$A\underline{x}=\underline{b}$ $\boxed{\text{線代}}$

$[A\,|\,\underline{b}]=[U\,|\,\underline{c}]$

U ↓ Upper-triangular matrix

無解 → singular case (unsolvable)

Elememary (Elimination) Matrix

$\begin{bmatrix}1&0&0\\-k&1&0\\0&0&1\end{bmatrix}$: substract k times row 1 from row 2

↳ E_{21}

$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$: exchange rows 2 and 3

↳ P_{23}

permutation (row exchange) Matrix

Inverse Matrices

$A\underline{x}=\underline{b} \Rightarrow \underline{x}=A^{-1}\underline{b}$

↳ invertible

if $\exists!\, B$ s.t. $AB=BA=I \Rightarrow A^{-1}=B$

↳ unique

proof: suppose A^{-1} has B and C

$\Rightarrow \underline{B}=BI=B(AC)=(BA)C=IC=\underline{\underline{C}}$

$BA=AC=I$: then $B=C=A^{-1}$

"left inverse" B = "right inverse" C

If $A\underline{x}=\underline{0}$ 有非0解, then A^{-1} DNE.

proof: If A^{-1} exists, $A\underline{x}=\underline{0}$

$\Rightarrow A^{-1}A\underline{x}=A^{-1}\underline{0} \Rightarrow \underline{x}=\underline{0}$

Diagonal matrix has an inverse, 其也為 diagonal matrix

$A=\begin{bmatrix}d_1 & & 0\\ & \ddots & \\ 0 & & d_n\end{bmatrix}$, then $A^{-1}=\begin{bmatrix}\frac{1}{d_1} & & 0\\ & \ddots & \\ 0 & & \frac{1}{d_n}\end{bmatrix}$

If A and B are invertible, then so is AB.

$(AB)^{-1}=B^{-1}A^{-1}$, $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$

Gauss-Jordan Elimination

Gauss process: 形成 row echelon form

Jordan process: 形成 reduced row echelon form $=R$ (RRE)

find A^{-1}: $\rightarrow [A|I]=[I|A^{-1}]$

$A^{-1}=[\underline{x}_1,\underline{x}_2,\underline{x}_3]$, $AA^{-1}=I$

$A[\underline{x}_1\,\underline{x}_2\,\underline{x}_3]=[\underline{e}_1\,\underline{e}_2\,\underline{e}_3]$

$\Rightarrow A\underline{x}_1=\underline{e}_1, A\underline{x}_2=\underline{e}_2, A\underline{x}_3=\underline{e}_3$

$[A\,|\,\underline{e}_1]=\left[I\,\middle|\,\begin{matrix}u\\v\\w\end{matrix}\right] \Rightarrow I\underline{x}_1=\begin{bmatrix}u\\v\\w\end{bmatrix}=\underline{x}_1$

$\Rightarrow [A\,|\,I]=[I\,|\,\underline{x}_1\,\underline{x}_2\,\underline{x}_3]=[I\,|\,A^{-1}]$

$A_{n\times n}$ is nonsingular if it has $\underline{\underline{n}}$ pivots.

A^{-1} exists if and only if A is nonsingular.

proof: ($\Rightarrow$) and ($\Leftarrow$) 皆可用 $[A|I]=[I|A^{-1}]$ 證

$A=LU$ $\quad E_1\cdot E_2\cdot E_3\cdot A=U$

$A=E_3^{-1}\cdot E_2^{-1}\cdot E_1^{-1}\cdot U=L\cdot U$

U: upper triangle matrix $[{}_{0}\!\!\diagdown]$

L: lower triangle matrix $[\diagdown^{0}]$

row echelon form → U

elimination matrix (S' product) → L

Note: $A=LU$ if no row exchanges are required.

$A=LDU$ 對角matrix pivot 全$=1$

if no row exchanges are required.

