Multivariable Calculus
Don Shimamoto

The book was last updated on November 9, 2021.

There are two pdf versions of the book. In one, the hyperlinks are enabled, and, in the other, they are not. The first may be a more suitable format for onscreen reading, the second more suitable for printing.

Main changes since the January 4, 2021 version
(not including changes in punctuation or minor changes in wording)

- **p. 41, Proposition 2.23.** The cross product is defined only for vectors in $\mathbb{R}^3$, so part 2 of the proposition should include a stipulation that α and β are paths in $\mathbb{R}^3$:

  . . .

  2. (Cross product) *For paths* α, β *in* $\mathbb{R}^3$, $(\alpha \times \beta)' = \alpha' \times \beta + \alpha \times \beta'$.

  . . .

- **p. 283, remarks before Exercise 4.19.** There is a typographical error three lines up from where the exercise begins. According to our notational convention, the integral of the differential form $f\, dy \wedge dx$ should have one integral sign, not two, so the correct statement is: " . . . as an integral of a differential form, $\int_D f\, dy \wedge dx$ is equal to the Riemann integral $\iint_D f(x,y)\, dx\, dy$."

Main changes since the October 15, 2020 version
(not including changes in punctuation or minor changes in wording)

- **p. 32, Figure 2.10.** A footnote has been added to the caption:

  [1]From now on, we won't always state that a vector has been translated. Any vector that starts at a point other than the origin should be understood to be a translated copy whether there is an explicit note to that effect or not.

  This increases the numerical labels of all the other footnotes by 1.

- **p. 153, Exercise 6.4.** The wording has been clarified to indicate that the xy-plane itself, $z = 0$, is also one of the surfaces that bounds W:

  . . . bounded by the surface $y = x^2$ and the planes $z = y$, $y = 1$, and $z = 0$.

- **p. 187, equation (7.5).** The lower endpoint of the innermost integral has been corrected. It should be $-\sqrt{1-x_1^2-x_2^2-x_3^2}$, not $-\sqrt{1-x_2^2-x_2^2-x_3^2}$:

$$\begin{aligned} \mathrm{Vol}\,(W) &= \iiint_B \left(\int_{-\sqrt{1-x_1^2-x_2^2-x_3^2}}^{\sqrt{1-x_1^2-x_2^2-x_3^2}} 1\, dx_4 \right) dx_1\, dx_2\, dx_3 \qquad (7.5) \\ &= \dots \end{aligned}$$

- **p. 218, Example 9.12.** In the first set of displayed equations, the middle equation should have two integral signs:

$$\begin{aligned} &\dots \\ &= \iint_D \big(y-(-2y)\big)\, dx\, dy \\ &\dots \end{aligned}$$

- **p. 285, answer to Section 1.2, Exercise 2.2.** The answer begins with a typo: "(y_1+y_2)" should be "(y_1, y_2)." Perhaps it's better anyway to write the solution as follows instead:

  2.2 $T(\mathbf{x}+\mathbf{y}) = T(x_1+x_2, y_1+y_2) = (x_1+y_1, x_2+y_2, 0) = (x_1, x_2, 0) + (y_1, y_2, 0) = T(\mathbf{x}) + T(\mathbf{y})$
  $T(c\mathbf{x}) = T(cx_1, cx_2) = (cx_1, cx_2, 0) = c(x_1, x_2, 0) = c\,T(\mathbf{x})$

Main changes since the July 20, 2020 version

(not including changes in punctuation or minor changes in wording)

- **p. 65, Example 3.16.** In the definition of the set K, the plainface x should be a boldface $\mathbf{x}$: "Let $K = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \le 1\}$. ..."

- **p. 104, first paragraph.** The operator $\nabla \cdot \mathbf{v}$ is better written as $\mathbf{v} \cdot \nabla$, so the notation has been changed:

  ... each additional derivative of g corresponds to applying the "operator" $\mathbf{v} \cdot \nabla = (h,k) \cdot (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$. ...

