HH0091 Energy, Energy, Energy

In previous lecture, we derived the E-conservation for the one-particle system:

$$\boxed{\Delta(K+U) = W_{nc}}$$

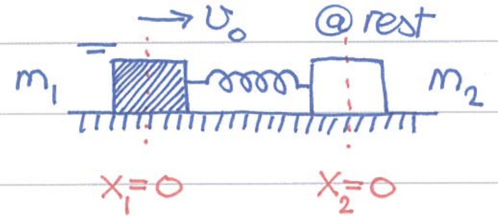
When $W_{nc}=0$, the total energy $K+U=\text{const}$.

Can we generalize the above E-conservation to many-particle systems? Yes. The conservation of total energy takes the form,

$$\boxed{\Delta\left(\frac{1}{2}Mv_{cm}^2 + U_{ex} + E_{in}\right) = W_{nc}}$$

U_{ex} is the potential energy of the external conservative force. $E_{in} = K_{in} + U_{in}$ is the internal energy of the system.

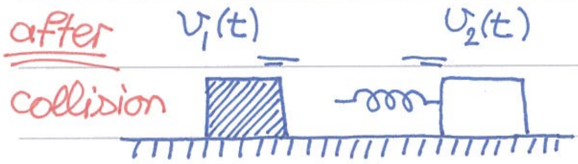
⊘ Elastic collision revisited. Write down the EOM's first



$$\boxed{m_1\frac{dv_1}{dt} = f_{12}, \quad m_2\frac{dv_2}{dt} = f_{21}}$$

Here $f_{12} = -f_{21} = -k(x_1-x_2)$.

Integrate the EOM's:



$$\Delta K_1 = \int_1^2 f_{12}\,dx_1$$

$$\Delta K_2 = \int_1^2 f_{21}\,dx_2 = -\int_1^2 f_{12}\,dx_2$$

Add these equations up,

$$\Delta(K_1+K_2) = \int_1^2 f_{12}\,dx_1 - \int_1^2 f_{12}\,dx_2 = \int_1^2 f_{12}\,d(x_1-x_2)$$

Because f_{12} is conservative, one can introduce the potential energy

$$\Delta U_{12} \equiv -\int_1^2 f_{12}\,d(x_1-x_2) = \frac{1}{2}k(x_1-x_2)^2\Big|_1^2$$

In this case, the potential energy U_{12} is rather simple,

$$\boxed{U_{12} = U_{12}(x_1-x_2) = \frac{1}{2}k(x_1-x_2)^2 + \text{const}}$$

The E-conservation for the two-particle system is

$$\Delta(K_1+K_2) = \int_1^2 f_{12}\, d(x_1-x_2) = -\Delta U_{12}$$

$$\Rightarrow \boxed{\Delta(K_1+K_2+U_{12}) = 0}$$

Sounds reasonable, but not done yet.

We can separate the kinetic energy into two parts. Rewriting the velocity with respective to the center of mass $v_i = v_i' + v_{cm}$

$$K = K_1 + K_2 = \frac{1}{2}m_1(v_1'+v_{cm})^2 + \frac{1}{2}m_2(v_2'+v_{cm})^2$$

$$= \underbrace{\left(\frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2\right)}_{\text{internal kinetic energy}} + \underbrace{\frac{1}{2}(m_1+m_2)v_{cm}^2}_{\text{cm motion}} + \underbrace{\cancel{(m_1v_1'+m_2v_2')}}_{\text{by definition}}{}^{\nearrow 0}\, v_{cm}$$

Thus, the total kinetic energy can be written as

$$\boxed{K = \frac{1}{2}Mv_{cm}^2 + K_{in}} \quad \text{where } K_{in} = \sum_{i=1}^{2}\frac{1}{2}m_i v_i'^2$$

The above decomposition can be generalized to N-particle systems easily and I shall not bore you with the details.

For two-particle systems, the internal kinetic energy Kin is particularly simple ☺

$$v_1' = v_1 - v_{cm} = v_1 - \frac{m_1v_1+m_2v_2}{m_1+m_2} = \frac{m_2}{m_1+m_2}(v_1-v_2) = \frac{\mu}{m_1}v_{12}$$

where the reduced mass $\mu = m_1m_2/m_1+m_2$. Similarly, one finds that $v_2' = \frac{\mu}{m_2}v_{12}$ substitute into K_{in} ☺

The internal kinetic energy now is truly simple.

$$\boxed{K_{in} = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 = \frac{1}{2}\mu^2 v_{12}^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right) = \frac{1}{2}\mu v_{12}^2}$$

Collect all pieces together, the E-conservation for the elastic collision is

$$\boxed{\Delta\left(\frac{1}{2}Mv_{cm}^2 + K_{in} + U_{12}\right) = 0}$$ almost done, but not yet.

