U$. Then F_n is

Double A

a nonempty closed and bounded set, $\forall n \in \mathbb{N}$

$\Rightarrow F_n$ is nonempty compact set, $\forall n \in \mathbb{N}$.

$\because E_{n+1} \subset E_n$

$\therefore F_{n+1} \subset F_n$

$\Rightarrow$ By nested set property,

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

$$\Rightarrow \bigcap_{n=1}^{\infty} E_n \neq \emptyset.$$

$$\Rightarrow \exists\ x_0 \in \bigcap_{n=1}^{\infty} E_n \subset A \quad \text{s.t. } (g_n)_\mu(x) \geq \frac{2\lambda}{5v(A)},$$

$\forall n \in \mathbb{N}$.

$\because g_n \geq (g_n)_\mu$

$\therefore g_n(x_0) \geq \frac{2\lambda}{5v(A)}$, $\forall n \in \mathbb{N}$

$\Rightarrow g_n(x_0) \not\to 0$

$\Rightarrow g_n \not\to 0$ $\quad \to\leftarrow$

$\Rightarrow \lambda = 0$

Q.E.D.

Cor 1. Let $f_n, f : A \subset \mathbb{R}^n \to \mathbb{R}$ be nonnegative integrable functions on the set A with $v(A) > 0$. Suppose that $0 \leq f_n \leq f_{n+1} \leq \dots \leq f$, $\forall n \in \mathbb{N}$ and $f_n \to f$ (i.e., $f_n \nearrow f$ on A). Then $\int_A f_n \to \int_A f$

pf. $\because f_n \nearrow f$ and f_n and f are integrable.

$\therefore$ $0 \le f - f_n \searrow 0$ and $f - f_n$ are integrable

$\because$ $v(A) > 0$

$\therefore$ By Lebesgue's monotone convergence thm, $\int_A (f - f_n) \to 0$

$\Rightarrow \int_A f - \int_A f_n \to 0$

$\Rightarrow \int_A f_n \to \int_A f$.

Q.E.D.

Remark. 由 Lebesgue's monotone convergence Thm + Cor1 可知道 nonnegative monotone sequence of integrable functions on the set A with $v(A) > 0$. 之積分有收斂性的充份條件為：

① 对 decreasing sequence，要遞減收斂到零函数

② 对 increasing sequence，limit function 可積

Cor2. If $f: A \subset \mathbb{R}^n \to \mathbb{R}$ is a nonnegative function on the set A with volume $v(A) > 0$ s.t. $\int_A f^2$ converges, then $\int_A (f - f_\mu)^2 \to 0$ as $\mu \to \infty$

pf. $(f - f_\mu)^2 = f^2 - 2 f f_\mu - (f_\mu)^2$

$\because f \ge 0$

$\therefore (f_\mu)^2 = (f^2)_{\mu^2}$.

$\therefore f^2$ is integrable

$\Rightarrow (f^2)_M$ is integrable, $\forall M>0$

$\Rightarrow (f_M)^2$ is integrable

$\because (f_M)^2$ is bounded and $bd(A)$ is measure zero

$\therefore f_M = \sqrt{(f_M)^2}$ is integrable by Lebesgue's Thm

$\because f^2$ is integrable

$\therefore$ The set of discontinuities of the extended f^2 is measure zero

$\because f \geq 0$

$\therefore$ The set of discontinuities of the extended f is also measure zero.

$\Rightarrow (ff_M)_M$ is integrable on bounded set A by Lebesgue's Thm.

$\because 0 \leq ff_M \leq f \cdot f = f^2$ and f^2 is integrable

$\therefore$ By amparison test B, ff_M is integrable

$\Rightarrow (f-f_M)^2$ is integrable, $\forall M>0$.

Notice that $0 \leq (f-f_M)^2 \searrow 0$ as $M \nearrow$

$\therefore 0 \leq \int_A (f-f_M)^2 \searrow$ as $M \nearrow \infty$

$\therefore \lim_{M\to\infty} \int_A (f-f_M)^2$ is finite.

On the other hand, by Lebesgue's monotone convergence thm,

$\int_A (f-f_N)^2 \to 0$, as $N \in \mathbb{N} \to \infty$

$\Rightarrow \int_A (f-f_\mu)^2 \to 0$ as $\mu \to \infty$

Q.E.D.

e.g. Prove that $\lim_{n\to\infty} \int_0^1 e^{-nx^2} x^p \, dx = 0$

if $p > -1$

-sol-. Consider $e^{-nx^2}x^p$ on $(0,1]$.

$0 \le e^{-nx^2}x^p \searrow 0$ as $n \to \infty$ on $(0,1]$.

Notice that $e^{-nx^2}x^p \to \infty$ as $x \to 0^+$ for $-1 < p < 0$

$\therefore (e^{-nx^2}x^p)$ is cont. on $(0,1]$

$\therefore (e^{-nx^2}x^p)_\mu$ is cont. on $(0,1]$

$\therefore (e^{-nx^2}x^p)_\mu$ is bounded and $(0,1]$ has volume

$\therefore$ By Lebesgue's thm, $(e^{-nx^2}x^p)_\mu$ is integrable.

Notice that $0 \le e^{-nx^2}x^p < x^p$ on $(0,1]$

$\therefore p > -1$

$\therefore \int_0^1 x^p \, dx$ converges

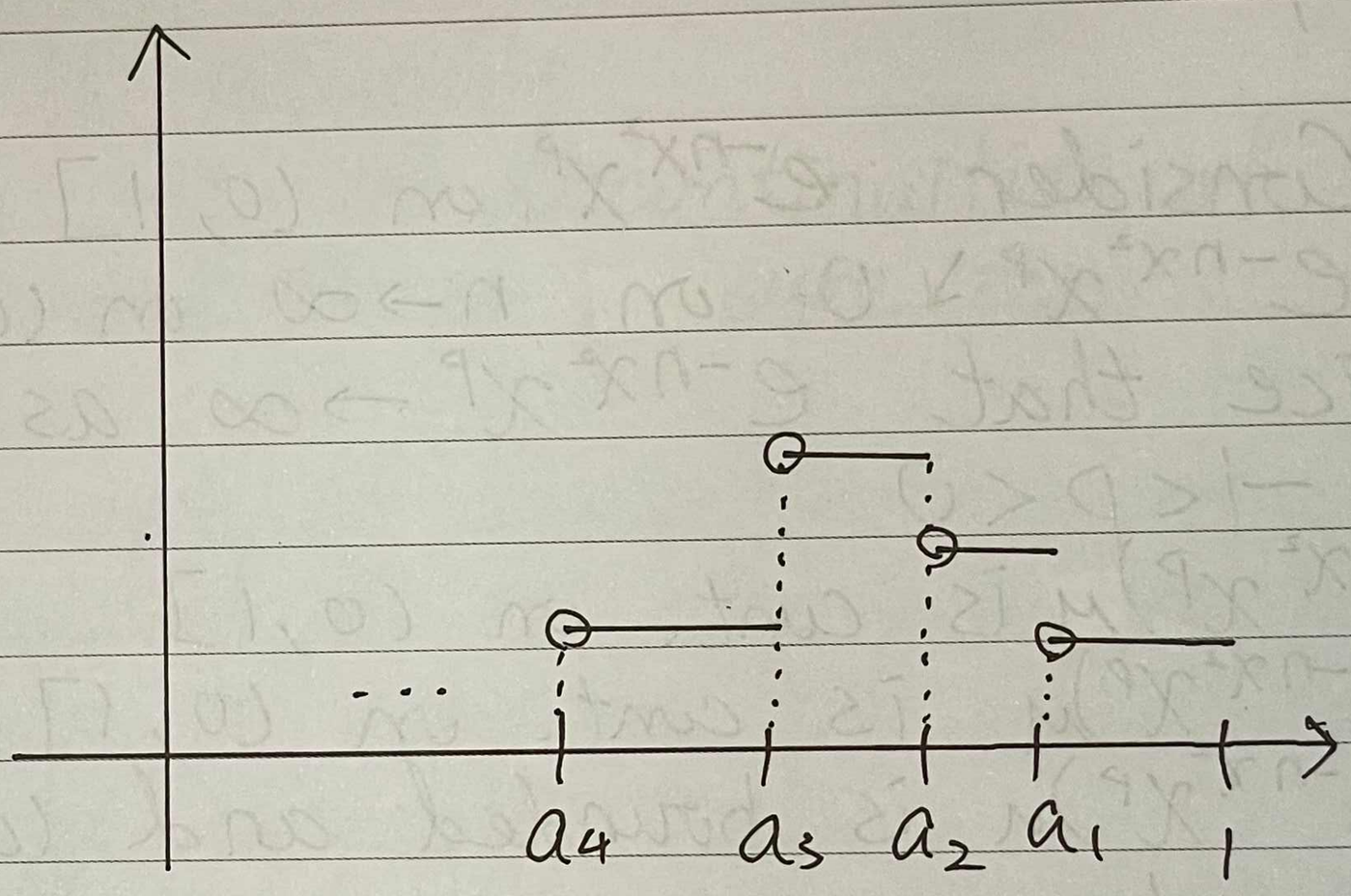
$\Rightarrow$ By comparison test B, $e^{-nx^2}x^p$ is integ. (and $\int_0^1 e^{-nx^2}x^p \, dx$ converges)

By Lebesgue's monotone convergence Thm,

$\int_0^1 e^{-nx^2}x^p \, dx \to 0$ #

step function with infinitely many steps

e.g. Let $a_1=1$ and $1 \geq a_i \geq 0$, $i=1,2,\dots$, be a decreasing seq. with $a_i \to 0$
Let $f_i \geq 0$ be a seq. of positive numbers and let $f(x)=f_i$ on the interval $(a_{i+1}, a_i]$



(Assume f is integrable) show that

$$\int_0^1 f(x)\,dx = \sum_{i=1}^{\infty} f_i(a_i - a_{i+1})$$

-sol-. Let $f_n(x) = \begin{cases} f(x), & \text{if } x > a_n \\ 0, & \text{if } 0 \leq x \leq a_n \end{cases}$

Then $0 \leq f_n \leq f_{n+1} \nearrow f$ as $n \to \infty$
$\because f_n, f$ are integrable (by Lebesgue's Thm)
$\therefore$ By Lebesgue's monotone convergence Thm, $\int_0^1 f_n(x)\,dx \longrightarrow \int_0^1 f(x)\,dx$

$$\because \int_0^1 f_n(x)\,dx = \sum_{i=1}^{n} f_i(a_i - a_{i+1})$$

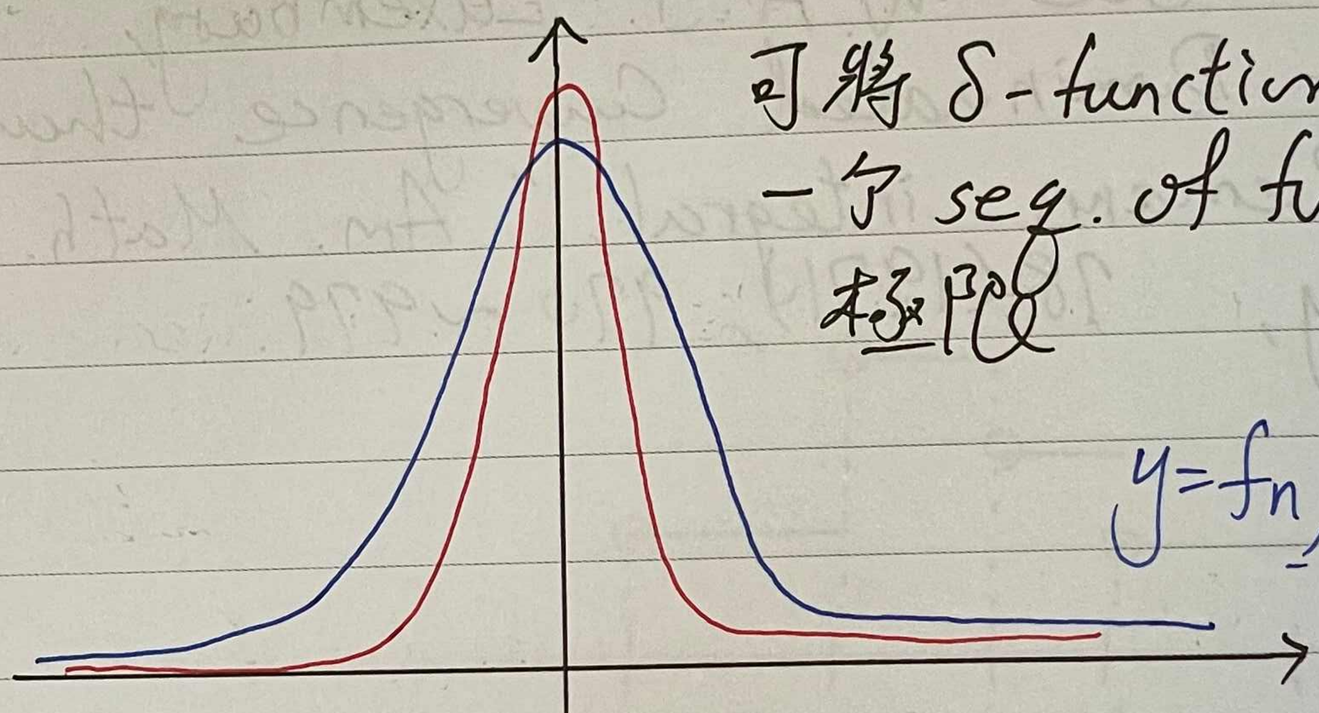
$$\therefore \int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{\infty} f_i(a_i - a_{i+1}) \quad \#$$