$\begin{bmatrix}2&1&1\\4&-6&0\\-2&7&2\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\-1&-1&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&-8&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&\frac12&\frac12\\0&1&\frac14\\0&0&1\end{bmatrix}$

L, U 對角元素全$=1$, D 對角 matrix

If $A=L_1D_1U_1=L_2D_2U_2$, where L, U 對角元素全$=1$, D 對角元素全$\neq 0$

then $L_1=L_2$, $D_1=D_2$, $U_1=U_2$

<唯一表法>

One square system = Two Triangular Systems

$A\underline{x}=\underline{b}$ 若"無"涉及"row exchange"

$A=LU \Rightarrow LU\underline{x}=\underline{b} \Rightarrow U\underline{x}=L^{-1}\underline{b}=\underline{c}$

We have $U\underline{x}=\underline{c}$, where $L\underline{c}=\underline{b}$

1. 對 $A=LU$ 做 Gauss Elimination
2. 解 $\underline{c}$ from $L\underline{c}=\underline{b}$
3. 解 $\underline{x}$ from $U\underline{x}=\underline{c}$

Order of complexity

Solve $A\underline{x}=\underline{b}$; $A_{n\times n}$

① $\underline{x}=A^{-1}\underline{b}$ ······ n^3

② $L\underline{c}=\underline{b}$, $U\underline{x}=\underline{c}$ ······ $\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{2}$

Transpose

- $(A^T)_{ij}=A_{ji}$
- $(A+B)^T=A^T+B^T$
- $(AB)^T=B^TA^T$

  └ $(ABC)^T=C^TB^TA^T$
- $(A^{-1})^T=(A^T)^{-1}$

symmetric: $A^T=A$

- R^TR, RR^T → symmetric
- A is symmetric, then $\underline{\underline{A}}=LDU=\underline{\underline{LDL^T}}$

There're $\underline{\underline{n!}}$ permutation matrices of order n.

- $P_{ab}^{-1}=P_{ab}$ <單步驟P_{ab}時成立>
- $PP^T=I \Rightarrow P^{-1}=P^T$

有row exchanges的 $A=LU$

$EPA=U$, $PEA=U$

$\Rightarrow \begin{cases} PA=E^{-1}U=LU \\ A=(PE)^{-1}U=E^{-1}P^{-1}U=L_1P_1U \end{cases}$

Inner product $\underline{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$, $\underline{y}=\begin{bmatrix}y_1\\y_2\end{bmatrix}$

$x_1y_1+x_2y_2=[x_1\;x_2]\begin{bmatrix}y_1\\y_2\end{bmatrix}=\underline{x}^T\underline{y}$

Vector Spaces and Subspaces

$\mathbb{R}^n$ = all column vectors with n (real) components

$\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\{(v_1,v_2,\dots,v_n): v_i\in\mathbb{R}, i=1,\dots,n\}$

If $\underline{v},\underline{w}\in\mathbb{R}^n$, $c\in\mathbb{R}$, then

① $\underline{v}+\underline{w}\in\mathbb{R}^n$

② $c\underline{v}\in\mathbb{R}^n$ // $\underline{0}=0\underline{v}\in\mathbb{R}^n$

Vector space // V: a set of vect

↳ 2 operations

8 rules

(1) $\underline{v}+\underline{w}=\underline{w}+\underline{v}$

(2) $\underline{u}+(\underline{v}+\underline{w})=(\underline{u}+\underline{v})+\underline{w}$

(3) There is a unique "zero" vector $\underline{0}$, s.t. $\underline{v}+\underline{0}=\underline{v}\ \forall\, \underline{v}\in V$.