During the collision, v_{cm} = const and so is $\frac{1}{2}Mv_{cm}^2$ — no change. Introduce the internal energy E_{in}

$$E_{in} \equiv K_{in} + U_{in} = \frac{1}{2}\mu v_{12}^2 + \frac{1}{2}kx_{12}^2 \qquad \begin{cases} v_{12} = v_1 - v_2 \\ x_{12} = x_1 - x_2 \end{cases}$$

The E-conservation takes the simple form,

$$\boxed{\Delta E_{in} = 0 \quad \longrightarrow \quad \Delta\left(\frac{1}{2}\mu v_{12}^2 + \frac{1}{2}kx_{12}^2\right) = 0}$$

Let's check whether the above conclusion is right.

$$v_1(t) = v_{cm} + \frac{m_2}{m_1+m_2}v_0\cos(\omega_0 t)$$

$$v_2(t) = v_{cm} - \frac{m_1}{m_1+m_2}v_0\cos(\omega_0 t)$$

$$v_{12}(t) = v_1 - v_2 = v_0\cos(\omega_0 t)$$

The above results come from solving the EOM's and $\omega_0 = \sqrt{\frac{k}{\mu}}$.

Note that $x_{12}(t=0) = x_1(0) - x_2(0) = 0$,

$$x_{12}(t) = \int_0^t v_{12}(t')\,dt' = \frac{v_0}{\omega_0}\sin(\omega_0 t)$$

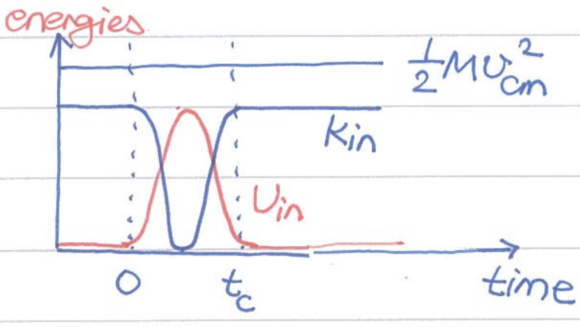
$t = \frac{\pi}{\omega_0} = t_c$, $x_{12}(t_c) = 0$!

We are ready to compute the internal energy

$$E_{in} = K_{in} + U_{in} = \frac{1}{2}\mu v_{12}^2 + \frac{1}{2}kx_{12}^2$$

$\omega_0^2 = k/\mu$

$$= \frac{1}{2}\mu v_0^2\cos^2(\omega_0 t) + \frac{1}{2}k\frac{v_0^2}{\omega_0^2}\sin^2(\omega_0 t) = \frac{1}{2}\mu v_0^2 \;!?$$



Plot all energies as shown on the left. It is clear that

$$K_{in} + U_{in} = \text{const}$$

⓪ Many-particle system. Let us start with the EOM's and carefully separate the forces into conservative and non-conservative parts.

$$\boxed{\begin{aligned} m_1 \frac{d\vec{v}_1}{dt} &= \vec{F}_1 + (\vec{f}_{12} + \vec{f}_{13} + \cdots + \vec{f}_{1N}) \\ &\vdots \\ m_N \frac{d\vec{v}_N}{dt} &= \vec{F}_N + (\vec{f}_{N1} + \vec{f}_{N2} + \cdots + \vec{f}_{NN-1}) \end{aligned}}$$

Integrate the EOM's to obtain various energies.

$$\Delta K_1 = \int_1^2 \vec{F}_1^c \cdot d\vec{r}_1 + \int_1^2 (\vec{f}_{12}^c + \vec{f}_{13}^c + \cdots + \vec{f}_{1N}^c) \cdot d\vec{r}_1 + W_1^{nc}$$

$$= -\Delta U_1 + W_1^{nc} + \sum_{j \neq 1} \int_1^2 \vec{f}_{1j}^c \cdot d\vec{r}_1$$

Adding all contributions together, the E-conservation reads

$$\boxed{\begin{aligned} \Delta(K_1 + \cdots + K_N) = -\Delta(U_1 + \cdots + U_N) + (W_1^{nc} + \cdots + W_N^{nc}) \\ + \sum_{\substack{(i,j) \\ \text{pairs}}} \int_1^2 \vec{f}_{ij}^c \cdot d(\vec{r}_i - \vec{r}_j) \end{aligned}}$$

Some reorganization is in order ☺

① $K_1 + K_2 + \cdots + K_N = \frac{1}{2} M v_{cm}^2 + K_{in}$ ← internal kinetic energy.

② $U_1 + U_2 + \cdots + U_N \equiv U_{ex}$ ← potential energy of external conservative force

③ $\Delta U_{ij} \equiv -\int_1^2 \vec{f}_{ij}^c \cdot d(\vec{r}_i - \vec{r}_j)$ ← internal potential energy.

Note that $\vec{f}_{ij}^c$ satisfies the criterion for the conservative force $\oint \vec{f}_{ij}^c \cdot d(\vec{r}_i - \vec{r}_j) = 0$!!