Remark. For a more complete discussion of convergence theorems in Riemann theory, see W. A. J. Luxemburg, "Arela's Dominated Convergence theorem for Riemann integral," Am. Math. Monthly, 78 (1971), 970 ~ 979.

§8.1 Introduction to distributions

Dirac, 1930, "Principles of Quantum mechanics"
→ δ-function is a function 具有這樣的性質:

$$\delta(x)=\begin{cases}0, & \text{if } x\neq 0\\ \infty, & \text{if } x=0\end{cases} \quad \text{and} \quad \int \delta(x)\,dx=1$$



a point charge 帶电荷 e, 其电荷密度 charge density $\sigma=e\delta$

但对数学家来說, δ之存在極不合理, 更不要說 Dirac 还对 δ 做微分。但這是一个很好的想法

如: $\int \delta(x)f(x)\,dx = f(0)\int \delta(x)\,dx = f(0)$

⇒發展出 distribution theory.

$\delta: C_b(\mathbb{R},\mathbb{R})\to\mathbb{R}$ is defined by

$\delta(f)=f(0)$. —(*)

我們知道 $C_b(\mathbb{R},\mathbb{R})$ 是一个 normed space，考虑上面的 sup norm，則在 sup norm 之下，δ map 是 anti.，且因是取 $f(0)$，δ 本身也是 linear

用 $\int \delta(x) f(x)\, dx = f(0)$ 来表示 $\delta(x)$

$g: C_b(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ 定義為 $f \mapsto \int_{-\infty}^{\infty} g(x) f(x)\, dx$

若此定義 well-defined，可知此 map 在 sup norm 之下連续.

那如何对 δ 微分呢？這想法来自 integration by part

If g is diff, $\boxed{\int_{-\infty}^{\infty} g(x) f(x)\, dx}$ $= -\int_{-\infty}^{\infty} g(x) f'(x)\, dx$

（假設存在 / 希望）

① 前面那項 (integ. by part 出來的第一項) 是 0

② 後 〃 存在

provideed that f is zero for large $|x|$.
and f is C^1

$$\Rightarrow \frac{dg}{dx}(f) = g\left(-\frac{df}{dx}\right)$$

所以 distribution 是一个 map，δ 是一个 distribution

δ' 也应該是

$$\to \delta'(f) = \delta(-f') = -f'(0).$$

MON TUE WED THU FRI

细节 ↓

Def. Let D denote the space of C^∞ functions that are identically zero outside some interval (D is called the space of test functions).
A distribution T is a linear map $T: D \to \mathbb{R}$ (Strictly speaking, one must require T to be cont. in the sense that if $f_n \to f$ uniformly on bounded sets and all derivatives of f_n converges uniformly to those of f on bounded sets and all the f_n vanish outsided fixed compact set, then $T(f_n) \to T(f)$).
The derivative T' is a function from D to R defined by $T'(f) = T(-f')$

Ch9. Fubini's Theorem and the Change of Variables Formula.

§9.2 Fubini's Thm

Thm. (Fubini's Thm)

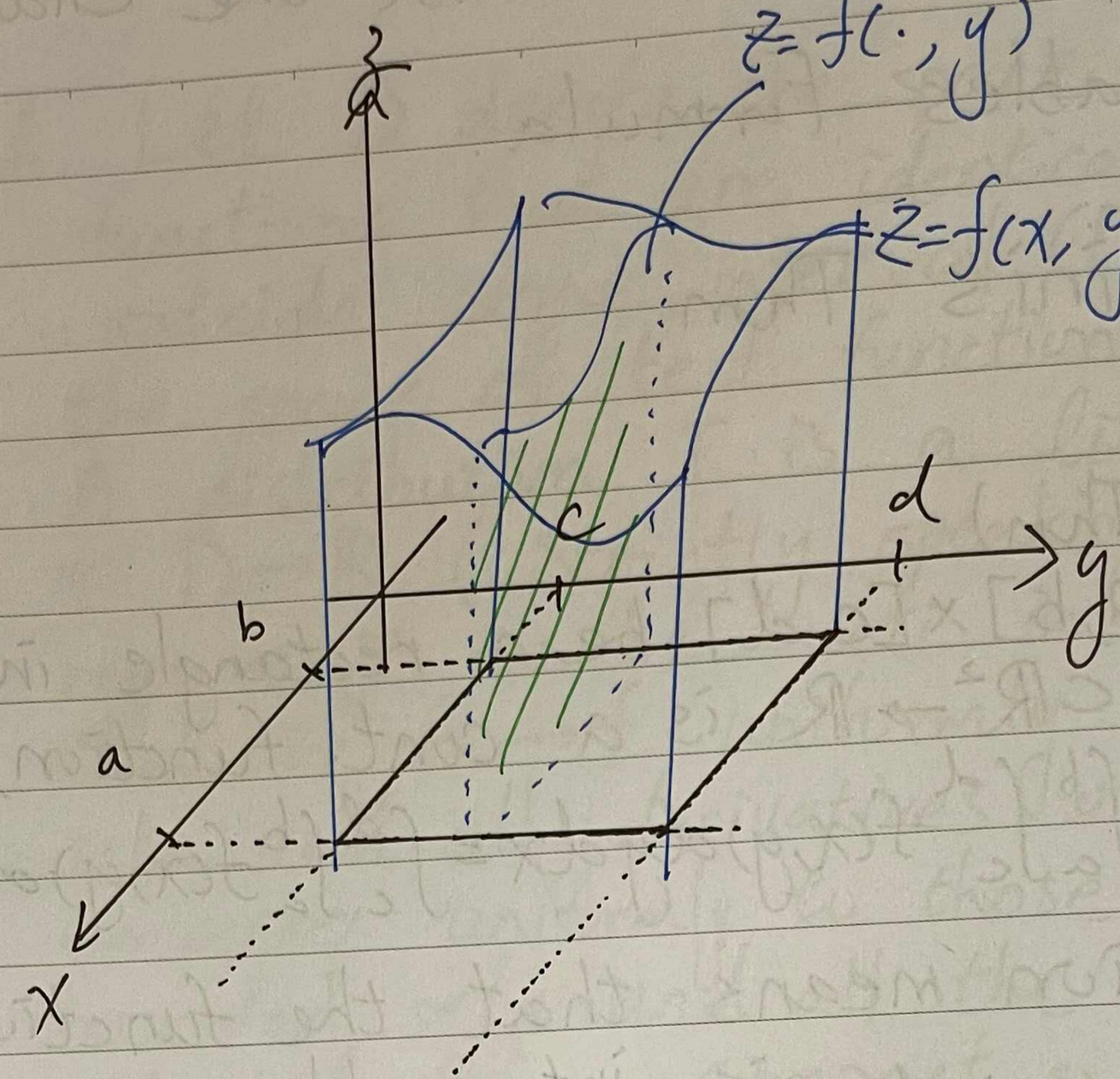
① Let $A=[a,b]\times[c,d]$ be a rectangle in $\mathbb{R}^2$. If $f: A\subset\mathbb{R}^2\to\mathbb{R}$ is a cont. function, then $\int_A = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$

The expression means that the function $g(x)=\int_c^d f(x,y)\,dy$ is integrable over $[a,b]$.

② Suppose that $f: A\subset\mathbb{R}^2\to\mathbb{R}$ is an integrable functim. If the function $f:[c,d]\to\mathbb{R}$ defined by $f_x(y)=f(x,y)$ is integrable for each fixed $x\in[a,b]$, then $\int_A f=\int_a^b\int_c^d f(x,y)\,dy\,dx$.

One can similarly assume that $\int_a^b f(x,y)\,dx$ exists for $y\in[c,d]$ and obtains $\int_A=\int_c^d\int_a^b f(x,y)\,dx\,dy$.

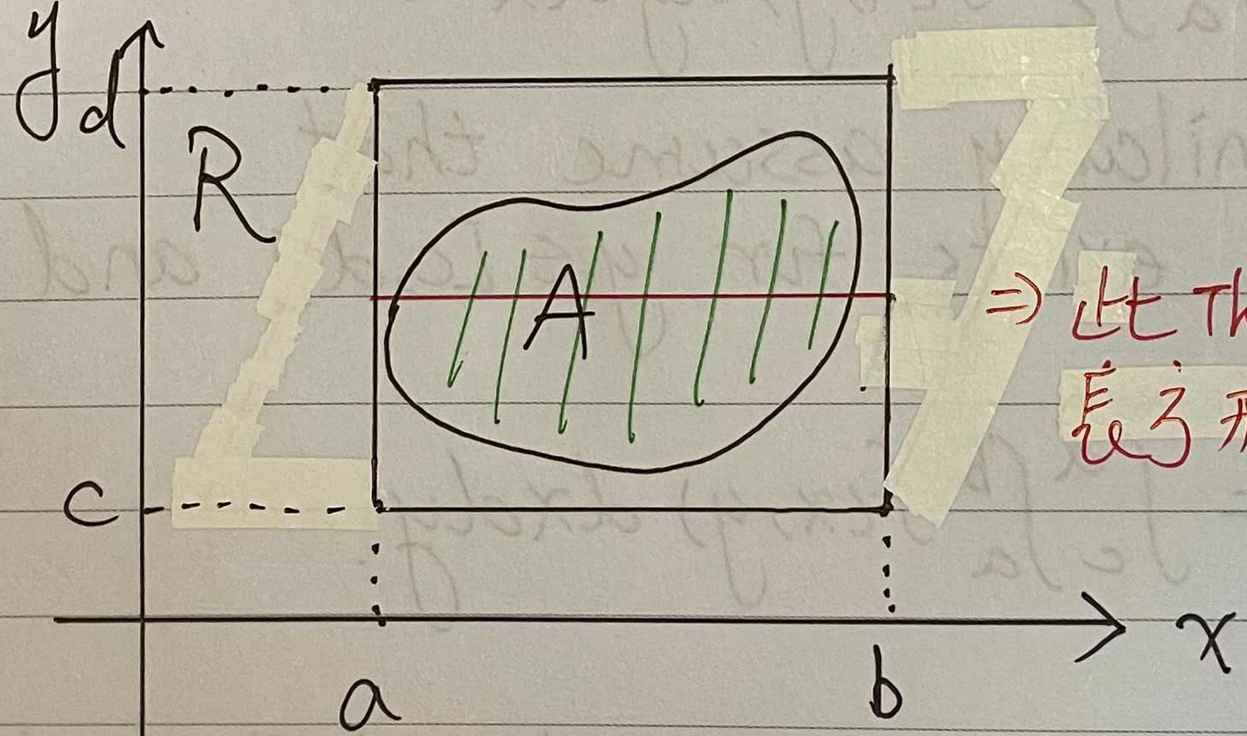
Note: $\int_A f$ 表 f 的图形与 A 所夹之广义面积

① 

$$\text{体積} \approx \sum \left(\int_a^b f(x,y)\,dx \right) \triangle y$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

② A is not rectangle but is bounded and closed.