(4) For each $\underline{v}$, there is a unique vector $-\underline{v}$ s.t. $\underline{v}+(-\underline{v})=\underline{0}$

(5) $1\cdot\underline{v}=\underline{v}$

(6) $(c_1c_2)\underline{v}=c_1(c_2\underline{v})$

(7) $c(\underline{v}+\underline{w})=c\underline{v}+c\underline{w}$

(8) $(c_1+c_2)\underline{v}=c_1\underline{v}+c_2\underline{v}$

example: is a vector space

1. $\mathbb{R}^n$ ✓
2. $M=\{$all real 2×2 matrices$\}$ ✓
3. $F=\{$all real function$\}$ ✓
4. $Z=\{\underline{0}\}$ ✓

- A subset W of a vector space V is a subspace if W is also a vector space
- every subspace contains the zero vector.

Column space : $C(A)$

The $C(A)$ of a matrix A consists of all 線性組合 of the columns of A

The combinations are all possible vectors $A\underline{x}=x_1\underline{a_1}+x_2\underline{a_2}+\dots+x_n\underline{a_n}$

- $A\underline{x}=\underline{b}$ soluable iff $\underline{b}\in C(A)$

Ex: $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $C(I)=\mathbb{R}^2$

$A=\begin{bmatrix}1&2\\2&4\end{bmatrix}$, $C(A)=\left\{x\begin{bmatrix}1\\2\end{bmatrix}: x\in\mathbb{R}\right\}$

$B=\begin{bmatrix}1&2&3\\0&0&4\end{bmatrix}$, $C(B)=\mathbb{R}^2$

All of the column spaces are subspaces of $\mathbb{R}^2$

- S = set of vectors in a vector space V.
- SS = the set of all linear combinations of vectors in S.

We call SS the "span" of S.

Then SS is a subspace of V, called the subspace "spanned" by S.

Ex:

$A=\begin{bmatrix}1&2\\2&3\end{bmatrix}$, $S=\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix}\right\}$, $SS=C(A)$

- $A_{m\times n}$, $C(A)$ is a subspace of $\mathbb{R}^m$

Nullspace of A: N(A)

Def: The nullspace of $A\to$ all solutions to $A\underline{x}=\underline{0}$ (a.k.a. the kernel of A)

$N(A)=\{\underline{x}: A\underline{x}=\underline{0}\}$

- $A_{m\times n}$, $N(A)$ is a subspace of $\mathbb{R}^n$

pf:

(i) if $A\underline{x}=\underline{0}$ and $A\underline{y}=\underline{0}$, then

$A(\underline{x}+\underline{y})=A\underline{x}+A\underline{y}=\underline{0}+\underline{0}=\underline{0}$

(ii) if $A\underline{x}=\underline{0}$, then

$A(c\underline{x})=c\cdot A\underline{x}=c\cdot\underline{0}=\underline{0}$

$\Rightarrow N(A)$ is a vector space

- $\mathbb{R}^a$ is a subspace of $\mathbb{R}^b$, where $a\le b$.

$R=\begin{bmatrix}1&0&2&0\\0&1&0&2\end{bmatrix}$, $R\underline{x}=\underline{0}$

$$\begin{matrix}x_1 & & +2x_3 & & =0\\ & x_2 & & +2x_4 & =0\end{matrix} \Rightarrow \begin{cases}x_1=-2x_3\\x_2=-2x_4\end{cases}$$

pivot variables, free variables

$$\underline{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-2x_3\\-2x_4\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}0\\-2\\0\\1\end{bmatrix}$$

$$\Rightarrow N(R)=\left\{\underline{x}: \underline{x}=x_3\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}0\\-2\\0\\1\end{bmatrix}; x_3,x_4\in\mathbb{R}\right\}$$

If $A_{m\times n}$ has r pivots, then there are $n-r$ free variables and $n-r$ special solutions.

$N(A)$ consists of all the linear combinations of these $n-r$ special solutions.

$\Rightarrow N(A)$ = the subspace spanned by these $n-r$ special solutions.

If $n>m$ ie. $A\underline{x}=\underline{0}$ has more unknowns than equations.

then $r\le m<n$, Hence $n-r>0$

,i.e. there must be 非$\underline{0}$解

Def:

The "rank" of A is the number of pivots.