- **p. 118, Exercise 11.3, second paragraph.** As in the previous item, $\nabla \cdot \mathbf{v}$ has been changed to $\mathbf{v} \cdot \nabla$: " ... the powers of the operator $\mathbf{v} \cdot \nabla = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$ ..."

- **p. 266, Exercise 3.5, parts (a) and (b).** The exercise has been reworded to resolve potential ambiguity about the names of the coordinates in $\mathbb{R}^3$, namely:

  3.5 (a) Find an example of a smooth vector field $\mathbf{F} = (F_1, F_2, F_3)$ defined on an open set U of $\mathbb{R}^3$ such that:

- its mixed partials are equal, i.e., $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, and $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$, and
- there exists a piecewise smooth oriented simple closed curve C in U such that $\int_C F_1\,dx + F_2\,dy + F_3\,dz \neq 0$.

(b) On the other hand, show that, if $\mathbf{F}$ is any smooth vector field on an open set U in $\mathbf{R}^3$ whose mixed partials are equal, then $\int_C F_1\,dx + F_2\,dy + F_3\,dz = 0$ for any piecewise smooth oriented simple closed curve C that is the boundary of an oriented surface S contained in U.

(c) ...

Main changes since the May 4, 2020 version

(not including changes in punctuation or minor changes in wording)

- **p. 151, Exercise 4.1(c) and p. 169, Exercise 4.3.** In these exercises, the spherical coordinate ϕ was typeset as "φ," which is inconsistent with the rest of the book. This has been corrected.

- **p. 252, definition of closed surface.** The definition now includes the assumption that the surface is *path-connected*. That is:

  Definition. Let S be a piecewise smooth surface in $\mathbb{R}^n$ such that every pair of points in S can be joined by a piecewise smooth curve in S. Then S is called **closed** if $\partial S = \emptyset$ (the empty set).

Main changes since the March 4, 2020 version

(not including changes in punctuation or minor changes in wording)

- **p. 68, top line.** The expression for $f(x,0)$ has been corrected: " ... $f(x,0) = \frac{x \cdot 0}{x^2+0^2} = 0$ ... "

- **p. 87, equation (4.6).** There are no cross-references to the equation, so its label has been removed. As a result, the labeled equations that follow in the rest of the chapter have their numbers decreased by 1, up to (4.30) on p. 117.

- **p. 101, Figure 4.9 and p. 115, Figure 4.14.** The viewpoint has been rotated so that x and y increase in the usual directions.

- **p. 139, sentence right before Example 5.16.** For the spherical coordinate ϕ, we use the interval $0 \leq \phi \leq \pi$ to describe all of $\mathbb{R}^3$, but the value $\phi = \pi$ was also included among the unnecessary redundant values. The corrected sentence is: "This is why values of ϕ in the interval $\pi < \phi \leq 2\pi$ are not needed—they would duplicate points already covered."

- **p. 247, equation (10.7).** This is another labeled equation to which there are no cross-references, so the label has been removed. The labeled equations that follow in the rest of the chapter have their numbers decreased by 1, up to (10.31) on p. 269.
- **p. 292, answer to Chapter 3, Exercise 2.7.** For the surface in the middle, the viewpoint has been rotated so that x and y increase in the usual directions.

Main changes since the January 8, 2020 version
(not including changes in punctuation or minor changes in wording)

- **p. 27, Example 2.4.** The direction vector $\mathbf{v}$ should be nonzero: "Given a point $\mathbf{a}$ and a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, ..."
- **p. 77, Exercise 6.1.** Change to a function of two variables: "Let $\mathbf{a} = (1, 2)$. Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ is continuous."

  The answer on p. 293 must be modified, too: "$f(x, y) = (1, 2) \cdot (x, y) = x + 2y$. We know that the projections x and y are continuous, so, as an algebraic combination of continuous functions, f is continuous as well."
- **p. 79, Exercise 8.3.** This exercise is new.

