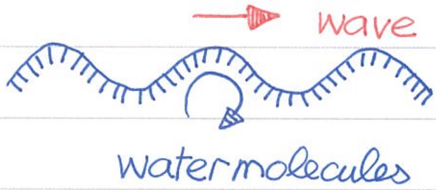
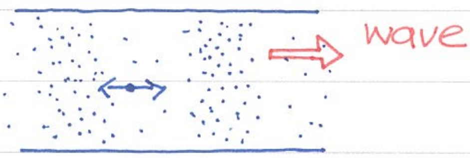
HH0099 What Is Propagating in Traveling Waves?

Water waves are commonly seen in daily life.

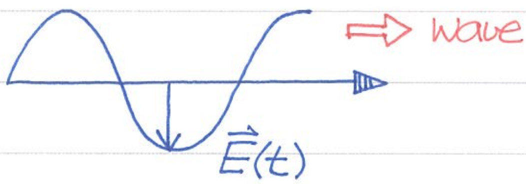


Sound waves and light waves are important because we have biological receptors for these waves. It is important to emphasize that the vibrational direction of microscopic molecules is not necessarily the same as that of wave propagation!



longitudinal wave

Vibrational direction of air molecules in sound waves is parallel to that of wave propagation $\rightarrow$ longitudinal.

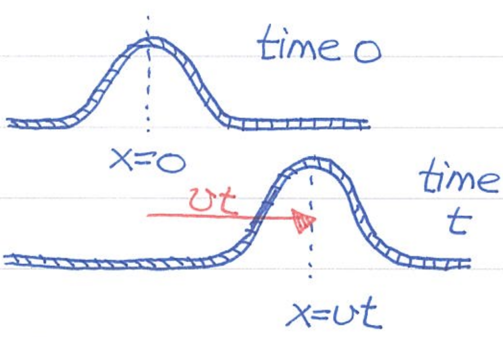


transverse wave.

The vibrating electric field $\vec{E}(t)$ is perpendicular to the wave number $\vec{k}$ $\rightarrow$ transverse wave!

Ø 1D traveling wave.

Consider a transverse wave traveling on a long stretched string. We can use a time-dependent scalar field $u(x,t)$ to describe its dynamics.



At $t=0$, $u(x,0)=f(x)$. At later time t, the wave travels the distance vt,

$$\rightarrow \boxed{u(x,t) = f(x-vt)}$$

Here, v is the wave speed

Similarly, one can perform the same analysis for left moving waves

$$\rightarrow \boxed{u(x,t) = g(x+vt)}$$

The field $u(x,t)$ contains all information about the wave!

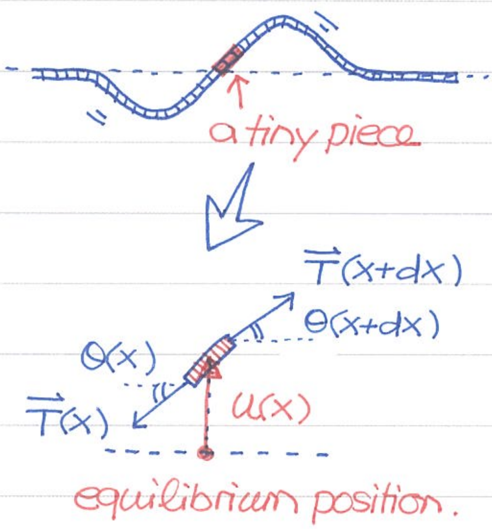
Because the 1D waves on a string can travel in either R or L directions, the general solution is

$$\boxed{u(x,t) = f(x-vt) + g(x+vt)}$$ d'Alembert solution (1747)

So, we got the solution without knowing the wave equation for propagation dynamics ☺ It is important to stress that a traveling wave has speed v, but it may not be a periodic wave and thus has NO definite wavelength λ or frequency f!