Pivot Column → Pivot 所在

Free Column → Free variable 所在

Special Solution → $\underline{v_i}$ where

$N(R)=\left\{\underline{x}: \underline{x}=\sum_{i=0}^{n} x_i\underline{v_i}\right\}$

$A\underline{x}=\underline{0}$ 代入 special solution

$\Rightarrow$ Free columns are 線性組合 of Pivot columns, and special solutions 描述其如何組成

Let $N=[\underline{n_1}\ \underline{n_2}\ \underline{n_3}]$ ($\underline{n_i}$為特殊解)

$\because A\underline{n_i}=\underline{0} \Rightarrow \underset{m\times n}{A}\ \underset{n\times(n-r)}{N}=O \leftarrow m\times(n-r)$ 的 $\underline{0}$

Variables 位置對調:

$R=\begin{bmatrix}1&3&0&2&-1\\0&0&1&4&-3\\0&0&0&0&0\end{bmatrix}$ $N=\begin{bmatrix}-3&-2&1\\1&0&0\\0&-4&3\\0&1&0\\0&0&1\end{bmatrix}$

$$\begin{array}{cc} & \begin{matrix}I & & F\end{matrix}\\ R'= & \begin{bmatrix}1&0&3&2&-1\\0&1&0&4&-3\\0&0&0&0&0\end{bmatrix}\end{array} \quad N'=\begin{bmatrix}-3&-2&1\\0&-4&3\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{matrix}-F\\ \\ I\end{matrix}$$

設前 r columns are pivot columns.

(R行行交換，同時N列列交換)

In general, $R=\begin{bmatrix}\underset{r\times r}{I}&\underset{r\times(n-r)}{F}\\0&0\end{bmatrix}\begin{matrix}\\ m-r\end{matrix} \Longleftrightarrow N=\begin{bmatrix}-F\\I\end{bmatrix}\begin{matrix}r\times(n-r)\\n-r\end{matrix}$

$A\underline{x}=\underline{0}\Rightarrow N(A)$

$A\underline{x}=\underline{b}$ Complete solution to $A\underline{x}=\underline{b}$

$\underline{x}=\underbrace{x_i[\ ]+x_j[\ ]}_{\text{general solution to } A\underline{x}=\underline{0}\ (\underline{x_n})}+[\ ]$ ← particular solution to $A\underline{x}=\underline{b}$ ($\underline{x_p}$)

$\Rightarrow \underline{x}=\underline{x_p}+\underline{x_n}$

- If $A\underline{x}=\underline{b}$, complete solution is $\underline{x}=\underline{x_p}+\underline{x_n}$
- Suppose $A_{m\times m}$, rank=m (可逆), what are $\underline{x_p}$ and $\underline{x_n}$?

→ The only particular solution to $A\underline{x}=\underline{b}$ is $\underline{x_p}=A^{-1}\underline{b}$, and the only solution to $A\underline{x}=\underline{0}$ is $\underline{x_n}=0$

$\Rightarrow \underline{x}=\underline{x_p}+\underline{x_n}=A^{-1}\underline{b}+\underline{0}=A^{-1}\underline{b}$

- If $A_{m\times n}$ has full column rank ($r=n$)
  (i) All columns of A are pivot columns.
  (ii) no free variable、special solutions.
  (iii) $N(A)=\{\underline{0}\}$
  (iv) If $A\underline{x}=\underline{b}$ 有解 (可能無解), then it has only one solution.
- If $A_{m\times n}$ has full row rank ($r=m$)
  (i) All rows have pivots (R has no zero rows)
  (ii) $A\underline{x}=\underline{b}$ is soluable $\forall\underline{b}$
  (iii) $C(A)=\mathbb{R}^m$
  (iv) There are $n-r=n-m$ special solution、free variables in $N(A)$

線代

- $A\underline{x}=\underline{b} \rightarrow R\underline{x}=\underline{c}$, $R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} I & F & c_i \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{x} = \underbrace{\begin{bmatrix} -F \\ I \end{bmatrix} \underline{x}_{\text{free}}}_{\underline{x_n}} + \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\underline{x_p}}$ pivot 個數, free