⇒ 此 Thm 可用在不只長方形 domain.

pf.
by
g
p
T
L
S
F
or
⇒

pf. ② ⇒ ①.

Let $g:[a,b]\to\mathbb{R}$ be a function defined by $g(x)=\int_c^d f(x,y)\,dx$. To show that g is integrable over $[a,b]$ st. $\int_a^b g(x)\,dx=\int_A f$. Let $P_{[a,b]}=\{a=x_0<x_1<\cdots<x_n=b\}$ and $P_{[c,d]}=\{c=y_0<y_1<\cdots<y_m=d\}$ be partitions of $[a,b]$, $[c,d]$ respectively. Then $P_A=P_{[a,b]}\times P_{[c,d]}$ be a partition of A.

Let $V_i=[x_{i-1},x_i]$ and $W_j=[y_{j-1},y_j]$, and $S_{ij}=[x_{j-1},x_i]\times[y_{j-1},y_j]$, $\forall i=1,\dots,n$, $j=1,\dots,m$

For fixed i, for any $x\in[x_{i-1},x_i]$,

$$m_{S_{ij}}(f)\le m_{W_j}(f_x)$$

$$\Rightarrow \sum_{j=1}^{m} m_{S_{ij}}(f)\,v(W_j)\le \sum_{j=1}^{m} m_{W_j}(f_x)\,v(W_j)$$

$$= L(f_x,P_{[c,d]})$$

$$\le \int_c^d f_x(y)\,dy$$

$$= g(x)$$

$\because x\in[x_{i-1},x_i]$

$$\therefore \sum_{j=1}^{m} m_{S_{ij}}(f)\,v(W_j)\le m_{V_i}(g)$$

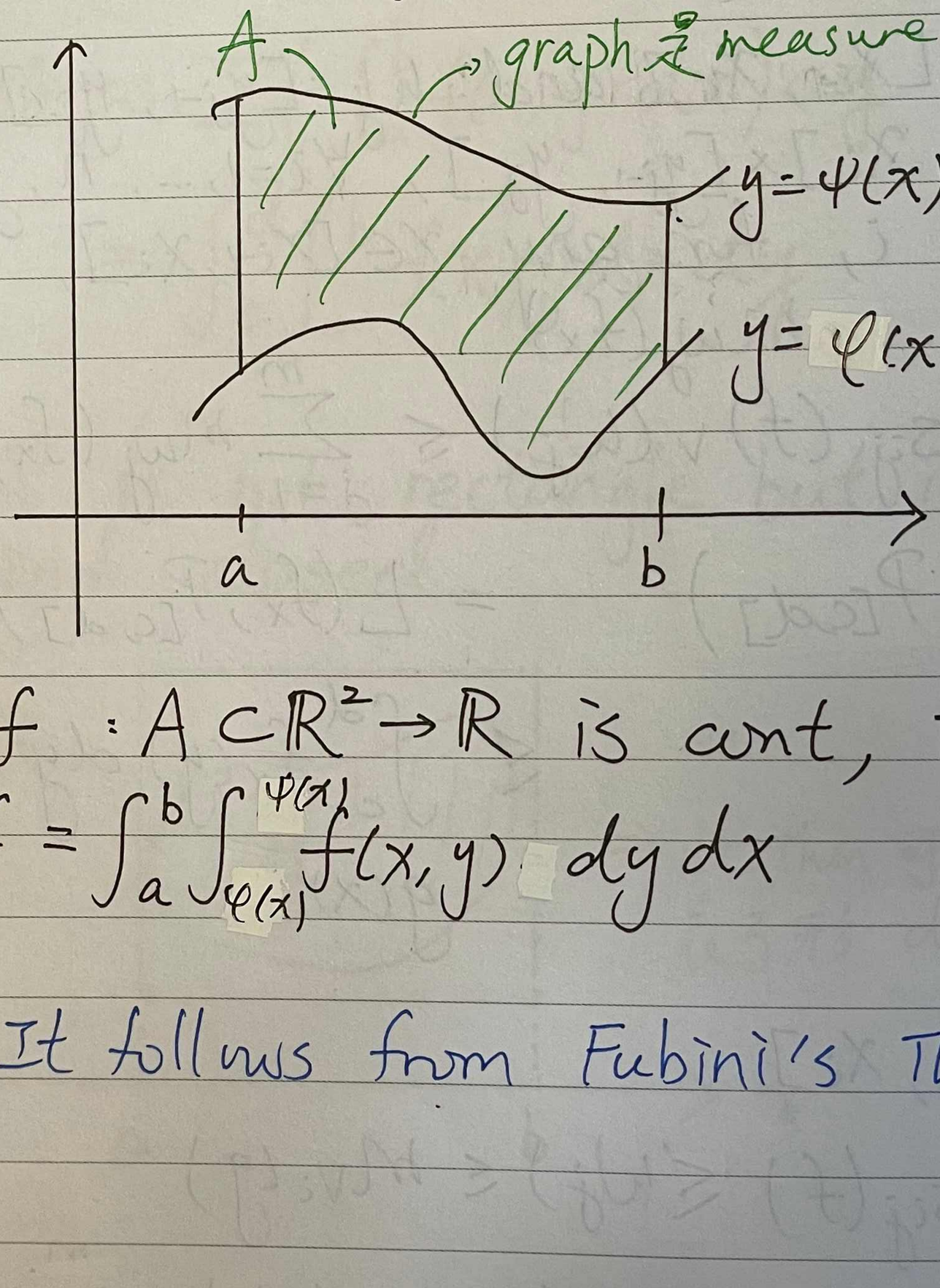
$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{m} m_{S_{ij}}(f)\,\underbrace{v(W_j)\,v(V_i)}_{v(S_{ij})}\le \sum_{i=1}^{m} m_{V_i}(g)\,v(V_i)$$

$\Rightarrow L(f, P_A) \leq L(g, P_{[a,b]})$

Similarly, $U(g, P_{[a,b]}) \leq U(f, P_A)$. So,
$L(f, P_A) \leq L(g, P_{[a,b]}) \leq U(g, P_{[a,b]}) \leq U(f, P_A)$
$\Rightarrow g$ is integrable and $\int_a^b f(x)dx = \int_A f$

Q.E.D.

Cor. Let $\varphi, \psi : [a,b] \to \mathbb{R}$ be cont. function s.t. $\varphi(x) \leq \psi(x)$, $\forall x \in [a,b]$, and let $A = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, \varphi(x) \leq y \leq \psi(x)\}$



If $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ is cont, then
$\int_A f = \int_a^b \int_{\varphi(x)}^{\psi(x)} f(x,y)\, dy\, dx$

pf. It follows from Fubini's Thm (2).

Remark. ① There is an analogous corollary with the role of x and y interchanged

② Fubini's idea of computing double integrals can be extended to multiple integrals as follows.

Thm. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be rectangles.

① Let $f: A \times B \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be cont. Define, for each $x \in A$ $f_x: B \subset \mathbb{R}^m \to \mathbb{R}$ by $f_x(y) = f(x,y)$. Then

$$\int_{A\times B} f = \int_A \int_B f(x,y)\, dx\, dy$$

② If f is integrable and f_x is integrable for each fixed $x \in A$, then

$$\int_{A\times B} f = \int_A \int_B f(x,y)\, dx\, dy$$

Note: In practice, this Thm may be used repeatedly to reduce a multiple integral to iterated one-dimensional integrals.

§9.3 Change of Variable Thm

Thm. (Change of variable Thm)
Let $A \subseteq \mathbb{R}^n$ be an open bounded set with volume and $g: A \to \mathbb{R}^n$ be a C^1 mapping that is 1-1. Assume $J_g(x) \neq 0$, $\forall x \in A$ (由 inverse function Thm 可知 $g: A \to \mathrm{Im}(A)$ is a C^1-diffeomurphism, 因此 g 可以看成 A 与 Im(A) 在 $\mathbb{R}^n$ 中的座标变换) and $|J_g(x)|$ and $\frac{1}{|J_g(x)|}$ are bounded on A. Let $B = g(A)$ and assume that B has volume. If $f: B \to \mathbb{R}$ is bounded and integrable, then $(f \circ g)|J_g|$ is integrable on A and

$$\int_B f = \int_A (f \circ g)|J_g|$$

That is,

$$\int_B f(y_1, \dots, y_n)\, dy_1 \cdots dy_n = \int_A f(g(x_1, \dots, x_n)) \left| \frac{\partial(g_1, \dots, g_n)}{\partial(x_1, \dots, x_n)} \right| dx_1 \cdots x_n$$

where $g = (g_1, \dots, g_n)$

pf. (Too long, omit)

Remark. ① 如 polar coordinate, $|J_g(x)| = r$.
Spherical coordinate, $|J_g(x)| = \rho^2 \sin\phi$
② f, $|J_g|$ and $1/|J_g|$ need not be assumed bounded ⇒ improper integral 亦可用 (i.e. 此定理在 improper integrals 之下亦成立)

MON TUE WED THU FRI SAT SUN DATE: / / NO.:

* 9.6

* 9.6, 9.7 自行閱讀

Ch 10 Fourier Analysis

§ 1-8 (Review) The complex numbers

Prop. The seq. $z_n \to z$ in $(\mathbb{C}, |\cdot|) \iff Re(z_n) \to Re(z)$ and $Im(z_n) \to Im(z)$.

pf. hint: $\max\{|Re(z)|, |Im(z)|\} \le |z| \le \sqrt{2}\max\{|Re(z)|, |Im(z)|\}$ → exercise

Cor. $(\mathbb{C}, |\cdot|)$ is complete

Prop. Let $f: A \subset (M, d) \to (C, |\cdot|)$ be a function, Then f is cont. at $x \in A \iff Ref$ and Imf are cont at $x \in A$
($f = Re(f) + i\,Im(f)$)

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§10.1 The inner product space

Def. Let V be a complex vector space; that is, a vector space over $\mathbb{C}$. An inner product on V is a mapping $\langle , \rangle : V \times V \to \mathbb{C}$ with the following properties:

(i) $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$, $\forall f, g, h \in V$, $\alpha, \beta \in \mathbb{C}$.

(ii) $\langle f, g \rangle = \overline{\langle g, f \rangle}$, $\forall f, g \in V$

(iii) $\langle f, f \rangle \geq 0$, $\forall f \in V$ and $\langle f, f \rangle = 0$ implies $f = 0$.

Note: ① $\langle h, \alpha f + \beta g \rangle = \overline{\langle \alpha f + \beta g, h \rangle}$

$= \overline{\alpha \langle f, h \rangle + \beta \langle g, h \rangle}$

$= \overline{\alpha} \overline{\langle f, h \rangle} + \overline{\beta} \overline{\langle g, h \rangle}$

$= \overline{\alpha} \langle h, f \rangle + \overline{\beta} \langle h, g \rangle$

(i.e. 对第二个位置是 conjugate linear)

② We say that f and g are orthogonal if $\langle f, g \rangle = 0$

Def. ① An inner product space $(V, \langle , \rangle)$ is a complex vector space V equipped with an inner product $\langle , \rangle$ on it.

