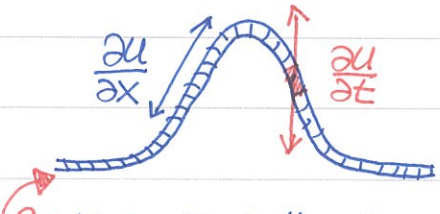
HH0100 Wave Equation

In previous lecture, we derived the wave equation for 1D string,

$$\boxed{\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}} \qquad \underbrace{v^2 = \frac{T}{\rho}}$$

T ← tension

ρ ← linear density.



In addition, the spatial derivative (shape) is related to the temporal derivative (local vibration),

$$\boxed{\frac{\partial u}{\partial t} = \mp v \frac{\partial u}{\partial x}}$$

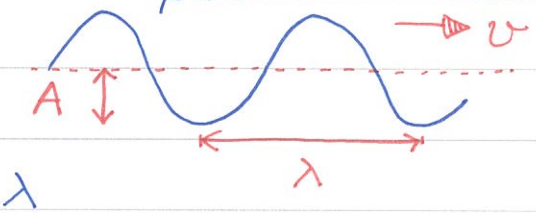
− : R-moving

+ : L-moving

Note that the horizontal tension is constant, $\underbrace{T_x = T}$!

⊘ Sinusoidal waves. The general solution for the wave eq. can be complicated. Thus, we start with the simplest solution with sinusoidal shape. Suppose the snapshot at t=0 is

$$\underbrace{u(x,0) = A\sin\left(\frac{2\pi x}{\lambda}\right)},$$



where A is the amplitude and λ is the wavelength. Note that $u(x+\lambda, 0) = u(x,0)$ and the shape repeats itself. If the wave propagates along the +x direction, i.e. right moving, the dynamics is obtained by $x \rightarrow x - vt$,

$$\boxed{u(x,t) = A\sin\frac{2\pi}{\lambda}(x - vt) = A\sin(kx - \omega t)}$$

← sinusoidal wave.

First of all, $\underbrace{v = f\lambda}$ for a sinusoidal wave as taught in high school. Two important quantities: wave number k and angular frequency ω.

$$\underbrace{k = \frac{2\pi}{\lambda}} \quad \text{and} \quad \underbrace{\omega = 2\pi f = \frac{2\pi}{T}}$$

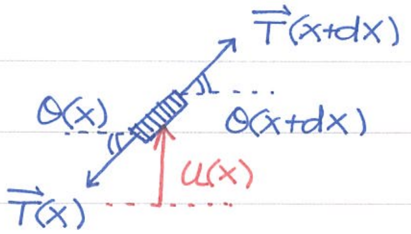
The relation between ω and k is called dispersion relation:

$$\boxed{\omega = 2\pi f = 2\pi \frac{v}{\lambda} = vk}$$

It is interesting to observe the local oscillatory motion of a tiny segment. For instance, at x = 0,

$$u(0,t) = A\sin(k\cdot 0-\omega t) = \underline{-A\sin\omega t}.$$

It is simple harmonic motion! We can also check the local EOM — the force in the y direction is



$$F_y = T_y(x+dx) - T_y(x) = \underline{\frac{dT_y}{dx}\cdot dx}$$

Making use of $T_y = T_x\tan\theta = T\frac{\partial u}{\partial x}$,

$$\rightarrow F_y = T\frac{\partial^2 u}{\partial x^2}\cdot dx = T\,dx\,\frac{\partial^2}{\partial x^2}\left[A\sin(kx-\omega t)\right]$$

$$F_y = -k^2T\,dx\cdot A\sin(kx-\omega t) \rightarrow \boxed{F_y = -(k^2T\,dx)\,u}$$

← Hooke's law

Following the same analysis, the natural frequency for the simple harmonic motion is

$$\boxed{\omega_0^2 = \frac{k^2T\cancel{dx}}{\rho\,\cancel{dx}} = k^2v^2 = \omega^2}$$

It is the same as the wave oscillation — reasonable.

We have assumed u is zero at (x,t) = (0,0). In general, this does not need to be the case. The general expression for a R-moving sinusoidal wave takes the form,

$$\boxed{u(x,t) = A\sin(kx-\omega t+\phi)}$$

$kx-\omega t+\phi$ is referred as the PHASE of the wave.

