SVD as change of bases

- $\underset{m\times n}{A} = \underset{m\times m}{U}\overset{m\times n}{\Sigma}\underset{n\times n}{V^T}$, U、V orthogonal
- β the standard basis for $\mathbb{R}^n$, β' the standard basis for $\mathbb{R}^m$

Let $A = [T]_\beta^{\beta'}$ for some linear transformation

Note $\alpha = \{\underline{v_1}\cdots\underline{v_n}\}$ is an orthonormal basis for $\mathbb{R}^n$

$\gamma = \{\underline{u_1}\cdots\underline{u_m}\}$ is an orthonormal basis for $\mathbb{R}^m$

Then $T = ITI \Rightarrow [T]_\beta^{\beta'} = [I]_\gamma^{\beta'}[T]_\alpha^\gamma[I]_\beta^\alpha$

We have $[I]_\beta^\alpha = \left([I]_\alpha^\beta\right)^{-1} = \left([\underline{v_1}\cdots\underline{v_n}]\right)^{-1} = V^{-1} = V^T$

$[I]_\gamma^{\beta'} = [\underline{u_1}\cdots\underline{u_m}] = U$

Since $T(\underline{v_i}) = \begin{cases} \sigma_i\underline{u_i} , & i = 1\cdots r \\ \underline{0} , & i = r+1\cdots n \end{cases}$

$(A\underline{v_i} = \sigma_i\underline{u_i})$

$[T]_\alpha^\gamma = \begin{bmatrix} \sigma_1 & & \\ & \ddots \sigma_r & \\ & & 0 \end{bmatrix} = \Sigma \Rightarrow$ This corresponds exactly to $A = U\Sigma V^T$

Polar Decomposition

Claim Every real 方陣 can be factored into $A = QH$ where Q is orthogonal and H is 半正定. If A 可逆, then H is 正定

Remark 1 $a+ib = \underset{H}{r}\,\underset{Q}{e^{i\theta}}$ where $r = \sqrt{a^2+b^2} \geq 0$ and $|e^{i\theta}| = 1$ $(\theta = \tan^{-1}\frac{b}{a})$

pf: Recall in SVD, $A = U\Sigma V^T = UV^TV\Sigma V^T = (UV^T)(V\Sigma V^T) = QH$

Then $Q = UV^T$ is orthogonal since $QQ^T = (UV^T)(UV^T)^T = UV^TVU^T$

$= UU^T = I$ and

$H = V\Sigma V^T = V\begin{bmatrix} \sigma_1 & & \\ & \ddots \sigma_r & \\ & & 0 \end{bmatrix}V^T$ is 半正定 since H 與半正定矩陣 Σ 相似

If A 可逆, then rank $= r = n$ and $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$ is 正定

, therefore H is 正定

Remark 2 $A = U\Sigma V^T = (U\Sigma U^T)(UV^T) = KQ$ where $K = U\Sigma U^T$ is 半正定 and $Q = UV^T$ is orthogonal, which is a polar decomposition in the reverse order

Pseudo Inverse

Recall in SVD, $\underset{m\times n}{A} = \underset{m\times m}{U}\,\overset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$

$\underline{v_1} \cdots \underline{v_r}$ form a basis for $C(A^T)$, $\underline{v_{r+1}} \cdots \underline{v_n}$ form a basis for $N(A)$

$\underline{u_1} \cdots \underline{u_r}$ form a basis for $C(A)$, $\underline{u_{r+1}} \cdots \underline{u_m}$ form a basis for $N(A^T)$

The pseudo inverse $A^+_{n\times m}$ given by $\underset{n\times m}{A^+} = \underset{n\times n}{V}\,\overset{n\times m}{\Sigma^+}\,\underset{m\times m}{U^T}$, where $\underset{n\times m}{\Sigma^+} = \begin{bmatrix} \sigma_1^{-1} & & & \\ & \ddots & & \\ & & \sigma_r^{-1} & \\ & & & 0 \end{bmatrix}$

We can then have $A^+ \underline{u_i} = \begin{cases} \frac{1}{\sigma_i} \underline{v_i}, & \text{for } i = 1 \cdots r \\ \underline{0}, & \text{for } i = r+1 \cdots m \end{cases}$ $\left(A\underline{v_i} = \begin{cases} \sigma_i \underline{u_i}, & i = 1 \cdots r \\ \underline{0}, & i = r+1 \cdots n \end{cases} \right)$

Hence, A^+ sends every vector in the $C(A)$ to the $C(A^T)$, and every vector in the $N(A^T)$ to $\underline{0}$

We can also have $AA^+\underline{u_i} = A(\frac{1}{\sigma_i}\underline{v_i}) = \frac{1}{\sigma_i}(A\underline{v_i}) = \frac{1}{\sigma_i}(\sigma_i \underline{u_i}) = \underline{u_i}$ for $i = 1 \cdots r$

Therefore $AA^+\underline{p} = \underline{p}$ $\forall\, \underline{p} \in C(A)$ and $AA^+\underline{e} = \underline{0}$ $\forall\, \underline{e} \in N(A^T)$

