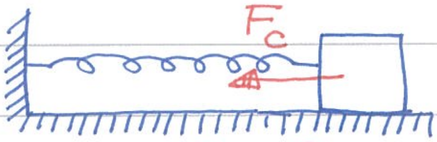
HH0090 Conservative Force and Potential Energy.

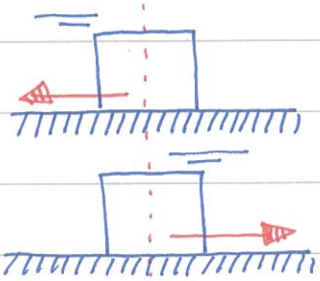
Let us consider 1D motion first. The external force can be classified into two categories: conservative and non-conservative forces.



Conservative force →

$$\boxed{F_c = F_c(x) \text{ only depends on the position.}}$$

On the other hand, friction is non-conservative. Depending on the velocity, the frictional force at the same position may be different. When we apply some force on the system, it is also considered as non-conservative because it depends on many factors.



the same position

For a conservative force, we can introduce the corresponding potential energy

$$\boxed{\Delta U = U(x_2) - U(x_1) \equiv -\int_1^2 F_c(x)\,dx}$$

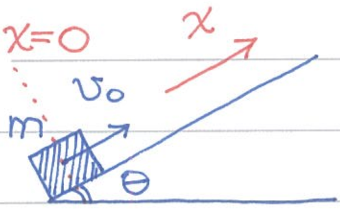
We will see immediately why potential energy is defined this way. The force can be separated into conservative and non-conservative parts:

$$F_c(x) + F_{nc} = m\frac{dv}{dt} \longrightarrow \int_1^2 F_c(x)\,dx + \int_1^2 F_{nc}\,dx = \Delta K$$

Making use of definition of potential energy,

$$-\Delta U + W_{nc} = \Delta K \longrightarrow \boxed{\Delta(K+U) = W_{nc}}$$

In the absence of W_{nc}, the sum of K and U remains constant at all times. This is the simplest version of energy conservation. Again, it can be derived from EOM, just like momentum conservation.



Consider the 1D motion on the left. The EOM is

$$\boxed{-mg\sin\theta - f = m\frac{d^2x}{dt^2}}$$

The gravitational force is a constant pointing in $-x$ direction — a trivial function of position x.

$$U(x_2)-U(x_1) = -\int_1^2 F_c(x)\,dx = -\int_1^2 (-mg\sin\theta)\,dx$$

$$= mg\sin\theta\,(x_2-x_1) \longrightarrow \boxed{U(x) = mg\sin\theta\cdot x}$$

set $U=0$ @ $x=0$.

Note that $x\sin\theta = z$ is just the vertical height. The potential energy we computed here is just the familiar $U = mg\sin\theta\, x = mgz$!

Can we do the same thing for the frictional force? Nope. When moving upward, f points in $-x$ direction. But, when moving downward, f points in x direction. It does not just depend on the position and the potential energy cannot be defined. ☺

Apply the integrated EOM: $W_{nc} = \Delta(K+U)$

$$-f\ell = (0 + mg\sin\theta\,\ell) - (\tfrac{1}{2}mv_0^2 + 0)$$

where ℓ is the position when the block comes to rest before sliding down.

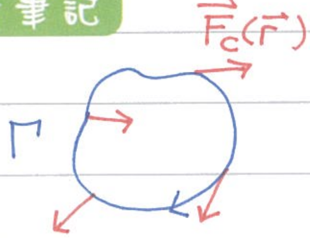
$$mg\sin\theta\,\ell + f\ell = \tfrac{1}{2}mv_0^2$$

$$\longrightarrow \boxed{\ell = \frac{\frac{1}{2}mv_0^2}{mg\sin\theta + f} < \frac{v_0^2}{2g\sin\theta}}$$

Friction makes the distance shorter.