ϕ is the phase constant. The oscillations at different positions

$$\boxed{u = -A\sin\left[\omega t-\phi(x)\right]}$$

are SHM of the same ω but with different phases $\phi(x)$.

where $\underline{\phi(x) = kx+\phi}$

Q: Try to write down L-moving sinusoidal waves

⑪ Energy transfer. We are ready to compute the energy transfer rate (power) in a sinusoidal wave. According to HH0099, the power for R-moving wave is

$$P = \frac{dE}{dt} = T v \left(\frac{\partial u}{\partial x}\right)^2 \leftarrow \frac{\partial u}{\partial x} = kA\cos(kx-\omega t)$$

$$\rightarrow P = v \cdot T \cdot k^2 A^2 \cos^2(kx-\omega t) = v \cdot \rho\omega^2 A^2 \cos^2(kx-\omega t)$$

Let us compute the time-average of the above result.

$$\langle \cos^2(kx-\omega t)\rangle \equiv \frac{1}{T}\int_0^T \cos^2(kx-\omega t)\,dt = \frac{1}{T}\int_0^T \frac{1}{2} + \frac{1}{2}\cos(2kx-2\omega t)\,dt$$

$$= \frac{1}{T}\left[\frac{1}{2}T + \frac{1}{4\omega}\cancelto{0}{\sin(2\omega t-2kx)\Big|_0^T}\right] = \frac{1}{2} \,!$$

The above result is rather general for sinusoidal waves — $\langle\cos^2\rangle = \langle\sin^2\rangle = \frac{1}{2}$. The average energy transfer rate in a sinusoidal wave is

$$\boxed{\langle P\rangle = v\rho\omega^2A^2\langle\cos^2(kx-\omega t)\rangle = \frac{1}{2}\rho\omega^2A^2\cdot v}$$

Now, we would like to link $\langle P\rangle$ and average energy density $\langle\varepsilon\rangle$ together. First of all, the kinetic energy density is

$$\varepsilon_K \equiv \frac{dK}{dx} = \frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 = \frac{1}{2}\rho\omega^2A^2\cos^2(kx-\omega t).$$

$$\rightarrow \langle\varepsilon_k\rangle = \frac{1}{2}\rho\omega^2A^2\langle\cos^2(kx-\omega t)\rangle = \frac{1}{4}\rho\omega^2A^2$$

Because the local oscillation is described by the simple harmonic motion, the potential energy density gets the same time-average,

$$\boxed{\langle\varepsilon_u\rangle \equiv \left\langle\frac{dU}{dx}\right\rangle = \left\langle\frac{dK}{dx}\right\rangle = \langle\varepsilon_k\rangle}$$

$$\langle\varepsilon_u\rangle = \langle\varepsilon_k\rangle = \frac{1}{4}\rho\omega^2A \,!!$$

The average energy density $\langle \varepsilon \rangle$ is

$$\boxed{\langle \varepsilon \rangle = \langle \varepsilon_k \rangle + \langle \varepsilon_u \rangle = 2\langle \varepsilon_k \rangle = \frac{1}{2}\rho\omega^2 A^2}$$

We can rewrite the average power in a very suggestive way

$$\langle P \rangle = \frac{1}{2}\rho\omega^2 A^2 \cdot v \longrightarrow \boxed{\langle P \rangle = \langle \varepsilon \rangle v}$$

simple!

This simple relation tells us that the average power can be viewed as the average energy density moving at the wave speed!

⑩ The principle of superposition. Now we understand the simple sinusoidal waves. But, can we understand more complex waves in systematic ways? We need to reveal some interesting properties of the wave equation. Suppose $u_1(x,t)$ and $u_2(x,t)$ satisfy the wave equation. We would like to show that

$$\boxed{u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t)}$$

is also a solution!!

c_1, c_2 are arbitrary constants.

The proof is trivial....

$$\frac{\partial^2 u}{\partial t^2} = c_1\frac{\partial^2 u_1}{\partial t^2} + c_2\frac{\partial^2 u_2}{\partial t^2} = c_1 v^2\frac{\partial^2 u_1}{\partial x^2} + c_2 v^2\frac{\partial^2 u_2}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = v^2\left(c_1\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{\partial^2 u_2}{\partial x^2}\right) = v^2\frac{\partial^2 u}{\partial x^2}$$

So $u = c_1u_1 + c_2u_2$ is also a soln.