② Define the norm $\|f\|$ by $\|f\| = \sqrt{\langle f, f \rangle}$, $\forall f \in V$

Prop. (Cauchy-Schwarz ineq.)
Let $f, g \in V$, then $|\langle f, g\rangle| \le \|f\|\|g\|$

pf. If $g=0$, then the result holds.
If $g \neq 0$, then $\|g\| \neq 0$

$$0 \le \|f - \langle f, g\rangle g\|^2$$
$$= \langle f - \langle f, g\rangle g,\ f - \langle f, g\rangle g\rangle$$
$$= \langle f, f\rangle - \overline{\langle f, g\rangle}\langle f, g\rangle - \langle f, g\rangle\overline{\langle f, g\rangle} + \langle f, g\rangle\overline{\langle f, g\rangle}\langle g, g\rangle$$
$$= \|f\|^2 - 2|\langle f, g\rangle|^2 + |\langle f, g\rangle|^2\|g\|^2$$

Notice that $\|g\| = 1$, that $0 \le \|f\| - |\langle f, g\rangle|^2$.
$\Rightarrow |\langle f, g\rangle| \le \|f\| = \|f\|\|g\|$

$\because \left\| \frac{g}{\|g\|} \right\| = 1$

$\therefore \left| \left\langle f, \frac{g}{\|g\|} \right\rangle \right| \le \|f\| \left\| \frac{g}{\|g\|} \right\|$

$\Rightarrow |\langle f, g\rangle| \le \|f\|\|g\|$

Q.E.D.

Cor. $\|\cdot\| = \sqrt{\langle\ ,\ \rangle}$ is a norm on the complex vector space V. Hence $(V, \|\cdot\|)$ is a norm space

pf. $\|f+g\|^2 = \|f\|^2 + \langle f, g\rangle + \langle g, f\rangle + \|g\|^2$
$\le \|f\|^2 + 2|\langle f, g\rangle| + \|g\|^2$

$$\leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2$$
$$= (\|f\| + \|g\|)^2$$

Thus $\|f+g\| \leq \|f\| + \|g\|$

Q.E.D.

Prop. Let $(V, \langle , \rangle)$ be an inner product space

① (Pythogurean Thm). If $\langle f, g \rangle = 0$, then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$

② (Polarization identity)

$$4\langle f, g \rangle = (\|f+g\|^2 - \|f-g\|^2) + i(\|f+ig\|^2 - \|f-ig\|^2)$$

③ (Parallelegram law)

$$2\|f\|^2 + 2\|g\|^2 = \|f+g\|^2 + \|f-g\|^2$$

④ $\|f+g\|\|f-g\| \leq \|f\|^2 + \|g\|^2$

pf. Exercise

Thm. The inner product $\langle , \rangle : V \times V \to \mathbb{C}$ is cont., where $V \times V$ is the product normed space

pf. <法I> By polarization identity.

<法II> Let $(f, g), (f_n, g_n) \in V \times V$ s.t. $\langle f_n, g_n \rangle \to \langle f, g \rangle$.

Let $\varepsilon > 0$,

$$|\langle f_n, g_n \rangle - \langle f, g \rangle| = |\langle f_n, g_n \rangle - \langle f_n, g \rangle| +$$

by C-R ineq.

$$|\langle f_n, g\rangle - \langle f_n, g_n\rangle| \leq \|f_n\|\|g_n - g\| + \|f_n - f\|\|g\|$$

$\because (f_n, g_n) \to (f, g)$

$\therefore f_n \to f$ and $g_n \to g$ in V.

Q.E.D.

Def. Let $f: [a,b] \to \mathbb{C}$ be a function. We say that f is integrable over $[a,b]$, if $Re\{f\}$ and $Im(f)$ are integ. over $[a,b]$, and we define

$$\int_a^b f(x)\,dx = \int_a^b Re(f)(x)\,dx + i\int_a^b Im(f)(x)\,dx$$

Prop. If $f, g: [a,b] \to \mathbb{C}$ are integ. functions and $\alpha, \beta \in \mathbb{C}$. Then

① f is integ. over $[a,b]$ and $\int_a^b \overline{f(x)}\,dx$

② $\alpha f + \beta g$ is integ. and $\int_a^b \alpha f + \beta g = \alpha\int_a^b f + \beta\int_a^b g$

③ $\int_a^b e^{i\alpha x}\,dx = \frac{1}{i\alpha} e^{i\alpha x}\Big|_a^b$, $\forall \alpha \in \mathbb{R}$

(可將 $e^{i\alpha x}$ 看成 e^x)

④ $\frac{d}{dx}(e^{i\alpha x}) = i\alpha e^{i\alpha x}$, $\forall \alpha \in \mathbb{R}$.

pf. ③ $\int_a^b e^{i\alpha x}dx = \int_a^b \cos(\alpha x)dx + i\int_a^b \sin(\alpha x)dx$ ∥

Thm A. The space V of cont. functions $f: [a,b] \to \mathbb{C}$. forms an inner product space if we define

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$$

pf. ① V is obvious a complex vector space

② $\langle \alpha f + \beta g, h\rangle = \int_a^b [\alpha f(x) + \beta g(x)]\overline{h(x)}dx$

$= \alpha\langle f, h\rangle + \beta\langle g, h\rangle$

③ $\overline{\langle f, g\rangle} = \overline{\int_a^b f(x)\overline{g(x)}dx}$

$= \int_a^b \overline{f(x)}\, g(x)dx$

$= \langle g, f\rangle$

④ $\langle f, f\rangle = \int_a^b |f(x)|^2 dx \geq 0$.

If $\int_a^b |f(x)|^2 dx = 0$. ∵ f is conti.

∴ $|f|^2 \geq 0$ is conti.

$\Rightarrow |f|^2 \equiv 0$

$\Rightarrow f \equiv 0$.

Q.E.D.

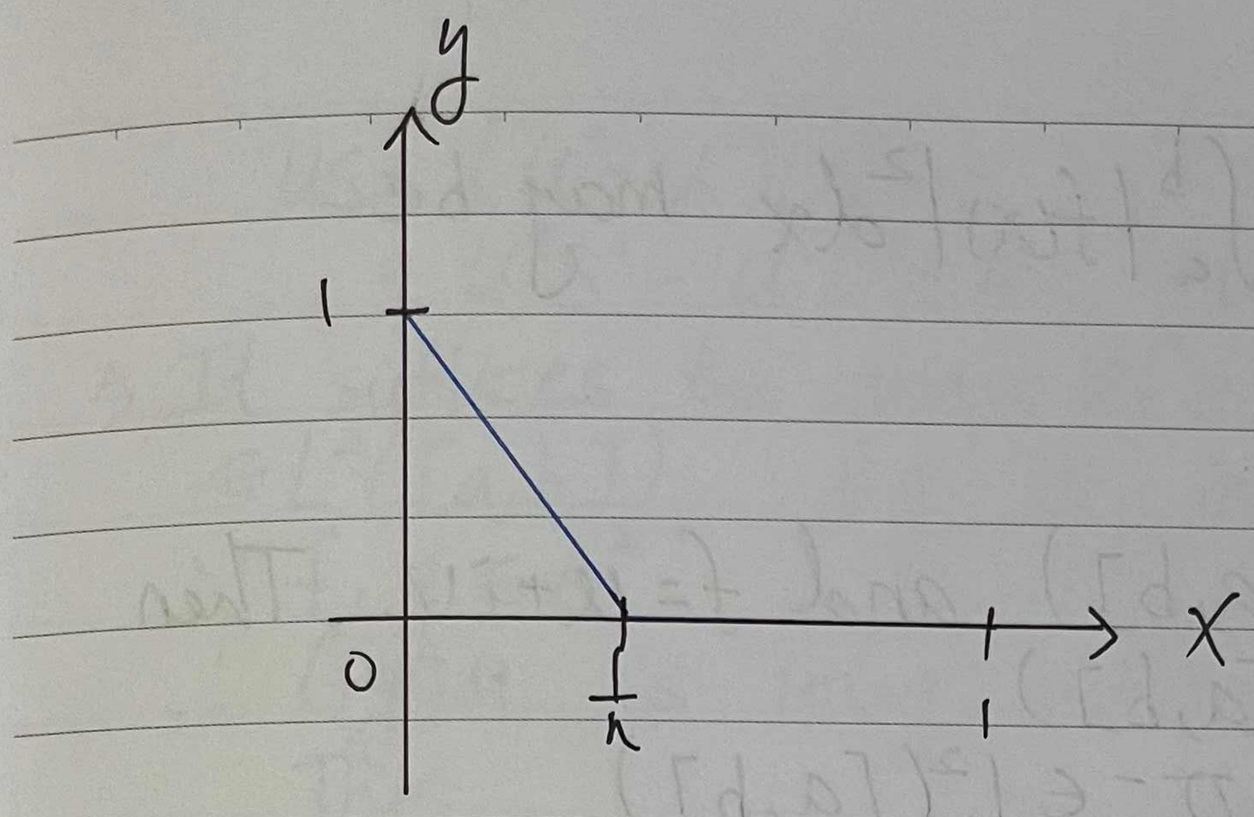
Cor 1. (Cauchy-Schwarz ineq.)
If $f, g: [a,b] \to \mathbb{C}$ are cont. functions,
then $\left|\int_a^b f(x)\overline{g(x)}\,dx\right|^2 \le \int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx$

Cor 2. (Minkowski inequality)
$$\left(\int_a^b |f(x)+g(x)|^2 dx\right)^{\frac{1}{2}} \le \left(\int_a^b |f(x)|^2 dx\right)^{\frac{1}{2}} + \left(\int_a^b |g(x)|^2 dx\right)^{\frac{1}{2}}$$

Def. With this inner product on the space V of functions, convergence is called convergence in the mean. So we write
$f_n \to f$ (in mean)
(亦將 f_n 可看成是 seq. of functions on $[a,b]$, $f_n \to f$ uniformly 與 $f_n \to f$ (pointwise) 作分別)

Remark. Notice that $f_n \to f$ (in mean)
$\iff \|f_n - f\| \to 0$.
$\iff \int_a^b |f_n(x) - f(x)|^2 dx \to 0$

For example, let $f_n(x) = \begin{cases} 1-nx, & \text{for } 0 \le x \le \frac{1}{n} \\ 0, & \text{for } \frac{1}{n} \le x \le 1 \end{cases}$
$\forall n \in \mathbb{N}$.



$$\because \int_0^1 |f_n(x)|^2 dx = \int_0^{\frac{1}{n}} (1-nx)^2 dx$$

$$= \int_0^{\frac{1}{n}} (1-2nx+n^2x^2)\, dx$$

$$= \cdots$$

$$= \frac{1}{3n} \to 0$$

as $n \to \infty$

$\therefore$ $f_n \to 0$ (in mean), but

$$f_n \to f(x) = \begin{cases} 0, & \text{if } x \in (0,1] \\ 1, & \text{if } x = 0 \end{cases} \neq 0 \text{ (pointwise)}$$

$\Rightarrow f_n \not\to 0$ (pointwise)

* Convergence in mean $\not\Rightarrow$ Pointwisely conv.

Def. $L^2([a,b]) = \{$ functions $f : [a,b] \to \mathbb{C}$ | f is conti. at every point except possibly on a set of measure zero and $\int_a^b |f(x)|^2 dx < \infty \}$, i.e., f is square integrable on $[a,b]$.

Notice that $\int_a^b |f(x)|^2 dx$ may be improper.

Prop. Let $f \in L^2([a,b])$ and $f = u + iv$. Then
① $u, v \in L^2([a,b])$
② $u^+, u^-, v^+, v^- \in L^2([a,b])$.

pf. ① Notice that f is cont. at x
$\Leftrightarrow u$ and v are cont. at x.
So $(u^2)_u$ is integ. by Lebesgue's Thm, $\forall u > 0$.

$\because |f|^2 = u^2 + v^2$
$\therefore 0 \le u^2 \le |f|^2$.
$\Rightarrow$ By comparison test B, u^2 is integ.
$\Rightarrow u \in L^2([a,b])$

② $u^2 = (u^+)^2 + (u^-)^2$. By similar argument in ①, $u^+ \in L^2([a,b])$

Q.E.D.