⊘ Wave equation from EOM. Consider the dynamics of a "tiny piece" and write down its EOMs. According to Newton's 2nd law,

x-direction → no motion.



$$T(x)\cos\theta(x) = T(x+dx)\cos\theta(x+dx)$$

That is to say, the x-component of the tension is constant

$$\rightarrow \boxed{T_x = T} \rightarrow \text{constant!}$$

In the y-direction, $\quad T_y(x+dx) - T_y(x) = (\rho\, dx)\dfrac{\partial^2 u}{\partial t^2}$

From the figure in above, $T_y = T_x \tan\theta = T \cdot \dfrac{\partial u}{\partial x}$. Meanwhile,

$$T_y(x+dx) - T_y(x) = \frac{dT_y}{dx}\cdot dx = T\frac{\partial^2 u}{\partial x^2}dx$$

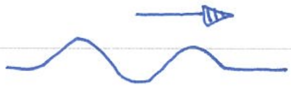
Substitute back to EOM in the y-direction:

$$\rightarrow T\frac{\partial^2 u}{\partial x^2}\cancel{dx} = \rho\cancel{dx}\frac{\partial^2 u}{\partial t^2} \rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}}$$ WAVE equation

It turns out the prefactor T/ρ is the wave speed square, i.e. $\frac{T}{\rho} = v^2 \longrightarrow \boxed{v = \sqrt{\frac{T}{\rho}}}$

Waves propagate faster for stronger tension T and lower linear density ρ.

Let's check d'Alembert solution indeed satisfies the wave eq..



$u(x,t) = f(x-vt)$ introduce $z = x - vt$

$$\frac{\partial u}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{d^2 f}{dz^2}\frac{\partial z}{\partial x} = \frac{d^2 f}{dz^2}.$$

compare

$$\frac{\partial u}{\partial t} = \frac{df}{dz}\frac{\partial z}{\partial t} = (-v)\frac{df}{dz}, \quad \frac{\partial^2 u}{\partial t^2} = (-v)\frac{d^2 f}{dz^2}\frac{\partial z}{\partial t} = v^2\frac{d^2 f}{dz^2}$$

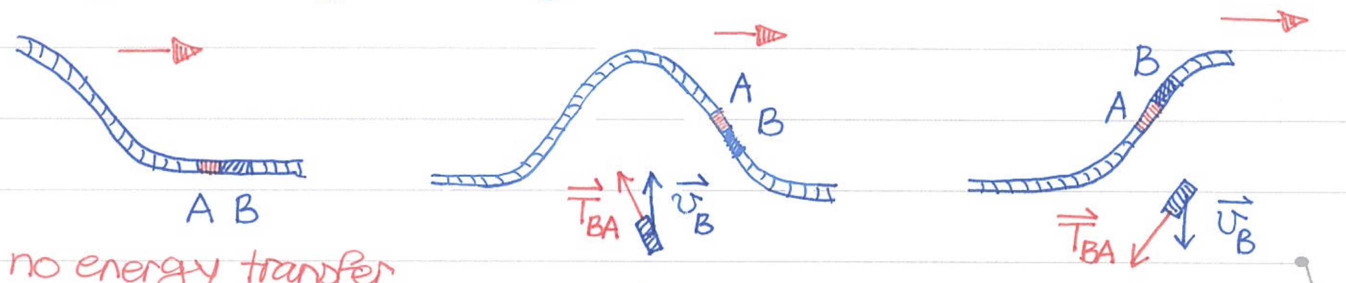
It is clear that the right-moving wave satisfies the wave eq.,

$$\boxed{\frac{\partial^2 u}{\partial t^2} = v^2\frac{\partial^2 u}{\partial x^2}}$$

In addition, it also shows that the wave speed $v = \sqrt{T/\rho}$ by comparison.

One can repeat the same calculation for left-moving wave $u(x,t) = g(x+vt)$ — it satifies the same wave equation.

⊘ Energy propagation. So, what is propagating in traveling waves? Energy is the first answer coming to mind. Consider a right-moving traveling wave



no energy transfer, yet

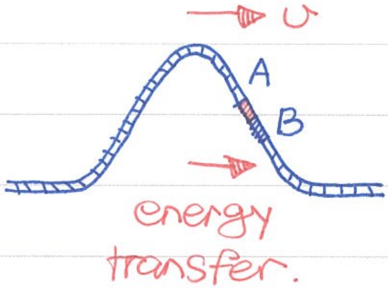
$P = \vec{T}_{BA} \cdot \vec{v}_B > 0$ $\qquad$ $P = \vec{T}_{BA} \cdot \vec{v}_B > 0$

