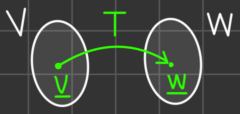
Linear transformations



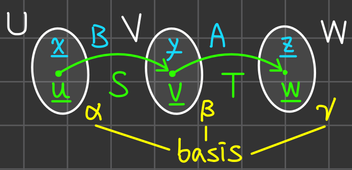
$\beta = \{\underline{v_1} \cdots \underline{v_n}\}$ basis of V , $\gamma = \{\underline{w_1} \cdots \underline{w_m}\}$ basis of W

T(<u>Vn</u>)在 γ 下的座標值

$$\underline{y} = A\underline{x} \Rightarrow \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- In general, if T rotates every vector by θ, then $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- Suppose $S: U \to V$ is linear and $T: V \to W$ is linear

  Then $TS: U \to W$ is linear from $\underline{u} \in U$ to $S(\underline{u}) \in V$ to $T(S(\underline{u})) \in W$

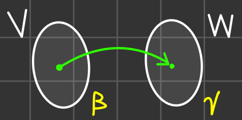


$\underline{y} = B\underline{x}$, $\underline{z} = A\underline{y} \longrightarrow \underline{z} = AB\underline{x}$

- Let $B = [S]_\alpha^\beta$ and $A = [T]_\beta^\gamma$. Then $\underset{m\times p}{[TS]_\alpha^\gamma} = [T]_\beta^\gamma [S]_\alpha^\beta = \underset{m\times n\ n\times p}{AB}$
- $T(\underline{v}) = \dfrac{dv}{dx}$ for $\underline{v} \in V$

  $\dfrac{d}{dx}(c\underline{v} + d\underline{w}) = c\dfrac{dv}{dx} + d\dfrac{dw}{dx} \to$ linear transformation

  ex:



$\beta = \{1, x, x^2, x^3\}$, $\gamma = \{1, x, x^2\}$

$$\begin{cases} T(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 \\ T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 \\ T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2 \\ T(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2 \end{cases}$$

Then $[T]_{\beta(4)}^{\gamma(3)} = A_{3\times 4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

- $S(\underline{w}) = \int_0^x \underline{w}\, dt \to$ linear transformation

  ex:

$$\begin{cases} S(1) = \int_0^x 1\, dt = 0\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ S(x) = \int_0^x t\, dt = 0\cdot 1 + 0\cdot x + \frac{1}{2}\cdot x^2 + 0\cdot x^3 \\ S(x^2) = \int_0^x t^2 dt = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + \frac{1}{3}\cdot x^3 \end{cases}$$

Then $[T]_{\gamma(3)}^{\beta(4)} = B_{4\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

- $AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 積再微 $BA = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 微再積

Change-of-basis Matrix (Matrix Representation for a Change of basis)

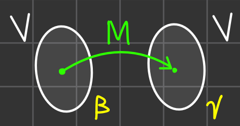
Suppose $\beta = \{\underline{v_1} \cdots \underline{v_n}\}$ basis for V, $\gamma = \{\underline{w_1} \cdots \underline{w_n}\}$ another basis for V

Then $\underline{v} = c_1\underline{v_1} + \cdots + c_n\underline{v_n} = d_1\underline{w_1} + \cdots + d_n\underline{w_n}$

Let $\underline{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ and $\underline{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$

Consider the identity transformation $I(\underline{v}) = \underline{v}$, then $\underline{d} = M\underline{c}$, $M = [I]_\beta^\gamma$

That is, $I(\underline{v_j}) = \underline{v_j} = m_{1j}\underline{w_1} + \cdots + m_{nj}\underline{w_n}$, $j = 1 \cdots n$



$\Rightarrow M = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix} = [m_{ij}]$: change-of-basis matrix

- Consider a linear operator T on V

  Let $\beta = \{\underline{v_1} \cdots \underline{v_n}\}$ basis for V, $\gamma = \{\underline{w_1} \cdots \underline{w_n}\}$ another basis for V

  Let $A = [T]_\beta$ and $B = [T]_\gamma$ Also $M = [I]_\gamma^\beta$

  Then $[T]_\gamma = [I]_\beta^\gamma [T]_\beta [I]_\gamma^\beta$, that is $B = M^{-1}AM$ (A.B similar)

- Matrix Representation of a linear transformation:

  change of basis = similarity transformation

- 以 $T(\underline{v_i})$ 為 columns 構成 轉換矩陣 A

- $[T]_\beta^\gamma$ > 換觀察座標

  改變 幾何本徵