Prop. If $f, g \in L^2([a,b])$, then $\int_a^b f(x)\overline{g(x)}\,dx < \infty$ (i.e. $f\bar{g}$ is integ.)

pf. Let $f = u + iv$ and $g = s + it$. Then
$f\bar{g} = (u+iv)(s-it)$
$= (us - vt) + i(vs - ut)$, and

$$uS = (u^+ - u^-)(S^+ - S^-)$$
$$= u^+S^+ - u^-S^+ - u^+S^- + u^-S^-$$

$\Rightarrow$ It suffices to show $\forall f \geq 0, g \geq 0$ and $f, g \in L^2([a,b])$, fg is integ.

$\because f, g \in L^2$

$\therefore (fg)_M$ is integ. $\forall M > 0$ by Lebesgue's Thm.

Notice that $0 \leq fg \leq \frac{f^2+g^2}{2}$ is integ.

By comparison test B, fg is integ. Q.E.D.

Cor. $L^2([a,b])$ is a vector space over $\mathbb{C}$

pf. 只需証 $f+g \in L^2([a,b])$.

$\because |f+g|^2 = (f+g)(\bar{f}+\bar{g})$

$= |f|^2 + f\bar{g} + \bar{f}g + |g|^2$

$\therefore$ By previous prop. $|f+g|^2$ is integ.

$\Rightarrow f+g \in L^2([a,b])$

Q.E.D.

Prop. Let $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$ be integ. functions. If $f = g$ except possibly on a set S of measure zero, then $\int_A f = \int_A g$.

pf. Let $h = f - g$, then h is integ. on A.

$\Rightarrow h \cdot 1_A$ is integ. on A

$\because h = 0$ on $A \setminus S$

$\therefore h \cdot 1_A = h \cdot 1_S$

$\Rightarrow h \cdot 1_S$ is integ. on S, and

$\int_A h \cdot 1_A = \int_A h \cdot 1_S = \int_S h \cdot 1_S = \int_S h$

$\Rightarrow \int_A h = \int_S h$

$\because S$ is measure zero

$\therefore \int_S h = 0$

$\Rightarrow \int_A h = 0.$

Q.E.D.

Note: Let $f \in L^2([a,b])$. Suppose $\int_a^b |f(x)|^2 dx = 0$,

$\because |f|^2$ is integ. and $|f|^2 \geq 0$

$\therefore \{x \in [a,b] \mid |f(x)|^2 \neq 0\}$ is measure zero

$\Rightarrow \{x \in [a,b] \mid f(x) \neq 0\}$ is measure zero.

It implies that $\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\,dx$ is not an inner product

Def. Let $f, g \in L^2([a,b])$. We say that $f \sim g$ if $f = g$ except on a set of measure zero.

Prop. $\sim$ is an equivalence relation

Def. Let $f \in L^2([a,b])$. Define $[f] = \{g \in L^2([a,b]) \mid f \sim g\}$

quotient space

and $L^2([a,b])/\sim = \{[f] \mid f \in L^2([a,b])\}$

Prop. Let $f_1, f_2, g_1, g_2 \in L^2([a,b])$ and $\alpha \in \mathbb{C}$

① If $f_1 \sim f_2$, then $\alpha f_1 \sim \alpha f_2$

② If $f_1 \sim f_2$ and $g_1 \sim g_2$, then $f_1 + g_1 \sim f_2 + g_2$ and $f_1 g_1 \sim f_2 g_2$

Def. Let $[f], [g] \in L^2([a,b])/\sim$, and $\alpha \in \mathbb{C}$. Define $\alpha[f] = [\alpha f]$, $[f] + [g] = [f+g]$ and $[f][g] = [fg]$. (By previous prop. the definitions are well-defined.) Then $L^2([a,b])/\sim$ is a vector space over $\mathbb{C}$.

Prop. Let $f, g, h, k \in L^2([a,b])$.

① If $f \sim g$, then $\int_a^b |f(x)|^2 dx = \int_a^b |g(x)|^2 dx$

② If $f \sim g$ and $h \sim k$, then $\int_a^b f(x)\overline{h(x)}\,dx = \int_a^b g(x)\overline{k(x)}\,dx$

pf. $fk = gk$ except on a set of measure zero

Q.E.D.

Cor. $(L^2([a,b])/\sim, \langle , \rangle)$ is an inner product space, where $\langle [f], [g] \rangle = \int_a^b f(x)\overline{g(x)}\,dx$.

We will still write $(L^2([a,b])/\sim, \langle,\rangle)$ as $(L^2([a,b]), \langle,\rangle)$.

Remark. $(L^2([a,b]), \langle,\rangle)$ is not complete. To make it complete, the Lebesgue integral is used.

Def. A complete inner product space is called a Hilbert space.

Def. A function $f: [a,b] \to \mathbb{C}$ is called sectionally continuous (or piecewise continuous) if $[a,b]$ has a finite partition $\{a = x_0 < x_1 < \cdots < x_n = b\}$ such that f is cont. and bounded on each open section (x_{i-1}, x_i) $\forall i = 1, \ldots, n$

Thm. The thm A is true if V is the space of sectionally cnti. functions $f: [a,b] \to \mathbb{C}$.

§10-2 Orthogonal families of functions

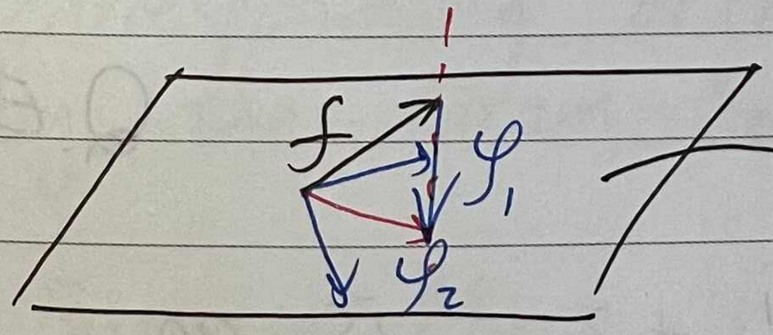
From now on, a vector space means a complex vector space.

Def. Let $(V, \langle , \rangle)$ be an inner product space. A seq. $\varphi_0, \varphi_1, \varphi_2, \cdots$ is called an orthonormal family if $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$, where $\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

→ 这就给代，直接内积就可以了

Thm. Let V be an inner product space and $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ be a set of orthonormal vectors in V. Then $\forall f \in V$, $\sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i$ and $f - \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i$ are orthogonal vectors. Hence

$$\|f\|^2 = \left\|f - \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i\right\|^2 + \left\|\sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i\right\|^2$$



→ 如同于到平面的最短距离

Thm. (Best Mean approximation)
Let $(V, \langle , \rangle)$ be an inner product space and $\{\varphi_0, \varphi_1, \cdots, \varphi_n\}$ be a set of orthonormal vectors in V. Then for each set of numbers $t_0, t_1, t_2, \cdots, t_n$,

$$\|f - \sum_{i=0}^{n} t_i \varphi_i\| \geq \|f - \sum_{i=0}^{n} \langle f, \varphi_i \rangle \varphi_i\|$$

Moreover, the eq. holds iff $t_k = \langle f, \varphi_k \rangle$, $\forall k = 0, 1, \ldots, n$.

pf. Let $c_k = \langle f, \varphi_k \rangle$, $s = \sum_{k=0}^{n} c_k \varphi_k$ and $h = \sum_{k=0}^{n} t_k \varphi_k$, then

$$\begin{aligned}
\|f-h\|^2 &= \langle f-h, f-h \rangle \\
&= \langle f, f \rangle - \langle h, f \rangle - \langle f, h \rangle + \langle h, h \rangle \\
&= \|f\|^2 - \sum_{k=0}^{n} t_k \bar{c}_k - \sum_{k=0}^{n} \bar{t}_k c_k + \sum_{k=0}^{n} |t_k|^2 \\
&= \|f\|^2 + \sum_{k=0}^{n} \left[|t_k - c_k|^2 - |c_k|^2 \right] \\
&= \underbrace{\|f\|^2 - \|s\|^2} + \sum_{k=0}^{n} |c_k - t_k|^2 \\
&\because f - s \perp s \\
&\therefore \|f\|^2 = \|f-s\|^2 + \|s\|^2 \\
&= \|f - s\|^2 + \sum_{k=0}^{n} |c_k - t_k|^2
\end{aligned}$$

Hence $\|f-h\| \leq \|f-s\|$

Q.E.D.

Prop. ① $\frac{e^{inx}}{\sqrt{2\pi}}$, $n = 0, \pm 1, \pm 2, \ldots$ is an orthonormal family of $L^2([0, 2\pi])$ or $L^2([-\pi, \pi])$

② $\frac{1}{\sqrt{2\pi}}$, $\frac{\sin(nx)}{\sqrt{\pi}}$, $\frac{\cos(nx)}{\sqrt{\pi}}$, $n = 1, 2, 3, \ldots$

is an orthonormal family of $L^2([0,2\pi])$ or $L^2([-\pi,\pi])$

pf. ① Let $n,m\in\mathbb{Z}$,

$$\left\langle \frac{e^{inx}}{\sqrt{2\pi}}, \frac{e^{imx}}{\sqrt{2\pi}} \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)x}\,dx$$

$$= \begin{cases} \frac{1}{2\pi i(n-m)} e^{i(n-m)x}\Big|_0^{2\pi}, & \text{if } n\neq m \\ 1, & \text{if } n=m \end{cases}$$

$$= \delta_{nm}$$

② 类似

Q.E.D.

Remark. Given an inner product space $(V,\langle\,\rangle)$ and linearly indep. vectors $g_0, g_1, \dots,$ in V, one can form a corresponding orthonormal family $\ell_0, \ell_1, \dots$ by the Gram-Smichidt process

Prop. Suppose that $\ell_0, \ell_1, \dots,$ are orthonormal functions on $[0,\ell]$, then the functions $\ell_n(x) = \sqrt{\frac{2\pi}{\ell}}\,\ell_n\left(\frac{2\pi x}{\ell}\right)$ are orthonormal functions on $[0,\ell]$ (By rescaling).

pf. $\int_0^l \varphi_n(x)\overline{\varphi_m(x)}\,dx = \frac{2\pi}{l}\int_0^l \varphi_n\left(\frac{2\pi x}{l}\right)\overline{\varphi_m\left(\frac{2\pi x}{l}\right)}$

$= \int_0^{2\pi} \varphi_n(u)\overline{\varphi_m(u)}\,du$

↓ Let $u = \frac{2\pi x}{l}$

$= \delta_{mn}$

Q.E.D.

Q: Let $\varphi_0, \varphi_1, \varphi_2, \dots$ be an orthonormal family in an inner product space $(V, \langle , \rangle)$. Can we write each $f \in V$ in the form $f = \sum_{k=0}^{\infty} c_k \varphi_k$ where $c_k \in \mathbb{C}$

A: Let $f \in V$ and $f = \sum_{k=1}^{\infty} c_k \varphi_k$. Let $S_n = \sum_{k=0}^{n} c_k \varphi_k$. Then $S_n \to f$

Let $i \in \mathbb{N} \cup \{0\}$. $\because \langle , \rangle$ is cont. on $V \times V$

$\therefore \langle S_n, \varphi_i \rangle \xrightarrow{n\to\infty} \langle f, \varphi_i \rangle$

As $n \geq i$, $c_i \to \langle f, \varphi_i \rangle \Rightarrow \langle f, \varphi_i \rangle = c_i$

Therefore the expansion of f is uniquely determined by $f = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$

Q: 什麼時候 $\sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$ converges, $\forall f \in V$?