$\forall\, \underline{b} \in \mathbb{R}^m$, $\underline{b} = \underline{p} + \underline{e}$ for some $\underline{p} \in C(A)$ and $\underline{e} \in N(A^T)$, and

$AA^+\underline{b} = AA^+(\underline{p} + \underline{e}) = \underline{p} + \underline{0} = \underline{p}$

$\therefore$ AA^+ is the projection matrix onto $C(A)$

Similarly, A^+A is the projection matrix onto $C(A^T)$

Note $\Sigma^+\Sigma = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix}$ (r)

Application to Least Squares

Recall $A^TA\hat{\underline{x}} = A^T\underline{b}$

If A 可逆 $\rightarrow$ $\hat{\underline{x}} = (A^TA)^{-1}A^T\underline{b}$

If A $(r<n)$, there may be many solutions to $A^TA\hat{\underline{x}} = A^T\underline{b}$

One solution is $\underline{x}^+ = A^+\underline{b}$ since $A^TA\underline{x}^+ = A^TAA^+\underline{b} = A^T(AA^+\underline{b}) = A^T\underline{p} = A^T(\underline{p} + \underline{e}) = A^T\underline{b}$

where $\underline{b} = \underline{p} + \underline{e}$, $\underline{p} \in C(A)$ and $\underline{e} \in N(A^T)$

Note $\underline{x}^+ = A^+\underline{b} = A^+(\underline{p} + \underline{e}) = A^+\underline{p} + \underline{0} = A^+\underline{p} \in C(A^T)$

orthogonal complement

Consider $\underline{x}' = \underline{x}^+ + \underline{x_n}$, where $\underline{x_n}$ is any vector in $N(A)$

We have $A^TA\underline{x}' = A^TA(\underline{x}^+ + \underline{x_n}) = A^TA\underline{x}^+ + A^T\underline{0} = A^TA\underline{x}^+ = A^T\underline{b}$

$\Rightarrow \underline{x}' = \underline{x}^+ + \underline{x}_n$ is the complete solution to $A^TA\hat{\underline{x}} = A^T\underline{b}$

But $\|\underline{x}'\| = \|\underline{x}^+ + \underline{x}_n\| \geq \|\underline{x}^+\|$, since $\underline{x}^+$ and $\underline{x}_n$ are orthogonal

Therefore $\underline{x}^+ = A^+\underline{b}$ is exactly the shortest least squares solution to $A\underline{x} = \underline{b}$

Remark 1 If A has full column rank, i.e. $r = n \leq m$, then A^TA is 可逆 and

$(A^TA)^{-1}A^T$ is a left inverse of A since $(A^TA)^{-1}A^TA = I$

Also $(A^TA)^{-1}A^T = A^+ \Leftrightarrow$ pseudo inverse = left inverse

Remark 2 If A has full row rank, i.e. $r = m \leq n$, then AA^T is 可逆 and

$A^T(AA^T)^{-1}$ is a right inverse of A since $AA^T(AA^T)^{-1} = I$

Also $A^T(AA^T)^{-1} = A^+ \Leftrightarrow$ pseudo inverse = right inverse

Remark 3 If A has full rank, i.e. $r = m = n$, then A is 可逆

$A^{-1} = (A^TA)^{-1}A^T = A^T(AA^T)^{-1} = A^+ \Leftrightarrow$ pseudo inverse = inverse

Complex Vectors and Matrices

Real	Complex		
$\mathbb{R}^n$	$\mathbb{C}^n$		
$\underline{x}^T\underline{y}$	$\underline{x}^H\underline{y}$, $\underline{x}^H = \overline{\underline{x}}^T$, H: Hermitian	Inner Product	
$\|\underline{x}\|^2 = x_1^2 + \cdots + x_n^2$	$\|\underline{x}\|^2 = \lvert x_1\rvert^2 + \cdots + \lvert x_n\rvert^2$	Norm	
$\underline{x}^T\underline{y} = 0$	$\underline{x}^H\underline{y} = 0$	Orthogonal	
$(A^T)_{ij} = A_{ji}$	$(A^H)_{ij} = \overline{A_{ji}}$	Transpose	
$(AB)^T = B^TA^T$	$(AB)^H = B^HA^H$		
$A^T = A$	$A^H = A$	Symmetric	Hermitian
$A^T = -A$	$A^H = -A$	Skew-Symmetric	Skew-Hermitian
$Q^T = Q^{-1}$	$U^H = U^{-1}$	Orthogonal	Unitary

For Symmetric $A = Q\Lambda Q^{-1} = Q\Lambda Q^{T}$ (orthogonal Q and real Λ)

For Hermitian $A = U\Lambda U^{-1} = U\Lambda U^{T}$ (unitary U and real Λ)

For orthogonal Q, $|\lambda| = 1$ and eigenvectors are orthonormal

For unitary U, $|\lambda| = 1$ and eigenvectors are orthonormal