Reasonable ☺

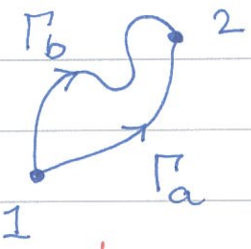
⊘ Criterion for conservative force. In 1D, as long as $F_c = F_c(x)$, potential energy can be defined and it is a conservative force. In 3D, the criterion for conservative force is more stringent,



$$\boxed{\text{For any closed contour, } \oint_\Gamma \vec{F}_c \cdot d\vec{r} = 0}$$

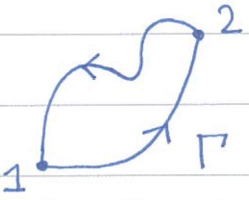
Conservative force!

The above criterion ensures the potential energy is well defined. Following the same spirit, we can define the potential energy as $\boxed{\Delta U = U(\vec{r}_2) - U(\vec{r}_1) \equiv -\int_1^2 \vec{F}_c \cdot d\vec{r}}$



But, we may run into big trouble because the integral may depend on the choice of contour.

$$\int_{\Gamma_a} \vec{F}_c \cdot d\vec{r} \overset{?}{=} \int_{\Gamma_b} \vec{F}_c \cdot d\vec{r}$$



a closed contour.

Choose a closed contour Γ, composed of Γ_a and reversed Γ_b.

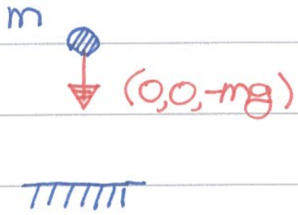
$$\boxed{\oint_\Gamma \vec{F}_c \cdot d\vec{r} = \int_{\Gamma_a} \vec{F}_c \cdot d\vec{r} - \int_{\Gamma_b} \vec{F}_c \cdot d\vec{r}}$$

For a conservative force, $\oint_\Gamma \vec{F}_c \cdot d\vec{r} = 0$, therefore,

$$\int_{\Gamma_a} \vec{F}_c \cdot d\vec{r} = \int_{\Gamma_b} \vec{F}_c \cdot d\vec{r} = -\Delta U \text{ is now well defined!}$$

One can see the potential energy for a conservative force in 3D can be defined in a similar fashion as in 1D. BUT! The criterion for conservative force in 3D is highly non-trivial!

Ⅶ Examples for potential energies

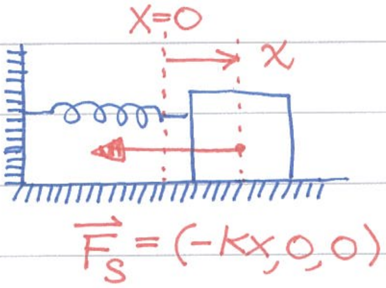


Gravity : $\vec{F}_G = (0,0,-mg)$

$$U(\vec{r_2}) - U(\vec{r_1}) = -\int_1^2 \vec{F}_G \cdot d\vec{r}$$

$$= -\int_1^2 (-mg)\, dz = mgz_2 - mgz_1$$

Thus, the gravitational potential energy is $\boxed{U_G = mgz + \text{const.}}$

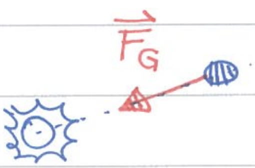


$$U(\vec{r_2}) - U(\vec{r_1}) = -\int_1^2 \vec{F}_S \cdot d\vec{r}$$

$$= -\int_1^2 (-kx)\, dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

$$\longrightarrow \boxed{U_S = \frac{1}{2}kx^2 + \text{const}}$$ elastic p-energy.