  8.3 (Sandwich principles.) Let U be an open set in $\mathbb{R}^n$, and let $\mathbf{a}$ be a point of U.

  (a) Let $f, g, h\colon U \to \mathbb{R}$ be real-valued functions such that $f(\mathbf{x}) \leq g(\mathbf{x}) \leq h(\mathbf{x})$ for all $\mathbf{x}$ in U. If $f(\mathbf{a}) = h(\mathbf{a})$—let's call the common value c—and if f and h are continuous at $\mathbf{a}$, prove that $g(\mathbf{a}) = c$ and that g is continuous at $\mathbf{a}$ as well.

  (b) Let f, g, h be real-valued functions defined on U, except possibly at the point $\mathbf{a}$, such that $f(\mathbf{x}) \leq g(\mathbf{x}) \leq h(\mathbf{x})$ for all $\mathbf{x}$ in U, except possibly when $\mathbf{x} = \mathbf{a}$. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} h(\mathbf{x}) = L$, prove that $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = L$, too.
- **p. 98, Section 4.9, last paragraph.** The discussion of differentiability, smoothness, and continuity when the domain D is not open is meant to apply to real-valued functions. The same comments remain valid for vector-valued functions $f\colon D \to \mathbb{R}^m$, but that would be getting ahead of ourselves at this point of the book. Also, the accommodation in the case of continuity is spelled out in the more customary way.

  > ... If D is a subset of $\mathbb{R}^n$, not necessarily open, we say that a function $f\colon D \to \mathbb{R}$ is differentiable at a point $\mathbf{a}$ in D if it agrees with a differentiable function near $\mathbf{a}$, that is, there is an open ball B containing $\mathbf{a}$ and a function $g\colon B \to \mathbb{R}$ such that $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x}$ in $D \cap B$ and g is differentiable at $\mathbf{a}$ in the sense previously defined. We make the analogous modification for smoothness. For continuity, we keep the definition as before but restrict our attention to points where f is defined: f is continuous at $\mathbf{a}$ if, given any open ball $B(f(\mathbf{a}), \epsilon)$ about $f(\mathbf{a})$, there exists an open ball $B(\mathbf{a}, \delta)$ about $\mathbf{a}$ such that $f(B(\mathbf{a}, \delta) \cap D) \subset B(f(\mathbf{a}), \epsilon)$. ...

The changes have the effect of pushing some of the material onto the next page, but, by p. 101, everything is back to normal.

- **p. 109, Exercise 1.14.** Change the point of interest from $(1,0)$ to $(1,\pi)$: "... Evaluate $\frac{\partial f}{\partial y}(1,\pi)$. ..."
- **p. 117, Exercise 10.10.** There are at least two data points: "Let $(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)$ be a collection of n distinct points in $\mathbb{R}^2$, where $n \geq 2$. ..."
- **p. 119, Exercise 11.5(b).** Add a hint: "... *Hint:* In the case that $C = 0$, too, consider the expansions of $(a \pm b)^2$."
- **p. 123, Example 5.1.** In the sequence of displayed equations in the middle of the page, the third equation should be with respect to y, not x:

$$\vdots$$
$$= \int_2^4 \big((12 - \frac{9}{2} + 6y) - (4 - \frac{1}{2} + 2y)\big)\, dy$$
$$\vdots$$

- **p. 167, Exercise 2.8.** Spherical coordinates were meant to be written consistently in the order (ρ,ϕ,θ):

  2.8 Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the spherical coordinate transformation from $\rho\phi\theta$-space to xyz-space: $f(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)$.

  (a) Find $Df(\rho,\phi,\theta)$. ...