The above theorem is the so-called the principle of superposition. It explains the interferences between different waves. Consider two sinusoidal waves moving in opposite directions: same A, same k, same ω

$$u_1 = A\sin(kx-\omega t), \quad u_2 = A\sin(kx+\omega t)$$

Adding them up leads to the "standing wave",

$$u = u_1 + u_2 = A\sin(kx-\omega t) + A\sin(kx+\omega t)$$

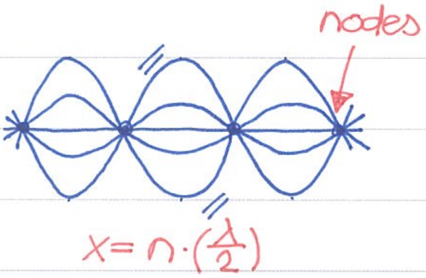
Making use of the identity for sinusoidal functions,

$$\sin X + \sin Y = 2\sin\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right).$$

The standing wave is thus described by the following form,

$$\boxed{u(x,t) = 2A\sin(kx)\cos(\omega t) = A(x)\cos\omega t}$$

A(x) is the local amplitude.



At $kx = n\pi$, the local amplitude $A(x) = 0$ — no oscillation at all.

$$\boxed{x = \frac{n\pi}{k} = \frac{1}{2}n\lambda}$$

These static points are called "nodes".

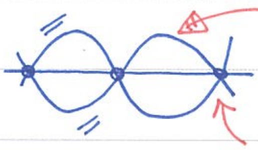
Because the standing wave is a superposition of R-moving and L-moving waves, it is expected that energy transfer rate is zero — that explains the name of "Standing".

$$\boxed{\langle P\rangle = \langle\varepsilon\rangle v - \langle\varepsilon\rangle v = 0}$$

In addition, the average energy density is not uniform anymore,

$$\langle\varepsilon\rangle = 2\langle\varepsilon_k\rangle = \cancel{2}\cdot\frac{1}{\cancel{2}}\rho\left\langle\left(\frac{\partial u}{\partial t}\right)^2\right\rangle = 4\rho\omega^2A^2\sin^2(kx)\langle\sin^2\omega t\rangle$$

$$\rightarrow \boxed{\langle\varepsilon\rangle = 2\rho\omega^2A^2\sin^2 kx}$$



$\langle\varepsilon\rangle_{max} = 2\rho\omega^2A^2$

$\langle\varepsilon\rangle = 0$ @ nodes

These standing waves turn out to be the natural solutions for a string with fixed ends.

Ø Standing waves on a string. Consider a string with both ends fixed, i.e. $u(0,t) = 0 = u(L,t)$

Choose the wave number k wisely so that the nodes coincide with the fixed ends.

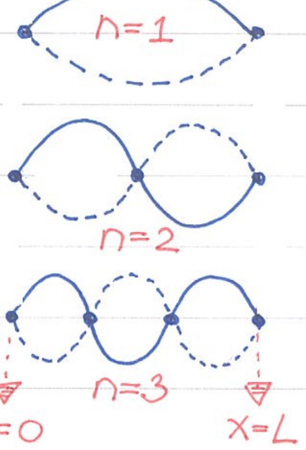
drop 2A here

$$\boxed{u_n(x,t) = \sin k_n x \cos \omega_n t}$$

$$u_n(L,t) = 0 \longrightarrow k_n L = n\pi$$

$$\boxed{k_n = \frac{n\pi}{L} \longleftrightarrow \lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}}$$

Standing waves on a string



The allowed frequencies on the string can be computed as well,

$n = 1, 2, 3, \cdots$

$$\boxed{f_n = \frac{\omega_n}{2\pi} = \frac{v}{2\pi} k_n = \frac{v}{2L} \cdot n}$$