A: Let $S_n = \sum_{k=0}^{n} \langle f, \varphi_k \rangle \varphi_k$. Then $(f - S_n) \perp S_n$,

$\forall n. \Rightarrow \|f\|^2 = \|f - S_n\|^2 + \|S_n\|^2$

$\Rightarrow \|S_n\|^2 \leq \|f\|^2, \ \forall n \in \mathbb{N}.$

$\because \|S_n\|^2 = \sum_{k=0}^{n} |\langle f, \varphi_k \rangle|^2, \ \forall n \in \mathbb{N}$

$\therefore \|S_n\|^2 \uparrow$ as $n \to \infty$

$\Rightarrow$ The seq. $\|S_n\|^2$ converges

$\Rightarrow$ The series $\sum_{k=0}^{\infty} |\langle f, \varphi_k \rangle|^2$ converges

and is less than $\|f\|^2$

$\forall n \geq m, \ \|S_n - S_m\| = \| \sum_{k=m+1}^{n} \langle f, \varphi_k \rangle \varphi_k \|$

$$= \sqrt{\sum_{k=m+1}^{n} |\langle f, \varphi_k \rangle|^2}$$

$\Rightarrow S_n$ is a Cauchy sequence.

Hence if V is a Hilbert space, then $\sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$ converges, $\forall f \in V$.

Q: 当 $\sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$ 收敛时，什麼時候它会收敛到 f 呢？(Fourier analysis 就是在討論这些東西.

何時

A: $S_n \to f$ as $n \to \infty$? $\Leftrightarrow \|S_n - f\| \to 0$ as $n \to \infty$?

$\because (f - S_n) \perp S_n$

$\therefore \|S_n - f\|^2 = \|f\|^2 - \|S_n\|^2$

Double A

$$\Rightarrow \lim_{n\to\infty} \|S_n - f\|^2 = \|f\|^2 - \sum_{k=0}^{\infty} |\langle f, \varphi_k\rangle|^2$$

Therefore $S_n \to f \iff \sum_{k=0}^{\infty} |\langle f, \varphi_k\rangle|^2 = \|f\|^2$

Thm. In an inner product space $(V, \langle\,\rangle)$, suppose that $f = \sum_{k=0}^{\infty} c_k \varphi_k$ for an orthonormal family in V for $f \in V$. Then $c_k = \langle f, \varphi_k\rangle$, $\forall k = 0, 1, 2, \ldots$

Thm. In any inner product space $(V, \langle,\rangle)$ with an orthonormal family $\varphi_0, \varphi_1, \ldots$ For any $f \in V$, then the series $\sum_{k=0}^{\infty} \langle f, \varphi_k\rangle \varphi_k$ is a Cauchy series in V. Hence if V is a Hilbert space, then $\sum_{k=0}^{\infty} \langle f, \varphi_k\rangle \varphi_k$ converges for $f \in V$.

Thm. (Bessel's inequality)
Let $\varphi_0, \varphi_1, \ldots$ be an orthonormal family in an inner product space $(V, \langle,\rangle)$. For any $f \in V$, the (real) series $\sum_{k=0}^{\infty} |\langle f, \varphi_k\rangle|^2$ converges and $\sum_{k=0}^{\infty} |\langle f, \varphi_k\rangle|^2 \le \|f\|^2$

Thm. (Parseval's thm)
Let $(V, \langle,\rangle)$ be an inner product space with an orthonormal family $\varphi_0, \varphi_1, \ldots$

Prop

Pf

Let $f \in V$. Then $f = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$ if and only if $\|f\|^2 = \sum_{k=0}^{\infty} |\langle f, \varphi_k \rangle|^2$.

Prop. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an orthonormal family in an inner product space $(V, \langle , \rangle)$, and $f \in V$. If $\sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$ converges, then $f - \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$ is orthogonal to φ_j, $\forall j = 0, 1, \ldots$ Moreover, $f - \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$ and $\sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$ are orthogonal vectors

pf. $\because f - \sum_{k=0}^{n} \langle f, \varphi_k \rangle \varphi_k \to f - \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$

and $\langle , \rangle$ is cont. on $V \times V$,

$\therefore \langle f - \sum_{k=0}^{n} \langle f, \varphi_k \rangle \varphi_k, \varphi_i \rangle$

$\to \langle f - \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k, \varphi_i \rangle$

$\because$ As $n > i$, $0 \to \langle f - \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k, \varphi_i \rangle$

$\therefore \langle f - \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k, \varphi_i \rangle = 0$, $\forall i = 0, 1, 2, \ldots$

Furthermore, $\langle f - \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k, \sum_{i=0}^{n} \langle f, \varphi_i \rangle \varphi_i \rangle$

$= 0$, $\forall n \in \mathbb{N}$.

Q.E.D.

Def. Let $\varphi_0, \varphi_1, \dots$ be orthonormal family is an inner product space $(V, \langle , \rangle)$

① Let $f \in V$. The series $\sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$

據此定義, is called the Fourier series of f with respect to $\varphi_0, \varphi_1, \dots$ and $\langle f, \varphi_k \rangle$ the k-th Fourier coefficient.

F.S.不一定收斂

② $\varphi_0, \varphi_1, \dots$ is called complete, if every $f \in V$ can be written

$$f = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$$

Prop. Let $(V, \langle , \rangle)$ be an inner product space with a complete orthonormal family $\varphi_0, \varphi_1, \dots$ Then

① Any proper subsequence of $\varphi_0, \varphi_1, \dots$ is not a complete orthonormal family

② If f is orthonormal to each φ_k, then $f = 0$.

pf. ① Let $\varphi_{k_0}, \varphi_{k_1}, \dots$ be a proper subsequence of $\varphi_0, \varphi_1, \dots$ Then $\exists\, \varphi_l \neq \varphi_{k_j}$

$\because \langle \varphi_l, \varphi_{k_j} \rangle = 0, \ \forall j = 0, 1, \dots,$

$\therefore$ The Fourier series of φ_l w.r.t. $\varphi_{k_0}, \varphi_{k_1}, \dots$ is 0

$\because \varphi_l \neq 0$

$\therefore \varphi_{k_0}, \varphi_{k_1}, \dots$ is not complete.

② $\because \langle f, \varphi_k \rangle = 0, \forall k = 0, 1, 2, \dots$, and $\varphi_0, \varphi_1, \dots$ is complete

$\therefore f = \sum_{k=0}^{\infty} 0 \cdot \varphi_k = 0$

Q.E.D.

Prop. Let $(V, \langle , \rangle)$ be an inner product space with an orthonormal family $\varphi_0, \varphi_1, \dots$. If V is a Hilbert space, then the following two statements are equivalent

① If $f \perp$ is orthogonal to each φ_k, then $f = 0$.

② $\varphi_0, \varphi_1, \dots$ is complete.

pf. ② $\Rightarrow$ ① has done. To show ① $\Rightarrow$ ②. Let $f \in V$. $\because V$ is Hilbert $\therefore \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$ converges.

$\Rightarrow \langle f - \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k, \varphi_i \rangle = 0, \forall i = 0, 1, 2, \dots$

By ①, $f - \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k = 0$

$\Rightarrow f = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$

Hence φ_0, φ_1 is complete.

Q.E.D.

e.g. Let $f \in L^2([0,2\pi])$. Show that
$\int_0^{2\pi} f(x)\sin(nx)\,dx \to 0$ and
$\int_0^{2\pi} f(x)\cos(nx)\,dx \to 0$

-sol- ① $\frac{\sin(nx)}{\sqrt{\pi}}$, $n=1,2,\dots$ is an orthonormal family in $L^2([0,2\pi])$

$\therefore$ By Bessel's ineq.,
$\sum_{n=1}^{\infty} |\langle f, \frac{\sin(nx)}{\sqrt{\pi}}\rangle|^2$ converges.

$\Rightarrow |\langle f, \frac{\sin(nx)}{\sqrt{\pi}}\rangle| \to 0$

$\Rightarrow \langle f, \frac{\sin(nx)}{\sqrt{\pi}}\rangle \to 0$ ⁂

e.g. Let $f:[0,\pi] \to \mathbb{C}$ be a cont. function. Prove the following ineq.
$|\int_0^{\pi} f(x)\sin(2x)\,dx|^2 + \dots + |\int_0^{\pi} f(x)\sin(2nx)\,dx|^2$
$\leq \frac{\pi}{2}\int_0^{\pi} |f(x)|^2\,dx$

-sol- ∵ $\frac{\sin(nx)}{\sqrt{\pi}}$, $n=1,2,\dots$ is an orthonormal family on $L^2([0,2\pi])$.

$\therefore \sqrt{\frac{2}{\pi}}\sin(2nx)$ $n=1,2,\dots$ is an orthonormal family for $L^2([0,\pi])$.

$\because f$ is cont. on $[0,\pi]$,
$\therefore f \in L^2([0,\pi])$.

By Bessel's ineq.,

$$\sum_{n=1}^{\infty} |\langle f, \sqrt{\frac{2}{\pi}}\sin(2nx)\rangle|^2 \leq \|f\|^2$$

$$\Rightarrow \sum_{n=1}^{\infty} |\langle f, \sin(2nx)\rangle|^2 \leq \frac{\pi}{2}\|f\|^2$$

$$\Rightarrow \left|\int_0^\pi f(x)\sin(2x)dx\right|^2 + \dots + \left|\int_0^\pi f(x)\sin(2nx)dx\right|^2$$

$$\leq \frac{\pi}{2}\int_0^\pi |f(x)|^2 dx \quad \#$$

§ 10.3 Completeness and Convergence Thm.

For $L^2([0,2\pi])$ or $L^2([-\pi,\pi])$, we deal primarily with the following two orthonormal family.

(a) Exponential system: $\frac{e^{inx}}{\sqrt{2\pi}}$, $n=0,\pm1,\pm2,\dots$

(b) Trigonometric system: $\frac{1}{\sqrt{2\pi}}$, $\frac{\sin(nx)}{\sqrt{2\pi}}$, $\frac{\cos(nx)}{\sqrt{2\pi}}$

Def. Let $f\in L^2([0,2\pi])$.

① The seq. of partial sum $S_N=\sum_{n=-N}^{N}\left\langle f,\frac{e^{inx}}{\sqrt{2\pi}}\right\rangle\frac{e^{inx}}{\sqrt{2\pi}}$

$=\sum_{n=-N}^{N}c_n e^{inx}$, where $c_n=\frac{1}{2\pi}\int_0^{2\pi}f(x)e^{-inx}dx$

and is called the nth Fourier coefficient, and the expnential Fourier series of f is the series $\sum_{n=-\infty}^{\infty}c_n e^{inx}=\lim_{N\to\infty}\sum_{n=-N}^{N}c_n e^{inx}$

② The sequence of partial sum S_N of the Fourier series of f w.r.t. the trigonometric system is defined by

$$S_N=\left\langle f,\frac{1}{\sqrt{2\pi}}\right\rangle\frac{1}{\sqrt{2\pi}}+\sum_{n=1}^{N}\left[\left\langle f,\frac{\cos(nx)}{\sqrt{\pi}}\right\rangle\frac{\cos(nx)}{\sqrt{\pi}}+\left\langle f,\frac{\sin(nx)}{\sqrt{\pi}}\right\rangle\frac{\sin(nx)}{\sqrt{\pi}}\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{N}\left(a_n\cos(nx)+b_n\sin(nx)\right),$$

where $a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos(nx)dx$,

$n=0,1,\dots$ and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$,

$n=1,2,\dots$ and a_n, b_n are called the Fourier coefficients. The trigonometric Fourier series of f is the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$= \lim_{N\to\infty}\left(\frac{a_0}{2} + \sum_{n=1}^{N}\left[a_n\cos(nx) + b_n\sin(nx)\right]\right)$$

Prop. Let $f \in L^2([0,2\pi])$ (or $L^2([-\pi,\pi])$) the Nth partial sum of the Fourier series of f equals the Nth exponential partial sum of the trignometric Fourier series of f. Hence the Fourier series of these two systems both either converge or diverge.

pf. $S_N = \sum_{n=-N}^{N} \frac{1}{2\pi}\langle f, e^{inx}\rangle e^{inx}$

$$= \frac{1}{2\pi}\langle f,1\rangle + \frac{1}{2\pi}\sum_{n=1}^{N}\Big(\langle f, e^{inx}\rangle e^{inx} + \langle f, e^{inx}\rangle e^{inx}\Big)$$

$$\begin{cases} \langle f, e^{inx}\rangle e^{inx} = \langle f, \cos(nx) + i\sin(nx)\rangle(\cos(nx) + i\sin(nx)) \\ \langle f, e^{-inx}\rangle e^{-inx} = \langle " \quad " \rangle(" \quad - \end{cases}$$

Thus $S_N = \frac{1}{2\pi}\langle f, 1\rangle + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[\langle f, \cos(nx)\rangle \cos(nx) + \langle f, \sin(nx)\rangle \sin(nx)\Big]$

Q.E.D.