$A\underline{x}=\underline{b}$

Case I. $r=m=n$, A: 可逆方陣

$R=[I]$ $A\underline{x}=\underline{b}$ 有一解: $\underline{x}=A^{-1}\underline{b}$

Case II. $r=m<n$ (full row rank)

A: short and wide $R=[I\ F]$

$A\underline{x}=\underline{b}$ has ∞ 解

Case III. $r=n<m$ (full column rank)

A: tall and thin $R=\begin{bmatrix} I \\ 0 \end{bmatrix}$

$A\underline{x}=\underline{b}$ has 0 or 1 解

Case IV. $r<m$, $r<n$

A: not full rank $R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

$A\underline{x}=\underline{b}$ has 0 or ∞ 解

$A\underline{x}=\underline{0} \rightarrow$ homogeneous solution

$A\underline{x}=\underline{b} \rightarrow$ particular solution + h.s.

= complete solution

Independence, Basis, and Dimension

Def: If $x_1\underline{v_1}+x_2\underline{v_2}+\cdots+x_n\underline{v_n}=0$ only happens when $x_1=x_2=\cdots=x_n=0$ then the vectors $\underline{v_1}.\underline{v_2}.\cdots.\underline{v_n}$ are LI (linearly independent), 否則為 LD (linearly dependent).

Rmk: special solutions in the nullspace of a matrix are LI.

pf: special solutions $R=\begin{bmatrix} -F \\ I \end{bmatrix} = [\overbrace{\underline{v_i}}^{i}]$

$\begin{bmatrix} -F \\ I \end{bmatrix}\underline{x}=\underline{0} \xrightarrow[\text{消去}]{\text{高斯}} \begin{bmatrix} 0 \\ I \end{bmatrix}\underline{x}=\underline{0}$

$\xrightarrow[A\underline{x}=\underline{b},\ \underline{b}=\underline{0}]{\text{Case III }\&}$ 唯一解 $(\underline{x}=\{\underline{0}\})$

$\rightarrow \forall x_i=0 \rightarrow \{\underline{v_i}\}$ is LI ※

- the columns of A LI exactly when $r=n$ (full column rank)

$R=\begin{bmatrix} I \\ 0 \end{bmatrix} \Rightarrow n$ pivots & no free variables

$\Rightarrow N(A)=\{\underline{0}\}$

Claim: Any set of n vectors in $\mathbb{R}^m$ must LD if $n>m$

pf: $A=[\underbrace{\underline{v_1}\,\underline{v_2}\cdots\underline{v_n}}_{n}]\}m$, $R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ $r\le m<n$, $N(A)$. ∞ 解

Def: A set of vectors spans a vector space if their 線性組合 fill the space.

Def: The row space of $A_{m\times n}$ is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

- Row space of A: $C(A^T)$

Def: A basis for a vector space is a sequence of vectors satisfying two properties:

(i) The basis vectors are LI.

(ii) They span the space.

- $AB=[A\underline{b_1}\ A\underline{b_2}\cdots A\underline{b_n}]$
- $AB=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} \underline{br_1} \\ \underline{br_2} \\ \underline{br_3} \end{bmatrix}$

$=\begin{bmatrix} a_{11}\underline{br_1}+a_{12}\underline{br_2}+a_{13}\underline{br_3} \\ a_{21}\underline{br_1}+a_{22}\underline{br_2}+a_{32}\underline{br_3} \\ a_{31}\underline{br_1}+a_{32}\underline{br_2}+a_{33}\underline{br_3} \end{bmatrix}$

$=\begin{bmatrix} \underline{ar_1}B \\ \underline{ar_2}B \\ \underline{ar_3}B \end{bmatrix}$

- $[A_1\ A_2]\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1B_1+A_2B_2$

$\updownarrow$ imply

- $AB=\sum_{j=i=1}^{N} A_jB_i$

ex:

$AB=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix}=\begin{bmatrix} aE+bG & aF+bH \\ cE+dG & cF+dH \end{bmatrix}$

$=\begin{bmatrix} a \\ c \end{bmatrix}[E\ F]+\begin{bmatrix} b \\ d \end{bmatrix}[G\ H]$

- 綜上，矩陣中小矩陣的相乘方式與單純數字乘法相同。

$AB=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}=\begin{bmatrix} A_1B_1+A_2B_3 & A_1B_2+A_2B_4 \\ A_3B_1+A_4B_3 & A_3B_2+A_4B_4 \end{bmatrix}$

- Claim: Any n independent vectors in $\mathbb{R}^n$ are a basis for $\mathbb{R}^n$.
- Claim: The vectors $\underline{v_1}.\underline{v_2}.\cdots.\underline{v_n}$ are a basis for $\mathbb{R}^n$ exactly when they are the columns of an $n\times n$ invertible matrix.
- Remark: $\mathbb{R}^n$ 有 ∞ 種 bases.

- Claim: any vectors in a vector space 可唯一表示為 basis vectors 的 linear combination.

pf: Let $\underline{v_1}.\underline{v_2}.\cdots.\underline{v_n}$ be a basis.

Suppose $\underline{v}=a_1v_1+a_2v_2+\cdots+a_nv_n$
$=b_1v_1+b_2v_2+\cdots+b_nv_n$

$\Rightarrow (a_1-b_1)v_1+(a_2-b_2)v_2+\cdots+(a_n-b_n)v_n=0$

$\Rightarrow a_i-b_i=0\ \forall i\in n \Rightarrow a_i=b_i\ \forall i\in n$

($\because \underline{v_1}.\underline{v_2}.\cdots.\underline{v_n}$ are L.I.)

- Recall that free columns of $\stackrel{R}{A}$ are 線性組合 of the pivot columns of $\stackrel{R}{A}$.

☆ pivot columns of $\stackrel{R}{A}$ are L.I.

$\Rightarrow$ pivot columns of $\stackrel{R}{A}$ are a basis for $\stackrel{C(R)}{C(A)}$.

- $C(R)$ 不一定 $= C(A)$ ＜當有 free 時＞

$C(R)=C(A)$ ＜當沒 free 時＞

- Claim: If $\underline{v_1}.\underline{v_2}.\cdots.\underline{v_m}$ and $\underline{w_1}.\underline{w_2}.\cdots.\underline{w_n}$ 皆為同個 vector space 的 basis, then $\underline{\underline{m=n}}$

pf: Suppose $n>m$. We have

$\underline{w_j}=a_{1j}\underline{v_1}+a_{2j}\underline{v_2}+\cdots+a_{mj}\underline{v_m}$, $j=1\sim n$

Consider $x_1\underline{w_1}+x_2\underline{w_2}+\cdots+x_n\underline{w_n}=\underline{0}$.

$\Rightarrow x_1(a_{11}\underline{v_1}+\cdots+a_{m1}\underline{v_m})+x_2(a_{12}\underline{v_1}+\cdots+a_{m2}\underline{v_m})+\cdots+x_n(a_{1n}\underline{v_1}+\cdots+a_{mn}\underline{v_m})=\underline{0}$

$\Rightarrow (a_{11}x_1+\cdots+a_{1n}x_n)\underline{v_1}+\cdots+(a_{m1}x_1+\cdots+a_{mn}x_n)\underline{v_m}=\underline{0}$

$\Rightarrow$ Since $\underline{v_1}\sim\underline{v_m}$ are L.I.,

$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Since $n>m$, ∃非0解 for $x_1\sim x_n$, imply $\underline{w_1}\sim\underline{w_n}$ are L.D., This is impossible $\because \underline{w_1}\sim\underline{w_n}$ are a basis. If $m>n$, 同理可證. 因此 $n=m$ ※