Because the wave motion is transverse, the power of trasmitted energy only depends on the y-component of the tension,

$$\boxed{P = \frac{dE}{dt} = \vec{T}_{BA} \cdot \vec{v}_B = T_y \frac{\partial u}{\partial t} > 0}$$

always positive

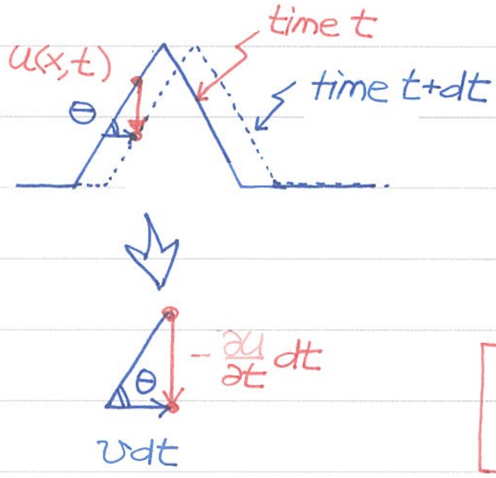
Because the power is always positive, we conclude that energy transfers from A to B in a right-moving wave.



① Try to repeat the same analysis for a left-moving wave. What do you expect to find ?

② We can relate the energy transmission rate P to the "shape" of the wave.

Consider the R-moving wave again. From the geometric relation,



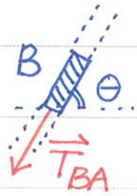
$$\tan\theta = \frac{\partial u}{\partial x} = \frac{-\partial u/\partial t \, \cancel{dt}}{v \, \cancel{dt}}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x}}$$

R-moving waves.

Furthermore, the tension acting on B is

$$\boxed{T_y = -T_x \tan\theta = -T \frac{\partial u}{\partial x}}$$



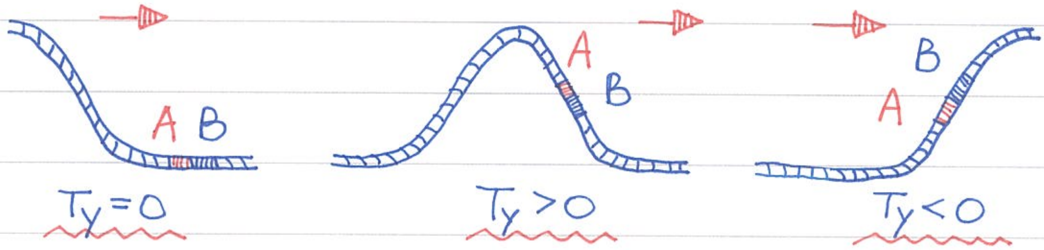
Here I use the important relation $T_x = T = \text{const}$ in a stretched string as studied before. Finally, the energy transmission rate for a R-moving wave is

$$\boxed{P = \frac{dE}{dt} = T_y \frac{\partial u}{\partial t} = T v \left(\frac{\partial u}{\partial x}\right)^2}$$

It is indeed positive at all times.

Note that $\partial u/\partial x$ is dimensionless and Tv gives the right dimension of energy power. The above relation tells us how P is related to the "shape" ($\partial u/\partial x$ here). Great!

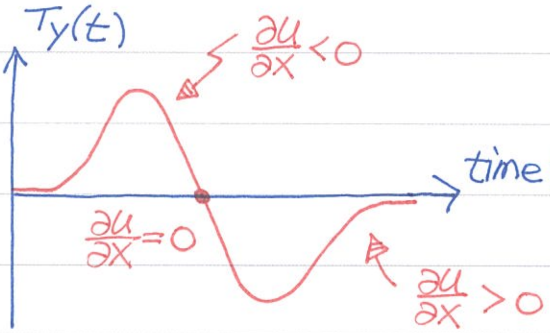
∅ Other "stuffs" propagated ... The traveling waves can propagate momentum as well.



The momentum P_y transfer per unit time is the force T_y,

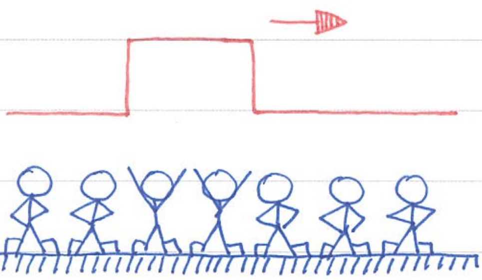
$$\boxed{\frac{dP_y}{dt} = T_y = -T_x \tan\theta = -T\frac{\partial u}{\partial x}}$$

The momentum transfer is more complicated.



The P_y momentum transfer rate is the same as the force that originally created the shape of the wave ☺ Thus, the traveling wave can propagate momentum P_y as well. Anything else?

Consider a row of people performing a series of collective motions as shown below. Apparently, this is not a mechanical wave. But, with proper training, the "signal" can be propagated. Or, one can say that some information is propagating through the row of people.



Is this a WAVE?

清大東院
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