Thm. (Mean complete theorem)
The exponential and trigonometric systems are complete in $L^2([0, 2\pi])$

Remark. Let $f \in L^2([0, 2\pi])$ (or $L^2([-\pi, \pi])$)

$\therefore f = \sum_{n=-\infty}^{\infty} \langle f, \frac{e^{inx}}{\sqrt{2\pi}}\rangle \frac{e^{inx}}{\sqrt{2\pi}}$ (in mean)

$= \sum_{n=-\infty}^{\infty} \left(\langle Re(f), \frac{e^{inx}}{\sqrt{2\pi}}\rangle \frac{e^{inx}}{\sqrt{2\pi}} + i\langle Im(f), \frac{e^{inx}}{\sqrt{2\pi}}\rangle \frac{e^{inx}}{\sqrt{2\pi}} \right)$

$\because f \in L^2([0, 2\pi])$

$\therefore Re(f), Im(f) \in L^2([0, 2\pi])$

By Mean Complete thm,

$$\begin{cases} Re(f) = \sum_{n=-\infty}^{\infty} \langle Re(f), \frac{e^{inx}}{\sqrt{2\pi}}\rangle \frac{e^{inx}}{\sqrt{2\pi}} \text{ (in mean)} \\ Im(f) = \sum_{n=-\infty}^{\infty} \langle Im(f), \frac{e^{inx}}{\sqrt{2\pi}}\rangle \frac{e^{inx}}{\sqrt{2\pi}} \text{ (in mean)} \end{cases}$$

$= \sum_{n=-\infty}^{\infty} \langle Re(f), \frac{e^{inx}}{\sqrt{2\pi}}\rangle e^{inx} + i\sum_{n=-\infty}^{\infty} \langle Im(f), \frac{e^{inx}}{\sqrt{2\pi}}\rangle \frac{e^{inx}}{\sqrt{2\pi}}$

⇒ 不需考虑複函數，只要考虑實函數就好了.
(real-valued)

∵可拆成2个实函数之 F.S.

Q. Let $f \in L^2([0, 2\pi])$ (or $[-\pi, \pi]$). By MCT, the exponential or trignometric FS of f conv. in the mean to f. Will the convergence be pointwise?

Thm. (L. Carleson)
If $f \in L^2([0, 2\pi])$, then its F.S. converges pointwise to f, except possibly on a set of measure zero.

pf. See Acta Math. 116 (1966) p135

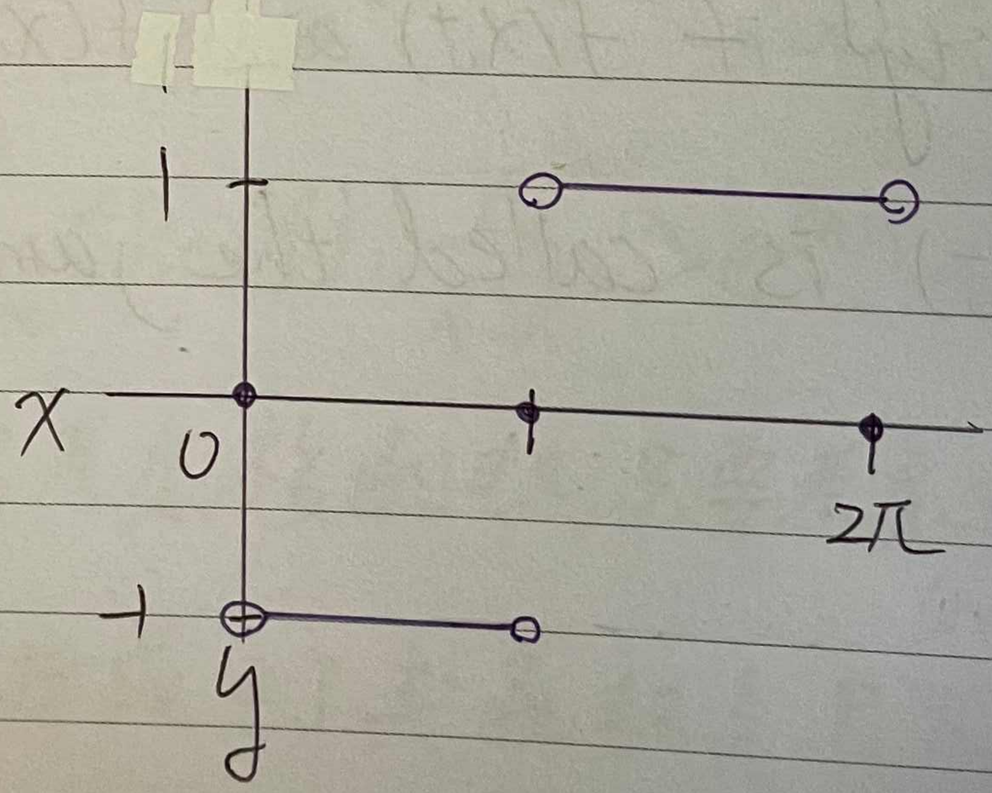
Def. Let $f: [0, 2\pi] \to \mathbb{R}$ be a function
① For $x_0 \in (0, 2\pi)$, define $f(x_0^+) = \lim_{x \to x_0^+} f(x)$
and $f(x_0^-) = \lim_{x \to x_0^-} f(x)$
② For $x_0 = 0$ or 2π, we extend f to be a 2π-periodic function by defining $f(x+2\pi) = f(x)$, $\forall x \in (0, 2\pi)$. Hence we can define $f(0^+)$, $f(0^-)$, $f(2\pi^+)$, $f(2\pi^-)$
③ A discontinuity x_0 is called a jump discontinuity if $f(x_0^+)$ and $f(x_0^-)$ both exist.
④ $f(x_0^+) - f(x_0^-)$ is called the jump off at x_0.

Thm. (Jordan)
Let $f:[0,2\pi] \to \mathbb{R}$ (or $[-\pi,\pi]$) be sectionally cont. and have a jump discontinuity at x_0 (extending f to periodic if necessary) and assume that $\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0^+)}{h}$ and $\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0^-)}{h}$ both exists, then the F.S. of f (in either exponential or trigonometric form) evaluated at x_0 converges to $\frac{f(x_0^+)+f(x_0^-)}{2}$

In particular, if f is differentiable at x_0, the F.S. of f converges at x_0 to $f(x_0)$

e.g. Find an example of a function f s.t. the F.S. of f converges in the mean to f and pointwise to f but not converges uniformly to f.

-sol-



§10.5 Computation of Fourier Series.

Let $f:[a,b]\to\mathbb{R}$ be a function s.t. $f\in L^2([a,b])$.

(1) The exponential F.S. on $[0,2\pi]$ (or $[-\pi,\pi]$): $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, where

$$c_n=\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx.$$

So the Parseval relation

$$\left(\int_0^{2\pi}|f(x)|^2dx=\sum_{k=0}^{\infty}|\langle f,e_k\rangle|^2\right):$$

$$\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2dx=\sum_{n=-\infty}^{\infty}|c_n|^2$$

eg. Find the exponential F.S. for $f(x)=x$ on $[0,2\pi]$.

-sol-. $c_0=\frac{1}{2\pi}\int_0^{2\pi}x\,dx=\frac{1}{4\pi}\cdot 4\pi^2=\pi$

$\forall n\neq 0$, $c_n=\frac{1}{2\pi}\int_0^{2\pi}\underbrace{x}_{\text{微}}\underbrace{e^{-inx}}_{\text{秩}}dx$

$$=\frac{1}{2\pi}\frac{1}{-in}2\pi+\frac{1}{2\pi in}\int_0^{2\pi}e^{-inx}\,dx$$

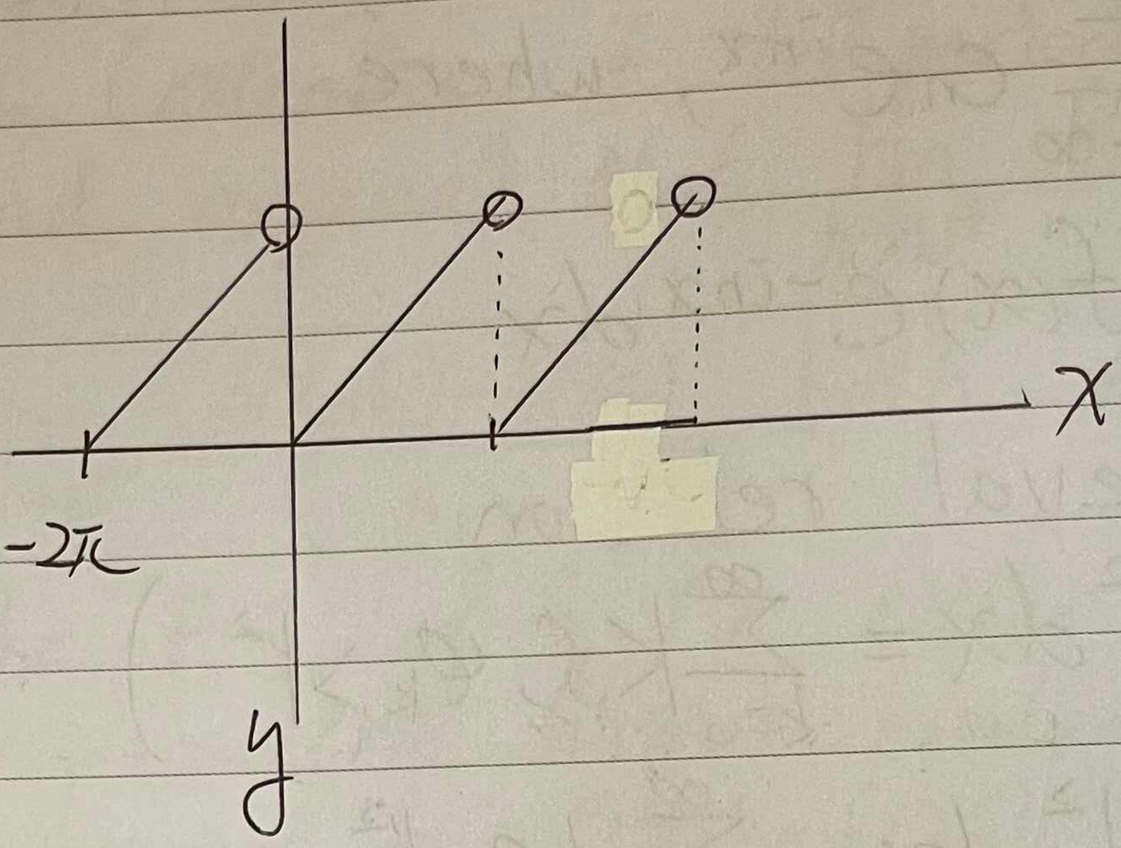
($\int_0^{2\pi}e^{-inx}dx=0$)

$$=\frac{-1}{in}$$

Hence the exponential F.S.

$$=\pi+\sum_{n=1}^{\infty}\left(\frac{-1}{in}e^{inx}+\frac{1}{in}e^{-inx}\right)$$

$$= \pi - \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$



By Pointwise Convergence Thm,
$\pi - 2\sum_{n=1}^{\infty} \sin(nx)$ is pointwise convergent to x on $(0, 2\pi)$ and convergent to π at 0 and 2π

e.g. Find the exponential F.S. of $f(x) = e^x$ on $[-\pi, \pi]$.