Let us try something harder ☺



$$\vec{F}_G = -\frac{GMm}{r^2}\hat{r} = -\frac{GMm}{r^3}(x, y, z)$$

According to the definition,

$$U(\vec{r_2}) - U(\vec{r_1}) = -\int_1^2 \vec{F}_G \cdot d\vec{r} = GMm \int_1^2 \frac{1}{r^3}(x, y, z) \cdot \overbrace{(dx, dy, dz)}^{d\vec{r}}$$

$$= GMm \int_1^2 \frac{1}{r^3}(x\,dx + y\,dy + z\,dz) \rightarrow d(x^2+y^2+z^2) \cdot \frac{1}{2}$$

Change variable to $u = r^2 = x^2 + y^2 + z^2$

$$U(\vec{r_2}) - U(\vec{r_1}) = GMm \int_1^2 \frac{1}{2} u^{-\frac{3}{2}} du$$

$$= GMm \cdot \frac{1}{2} \frac{1}{(-\frac{1}{2})} u^{-\frac{1}{2}} \Big|_1^2 \leftarrow u^{\frac{1}{2}} = r.$$

$$= -\frac{GMm}{r_2} + \frac{GMm}{r_1}$$

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The potential energy takes the famous form,

$$\boxed{U(\vec{r}) = -\frac{GMm}{r} + \text{const}} \qquad r = |\vec{r}|$$

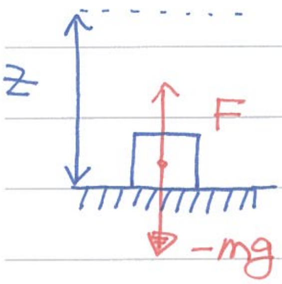
We often drop the constant by setting $U=0$ @ $r \to \infty$.

⊘ Energy conservation revisited. Let us try to derive the conservation of energy again but treat all vectors properly.

$$\vec{F}_{nc} + \vec{F}_c = m\frac{d\vec{v}}{dt} \longrightarrow \int_1^2 \vec{F}_{nc}\cdot d\vec{r} + \int_1^2 \vec{F}_c\cdot d\vec{r} = \int_1^2 m\frac{d\vec{v}}{dt}\cdot d\vec{r}$$

$\longrightarrow W_{nc} - \Delta U = \Delta K$ Finally, we get $\boxed{W_{nc} = \Delta(K+U)}$

Go back to the confusing example before. Lift the block from 0 to z by a force $F=mg$.



$$W_{nc} = F\cdot z = mgz$$

$$\Delta K = 0$$

$$\Delta U = mgz - 0$$

$W_{nc} = \Delta(K+U)$

correct

⊘ Relation between conservative force and potential energy.

In 1D, $U(x_2) - U(x_1) = -\int_1^2 F_c\, dx \longrightarrow \boxed{F_c = -\frac{dU}{dx}}$

examples: ① $U_G = mgz + \text{const}.$

$$F_G = -\frac{dU_G}{dx} = -mg \checkmark$$

② $U_S = \frac{1}{2}kx^2 + \text{const}$

$$F_S = -\frac{dU_S}{dx} = -kx \checkmark$$

But, can we generalize it to 3D?

Yes, we can. Let me just state the answer without proof.

$$\boxed{\vec{F}_c = \left(-\frac{\partial U}{\partial x},\ -\frac{\partial U}{\partial y},\ -\frac{\partial U}{\partial z}\right)}$$

The above relation is very useful. Once we know the potential energy (a scalar), we can find the corresponding conservative force (a vector).

example. Gravity. $U(x,y,z) = -\frac{GMm}{r}$, $r=\sqrt{x^2+y^2+z^2}$

$$F_x = -\frac{\partial U}{\partial x} = GMm\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= GMm\left(-\frac{1}{r^2}\right)\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} = -\frac{GMm}{r^3}\cdot x$$

Similarly, $F_y = -\frac{\partial U}{\partial y} = -\frac{GMm}{r^3}\,y$ and $F_z = -\frac{GMm}{r^3}\,z$

Collecting all components together,

$$\vec{F} = -\frac{GMm}{r^3}(x,y,z) = -\frac{GMm}{r^2}\,\hat{r}$$ Yes

Finally, we understand how to relate a conservative force with its potential energy by calculus!

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