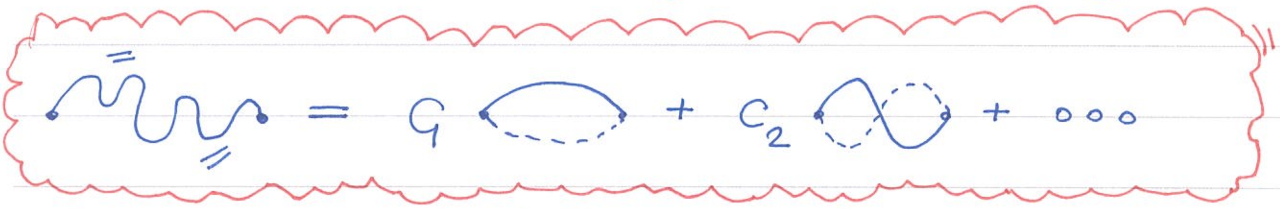
All frequencies are integer multiples of $f_1 = \frac{v}{2L}$

Making use of the principle of superposition again, combinations of standing waves are also solutions of the wave equation,

$$\boxed{\begin{aligned} u(x,t) &= c_1 u_1(x,t) + c_2 u_2(x,t) + c_3 u_3(x,t) + \cdots\cdots \\ &= \sum_{n=1}^{\infty} c_n \sin k_n x \cos \omega_n t \end{aligned}}$$

Fourier analysis.

It turns out that all possible waves on the string can be described by the above form — as long as the coefficients c_n are chosen properly. I won't bore you with the detail proof here. But, the message is very surprising.



Once we understand the sinusoidal waves, we can understand all kinds of wave dynamics!

⊘ Taylor expansion. We will revisit the wave equation from a different perspective. We need a powerful mathematical tool first. Given an analytic function f(x), it can be expressed as

$$\boxed{f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots}$$

↑ Taylor Expansion for f(x).

Here comes the proof...

Suppose an arbitrary function can be written in power series,

$$f(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n + \cdots.$$

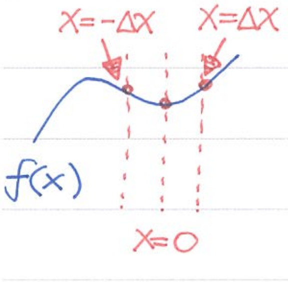
We would like to figure out C_n here ☺

① Plug in $x=0$ on both sides, $f(0) = C_0$. — a good start.

② Take one derivative and then plug in $x=0$ on both sides,

$$\rightarrow \frac{df}{dx}(0) = C_1 + \cancelto{0}{2C_2 \cdot 0} + \cancelto{0}{3C_3\, 0^2} + \cdots \quad \rightarrow \quad C_1 = f'(0)$$

③ Apply the same trick again and again. It shall be easy to show that $f^{(n)}(0) = n!\, C_n \longrightarrow C_n = \frac{f^{(n)}(0)}{n!}$ q.e.d.



We can apply Taylor expansion to estimate the difference between f(0) and its neighbors,

$$f(\pm\Delta x) = f(0) \pm f'(0)\Delta x + \frac{1}{2!}f''(0)(\Delta x)^2 + \cdots\cdots$$

The average value of its neighbors is approximately

$$\boxed{\langle f \rangle \equiv \frac{1}{2}\left[f(\Delta x) + f(-\Delta x)\right] \approx f(0) + \frac{1}{2!}f''(0)(\Delta x)^2}$$

The higher order terms can be ignored as long as Δx is extremely small. We thus find an interesting relation,

$$\boxed{f(0) - \langle f \rangle = -\frac{1}{2}f''(0)(\Delta x)^2}$$

← $f''(0)$ shows up ☺

⑪ Wave equation, revisited. Apply the same estimate on the 1D string. The deviation from local equilibrium $\langle u \rangle$ can be expressed in a similar way,

$$\boxed{u(x) - \langle u \rangle = -\frac{1}{2}(\Delta x)^2 \frac{\partial^2 u}{\partial x^2}}$$

$$\frac{1}{2}u(x+\Delta x) + \frac{1}{2}u(x-\Delta x) \equiv \langle u \rangle$$

If $\partial^2 u/\partial x^2 \neq 0$, the tiny segment is out of local equilibrium. In an elastic string, it tries to use the acceleration to restore the local equilibrium:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = -C\left[u(x) - \langle u \rangle\right] = \frac{C}{2}(\Delta x)^2 \frac{\partial^2 u}{\partial x^2}}$$

Can you see Hooke's law here?