-sol- ① $c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{2\pi}(e^{\pi} - e^{-\pi}) = \frac{\sinh \pi}{\pi}$

For $n \neq 0$, $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x \cdot e^{-inx} dx$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-in)x} dx$

$= \frac{1}{2\pi} \frac{1}{1-in} e^{(1-in)x} \Big|_{-\pi}^{\pi}$

$$= \frac{1}{2\pi} \frac{1}{1-in} \left(e^{\pi}(-1)^n - e^{-\pi}(-1)^n \right)$$

$$= \frac{\sinh \pi}{\pi} \frac{(-1)^n}{1-in}$$

Hence the exponential F.S. of f

$$= \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-in} e^{inx}$$

By pointwise convergence thm, the series is e^x on $(-\pi, \pi)$ and is $\frac{e^{-\pi}+e^{\pi}}{2} = \cosh \pi$

at $-\pi, \pi$.

(2) The trigonometric F.S. on $[0, 2\pi]$ (or $[-\pi, \pi]$): $\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$,

where $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx$, $n = 0, 1, 2, \ldots$

and $b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx$, $n = 1, 2, \ldots$

The Parseval relation:

$$\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2 \right) = \frac{1}{\pi} \int_0^{2\pi} |f(x)|^2 dx$$

e.g. Find the trigonometric series for $f(x) = x$ on $[0, 2\pi]$.

Double A

-sol-. $a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = 2\pi$.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\frac{1}{n} x\sin(nx)\Big|_0^{2\pi} - \frac{1}{\pi n}\int_0^{2\pi}\sin(nx)\,dx$$

$$= 0.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin(nx)\,dx$$

$$= -\frac{1}{\pi}\frac{1}{n} x\cos(nx)\Big|_0^{2\pi} + \frac{1}{\pi n}\int_0^{2\pi}\cos(nx)\,dx$$

$$= -\frac{2}{n}.$$

Hence the trigonometric F.S. is

$$\pi - 2\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}.$$

The Parseval relation is

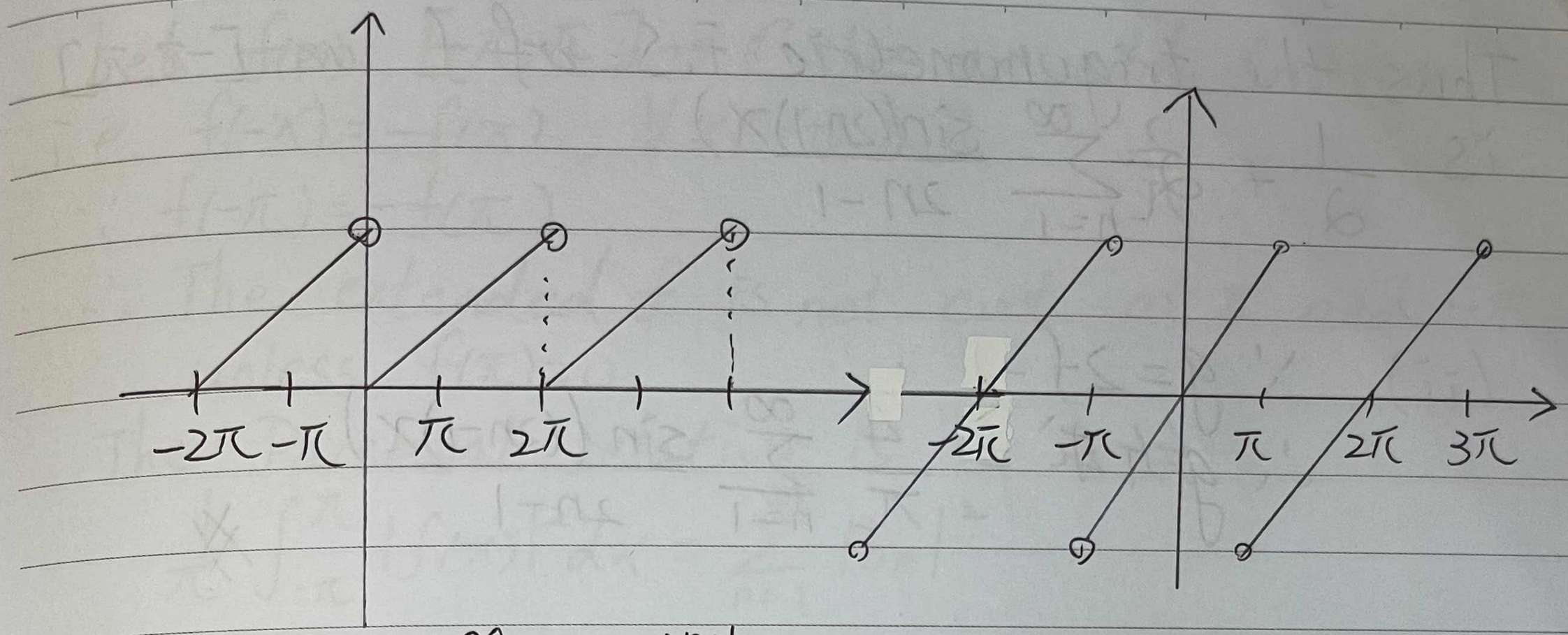
$$2\pi^2 + \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{8}{3}\pi^2$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi}{6} \quad \#$$

Prop. $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi}{6}$.

e.g. 3: Find the trigonometric F.S. for $f(x) = x$ on $[-\pi, \pi]$.

Double A



-sol- $2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$ ∀

e.g. ④ (i) Find the trigonometric F.S. for

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi \\ 0, & \text{if } -\pi \le x < 0 \end{cases}$$

(ii) Find the trigonometric F.S. for

$$g(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi \\ -1, & \text{if } -\pi \le x \le 0. \end{cases}$$

-sol- ① $a_0 = \frac{1}{\pi}\int_0^{\pi} dx = 1$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos(nx)\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(nx)\,dx$$

$$= \frac{-1}{\pi n}\cos(nx)\Big|_0^{\pi}$$

$$= \frac{1}{\pi n} - \frac{(-1)^n}{\pi n}$$

Thus the trigonometric F.S. of f on $[-\pi, \pi]$ is $\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}$

(ii) $\because g = 2f - 1$

$\therefore$ 所求 $= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}$ ※

(3) Fourier cosine series for even function on $[-\pi, \pi]$: $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$, where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

Let $f: [-\pi, \pi] \to \mathbb{C}$ be an even function;
$f(-x) = f(x)$, $\forall x \in (0, \pi]$.

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin(nx)}_{\text{odd function}}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(-x)\sin(-nx) = -b_n$

So $b_n = 0$.

The Parseval relation is $\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$

$= \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2$.

(4) Fourier sine series for odd functions on $[-\pi, \pi]$: $\sum_{n=1}^{\infty} b_n \sin(nx)$, where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Let $f(x): [-\pi, \pi] \to \mathbb{C}$ be an odd function;
i.e. $f(-x) = -f(x)$, $\forall x \in (0, \pi]$
$\because f(-\pi) = -f(\pi)$
$\therefore$ The extended f is not cont. at π and $-\pi$ unless $f(\pi) = 0$.

The Parseval relation is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=1}^{\infty} |b_n|^2$$

(5) Exponential Fourier series on $[-l, l]$:

$$\sum_{n=-\infty}^{\infty} c_n e^{in\frac{\pi}{l}x}, \text{ where } c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-in\frac{\pi}{l}x} dx$$

Notice that $\sqrt{\frac{2\pi}{2l}} e^{in\frac{2\pi}{2l}x} = \sqrt{\frac{\pi}{l}} e^{in\frac{\pi}{l}x}$,

$n = 0, \pm 1, \pm 2, \ldots$ is the orthonormal family for $L^2([0, 2l])$.

$\because \sqrt{\frac{\pi}{l}} e^{in\frac{\pi}{l}x}$ are $2l$-periodic functions

$\therefore \sqrt{\frac{\pi}{l}} e^{in\frac{\pi}{l}x}$ is the orthonormal family for $L^2([-l, l])$

The Parseval relation (in terms of c_n):

$$\frac{1}{2l}\int_{-l}^{l} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

(6) Trigonometric F.S. on $[-l, l]$:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(n \frac{\pi}{l} x\right) + b_n \sin\left(n \frac{\pi}{l} x\right) \right),$$

where $a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(n \frac{\pi}{l} x\right) dx, \quad n = 0, 1, 2, \ldots,$

and $b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(n \frac{\pi}{l} x\right) dx, \quad n = 1, 2, \ldots$

The Parseval relation is

$$\frac{1}{l} \int_{-l}^{l} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2 \right)$$

(7) Half-interval cosine series for $f: [0, l] \to \mathbb{C}$:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{l} x\right), \text{ where}$$

$$a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos\left(n \frac{\pi}{l} x\right) dx, \quad n = 0, 1, 2, \ldots$$

Extend f to even function on $[-l, l]$ by defining $f(-x) = f(x)$ for $x \in (0, l]$. So it has a Fourier cosine series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{l} x\right), \text{ where}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(n \frac{\pi}{l} x\right) dx$$

$$= \frac{2}{l} \int_{0}^{l} f(x) \cos\left(n \frac{\pi}{l} x\right) dx$$

Parseval relation is

$$\frac{2}{l} \int_{0}^{l} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2$$

Notice that the extended f is automatically a cont. $2l$-periodic function in $\mathbb{R}$ if f is cont. on $[0,l]$ ($\because f(-l)=f(l)$)

(8) Half-interval sine series for $f:[0,l]$

$\to$ ①:

$$\sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{l}x\right), \text{ where}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(n\frac{\pi}{l}x\right)dx$$

Extend f to an odd function on $[-l,l]$ by defining $f(-x)=-f(x)$ for $0<x\le l$. So it has Fourier sine series.

The Parseval relation is

$$\frac{2}{l}\int_0^l |f(x)|^2 dx = \sum_{n=1}^{\infty} |b_n|^2.$$

Notice that the extended f is not cont. at 0 or l, if $f(0)\neq 0$ and $f(l)\neq 0$.

e.g. Find the half-interval cosine series for $f(x)=x$ on $[0,\pi]$.

sol. $a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \pi$

$$a_n = \frac{2}{\pi}\int_0^\pi \underset{\text{微}}{x}\,\underset{\text{積}}{\cos(nx)}\,dx$$

$$= \frac{2}{\pi}\cdot\frac{1}{n}\, x\sin(nx)\Big|_0^\pi$$

$$= \frac{2}{\pi n^2}\cos(nx)\Big|_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

Hence the half-interval cosine series is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2}$$

(By Pointwise convergence thm, the series is x on $[0,\pi]$). #

Question: How about half-interval sine series for $f(x)=x$ on $[0,\pi]$?

Answer: $\because$ x is odd on $[-\pi,\pi]$ itself
$\therefore$ the half-interval sine series is the same as the Fourier sine series of x on $[-\pi,\pi]$.

e.g. For each $0<x<\pi$, prove that

$$\frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[1-(-1)^n\right]\cos(nx) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$$

-sol- By e.g.⑤, 左式是 half-interval cosine series for x on $[0,\pi]$ and e.g.③, 右式是 trigonometric F.S. for x on $[-\pi,\pi]$.
$\because$ Both series converges pointwise to x on $(0,\pi)$,
$\therefore$ They equal to each other. #

e.g. Let $f(x)=1$ on $[0,\pi]$.
① Find its half-interval sine series
② " cosine "

-sol- ① $b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx$

$$= \frac{-2}{\pi n}\cos(nx)\Big|_0^{\pi}$$

$$= \frac{2}{\pi n}\left(1-(-1)^n\right)$$

So the half-interval sine series is
$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)x]}{2n-1}$ (which converges to 1 on $(0,\pi)$ and is 0 at 0 and π).

② The even extension of 1 on $[-\pi,\pi]$ is 1.

By uniqueness of F.S., 1 is the half-interval cosine series on $[0,\pi]$. #

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