Finally, the E-conservation for a many-particle system takes the form:

$$\boxed{\Delta\left(\frac{1}{2}Mv_{cm}^2 + U_{ex} + E_{in}\right) = W_{nc}}$$

Major Result ☺

The "mysterious" internal energy of a many-particle system can be spelled out explicitly

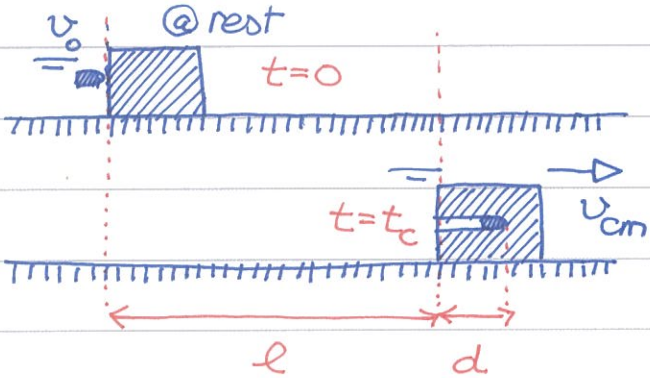
$$\boxed{E_{in} = K_{in} + U_{in} = \sum_{i=1}^{N} \frac{1}{2}m_i v_i'^2 + \sum_{i \neq j} U_{ij}(\vec{r}_i - \vec{r}_j)}$$

In the absence of W_{nc}, the total energy is conserved,

$$E_{total} = \frac{1}{2}Mv_{cm}^2 + U_{ex} + E_{in} = \text{const}\ !$$

But, this is not the end of the story yet.... We shall see in the following example how E_{total} depends on the choice of degrees of freedom (DOF).

Inelastic collision revisited. The kinetic energy of the two-particle system is



$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}\mu v_{12}^2$$

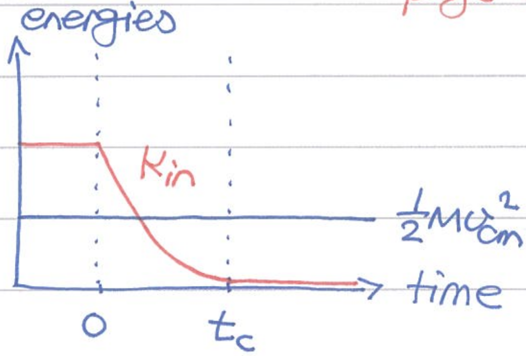
No U_{ex} and no internal potential energy U_{in}. Because v_{cm} is constant during collision, the E-conservation for this case is

$$\boxed{\Delta K_{in} = W_{nc}} \qquad \Delta K_{in} = \frac{1}{2}\mu 0^2 - \frac{1}{2}\mu v_0^2 = -\frac{1}{2}\mu v_0^2$$

Solving the EOM (as done in previous lecture) $\rightarrow d = \dfrac{\mu v_0^2}{2f}$

$$W_{nc} = f\cdot\ell + (-f)(\ell+d) = -f\cdot d = -\frac{1}{2}\mu v_0^2 \quad \text{yes ☺}$$

Plot all energies during the inelastic collision. →
It shall be clear that E_{total} decreases.



Now try to include all molecular motions during the inelastic collision. At molecular level, there is no such thing called "friction". Thus, we expect the total energy is conserved.

BEFORE

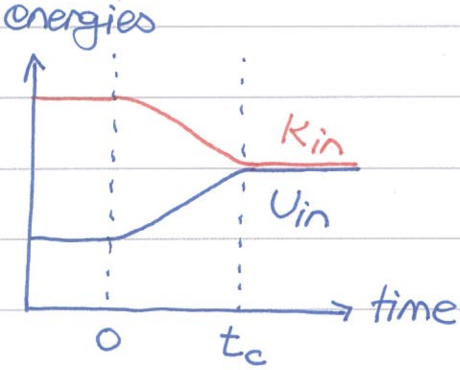
$$\boxed{\begin{aligned} K_{in} &= \varepsilon(T_1) + \tfrac{1}{2}\mu v_0^2 \\ U_{in} &= \varepsilon(T_1) \end{aligned}}$$

$\varepsilon(T_1)$ is the energy due to molecular vibrations.

After

$$\boxed{\begin{aligned} K_{in} &= \varepsilon(T_2) \\ U_{in} &= \varepsilon(T_2) \end{aligned}}$$

After the collision, the temperature increases, $T_2 > T_1$.



The total energy is conserved,

$$\boxed{E_{total} = \frac{1}{2}Mv_{cm}^2 + E_{in} = \text{const}}$$

The macroscopic $K_{in}(\text{macro}) = \frac{1}{2}\mu v_0^2$ is transformed into microscopic E_{in}.

The E-conservation looks rather different for DOF = 2 and DOF = many + MANY → Need thermodynamics

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