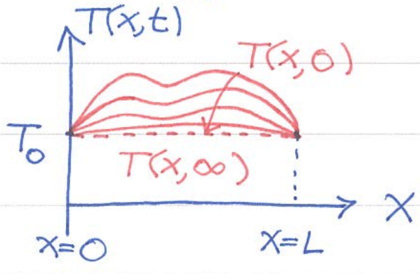
Set the constant $\frac{1}{2}C(\Delta x)^2 = v^2$ and the wave equation appears,

$$\rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}}$$

Hum, using "acceleration" to restore the local equilibrium is overreacting.

Therefore, overshoot occurs and the wave motion is oscillatory — aiming at reaching equilibrium but never staying there.

Wave dynamics is not the only choice. Consider a thermal rod with both ends held at constant temperature T_0. Given some strange profile $T(x,0)$ initially. Our common sense tells us that



$$\boxed{T(x,\infty) \equiv \lim_{t\to\infty} T(x,t) = T_0}$$

What happens here is diffusive dynamics — the system tries to restore local equilibrium by the corresponding "velocity".

$$\boxed{\frac{\partial T}{\partial t} = -C\left[T(x) - \langle T \rangle\right] = \frac{1}{2}C(\Delta x)^2 \frac{\partial^2 T}{\partial x^2}}$$

Set $\frac{1}{2}C(\Delta x)^2 = D$, and it leads to

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$$

diffusion equation

Using "velocity" to restore local equilibrium is very efficient. Thus, diffusive dynamics often leads to very boring and trivial final state.

We can push this idea to the dynamics of matter wave in quantum world. The system is trying to use "imaginary velocity" to restore the local equilibrium,

$$\boxed{i\frac{\partial\psi}{\partial t} = -C\left[\psi(x) - \langle\psi\rangle\right] = \frac{1}{2}C(\Delta x)^2\frac{\partial^2\psi}{\partial x^2}}$$

Re-inserting the Planck Constant in proper positions, we arrive at the Schrödinger equation for free particles,

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}}$$

It is natural to guess that local equilibrium won't be restored this way, ha

Indeed, quantum dynamics looks rather exotic as demonstrated in class lecture.

It is fun to visualize these different dynamics by animation,

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cos(n\pi v t) && \leftarrow \text{wave} \\ T(x,t) &= \sum_{n=1}^{\infty} a_n \sin(n\pi x)\, e^{-n^2\pi^2 D t} && \leftarrow \text{diffusion} \\ \psi(x,t) &= \sum_{n=1}^{\infty} a_n \sin(n\pi x)\, e^{-\frac{i}{2m}n^2\pi^2 t} && \leftarrow \text{quantum } (\hbar=1 \text{ here}) \end{aligned}}$$

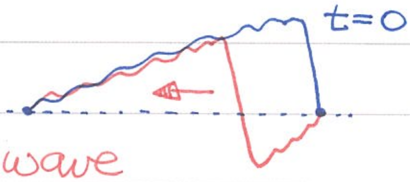
All dynamics starts with the same triangular profile,

$$a_n = \frac{(-1)^{n-1}}{n\pi}\cdot C$$

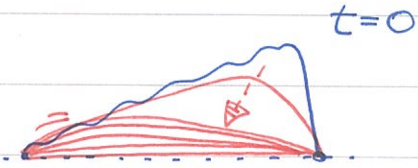
(The constant C=200 in the numerical program for visual clarity.)

note that $u(x,0) = T(x,0) = \psi(x,0)$.

In addition, the infinite sum is truncated at $n_{max}=20$. This gives rise to some wavy correction of the triangular shape.

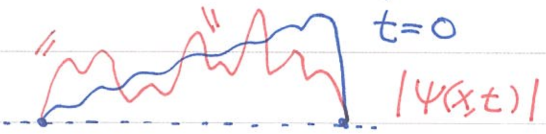


wave



diffusion

And, the quantum dynamics is pretty wild. It just keeps going with crazy dance. Never come back to the initial state like the elastic wave. Never go into the boring final state like the diffusion. It is the charm of quantum mechanics ☺



quantum dance.

Happy

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