- **p. 171, Exercise 4.12.** f should be a function of two variables, not three: "Let $f(x,y) = x^2 + y^2 + xy$. ..."
- **p. 207, equation (9.4).** A footnote has been added elaborating on reparametrization:

  We assume here that g is smooth. Indeed, equation (9.3) may be the best way to define what it means in the first place for α and β to parametrize the same curve C, namely, that there exists a smooth function $g\colon [c,d] \to [a,b]$ such that $\beta(u) = \alpha(g(u))$ and $g'(u) \neq 0$ for all u in $[c,d]$. Then, we say that α and β traverse C in the same direction if $g' > 0$ and in opposite directions if $g' < 0$.

- **p. 218, first displayed equation.** In the double integral on the right, a missing dy has been added:

$$\int_{C=\partial D} (x - y^2)\, dx + xy\, dy = \iint_D \big(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x - y^2)\big)\, dx\, dy \ \ \dots$$

- **p. 221, first displayed equation.** This equation has been given a number.

$$\int_{C_a} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\pi. \tag{9.10}$$

  There are cross-references to it in the appropriate places on p. 222 (below the second displayed equation) and p. 224 (on the next-to-last line). Also, the numbered equations that follow in the rest of the chapter all have their numbers increased by 1, up to (9.14) on p. 226.

- **p. 222, last sentence of Section 9.5.** To include the possibility of curves oriented clockwise, the sentence should be replaced with the following:

  > ... If C is oriented clockwise instead, then the change in orientation reverses the sign of the integral, so the corresponding values in the two cases are 0 and -2π, respectively. That there are exactly three possible values of the integral over all piecewise smooth oriented simple closed curves in $\mathbb{R}^2 - \{(0,0)\}$ may be somewhat surprising.

- **p. 227, Exercise 1.11.** In the 1-form that is being integrated, x^y and x^π have been changed to $x+y$ and π^x, respectively: "$\int_C e^{8xy}\,dx - \ln\big(\cos^2(x+y) + \pi^x y^{1{,}000{,}000}\big)\,dy$, where ..."

- **p. 228, Exercise 1.13.** In the second paragraph, the assumption that g is smooth is stated explicitly: " ... As in equation (9.3), there is a function $g\colon J \to I$, assumed to be smooth, such that $\beta(u) = \alpha(g(u))$ ..."

  Also, the first couple of sentences in the last paragraph have been rewritten:

  > In other words, if $\mathbf{x} = \beta(u) = \alpha(t)$ is any point of C other than an endpoint, then $\kappa_\beta(u) = \kappa_\alpha(t)$. We denote this common value by $\kappa(\mathbf{x})$, i.e., $\kappa(\mathbf{x})$ is defined to be $\kappa_\alpha(t)$ for any smooth parametrization α of C, where t is the value of the parameter such that $\alpha(t) = \mathbf{x}$. ...

- **p. 261, first sentence of Section 10.8.** The assumption in the definition of the surface integral that $\frac{\partial\sigma}{\partial s} \times \frac{\partial\sigma}{\partial t}$ is nonzero has been added as part of the setup:

  > Finally, we address the long-standing question of the extent to which our original definition of the surface integral, $\iint_S \mathbf{F}\cdot d\mathbf{S} = \pm\iint_D \mathbf{F}(\sigma(s,t)) \cdot \big(\frac{\partial\sigma}{\partial s} \times \frac{\partial\sigma}{\partial t}\big)\,ds\,dt$, depends on the smooth parametrization $\sigma\colon D \to \mathbb{R}^3$ of S, where D is a subset of the st-plane and, by assumption, $\frac{\partial\sigma}{\partial s} \times \frac{\partial\sigma}{\partial t} \neq \mathbf{0}$, except possibly on the boundary of D. ...

- **p. 264, Exercise 1.3.** The vector field $\mathbf{F}$ should be assumed to be continuous: "Let $\mathbf{F}$ be a continuous vector field on $\mathbb{R}^3$..."

- **p. 264, Exercise 1.4.** The vector field $\mathbf{F}$ may be assumed to be continuous: "... Let $\mathbf{F}$ be a continuous vector field on U. ..."