- In general $R^T \underline{y} = \underline{0}$ $\underline{y}^T R = \underline{0}^T$

$$R = m\left\{\begin{bmatrix} 1 & & & & & \\ 0 & \ddots & & & & \\ 0 & & 1 & & & \\ \vdots & & \vdots & & & \\ 0 & \cdots & \vdots & 0 & 0 & 0 \\ 0 & \cdots & & 0 & 0 & 0 \end{bmatrix}\begin{matrix} \}r \\ \\ \\ \}m-r \end{matrix}\right. \Rightarrow \underline{y}^T = [\underbrace{0 \cdots 0}_{r}\ \underbrace{y_{r+1} \cdots y_m}_{-r}]$$

$\therefore (0 \cdots 0 \underset{\substack{\uparrow \\ r+1}}{1} 0 \cdots 0),\ (0 \cdots 0 \underset{\substack{\uparrow \\ r+2}}{1} 0 \cdots 0), \cdots, (0 \cdots 0 \underset{\substack{\uparrow \\ m}}{1})$ form a basis for $N(R^T)$

In other words, the last $m-r$ rows of $I_{m \times m}$, form a basis for $N(R^T)$

$\therefore \dim(N(R^T)) = m-r$

In general, $EA = R$

$$E_{m\times m} A_{m \times n} = \begin{bmatrix} 1 & & & & \\ & \ddots & 1 & & \\ 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}\begin{matrix} \}r \\ \\ \}m-r \end{matrix}$$
$\qquad\qquad\qquad\qquad n$

$\downarrow$ last m-r row of E

$$[\equiv] A = O$$
$$\underline{y}^T \ A = \underline{0}^T$$

Since E 可逆, all rows of E L.I.

$\therefore \dim(N(A^T)) = m-r \Rightarrow$ The last $m-r$ rows of E form a basis for $N(A^T)$

$$\begin{bmatrix} \text{row} & r+1 \\ & \vdots \\ \text{row} & m \end{bmatrix} A = O \Rightarrow \underline{y} = y_1(\text{row } r+1) + \cdots + y_r(\text{row } m)$$

- Orthogonality 正交性

$\underline{v} \cdot \underline{w} = \underline{v}^T w$

$\underline{v} \cdot \underline{v} = \underline{v}^T v = v_1^2 + \cdots + v_n^2 \triangleq \|\underline{v}\|^2$

length of $\underline{v} \Rightarrow \|\underline{v}\| = (\underline{v}^T v)^{\frac{1}{2}}$

def: $\underline{v}$ and $\underline{w}$ orthogonal if $\underline{v}^T \underline{w} = 0$ $(\underline{v} \cdot \underline{w} = 0)$

Remark $\underline{v}^T \underline{w} = 0$ $(\underline{w}^T \underline{v} = 0) \Leftrightarrow \|\underline{v}\|^2 + \|\underline{w}\|^2 = \|\underline{v} + \underline{w}\|^2$ 勾股推廣

def: Two subspace V and W 正交 if $\underline{v}^T \underline{w} = 0$ $\forall$ $\underline{v} \in V$ and $\underline{w} \in W$

$$Ax = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Leftrightarrow \begin{matrix} (\text{row } 1) \cdot \underline{x} = \underline{0} \\ \vdots \\ (\text{row } m) \cdot \underline{x} = \underline{0} \end{matrix}$$

$\Rightarrow$ $N(A)$ and $C(A^T)$ are orthogonal subspaces of $\mathbb{R}^n$

$N(A) \perp C(A^T)$

Similarly, $A^T\underline{y} = \begin{bmatrix} (\text{column } 1)^T \\ \vdots \\ (\text{column } n)^T \end{bmatrix} \begin{bmatrix} \underline{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\therefore$ $N(A^T)$ and $C(A)$ are orthogonal subspaces of $\mathbb{R}^m$

$N(A^T) \perp C(A)$

def: The orthogonal complement $V^\perp$ "V perp" of a subspace contains every vector that is orthogonal to V

$V^\perp = \{\underline{w} : \underline{w}^T \underline{v} = 0 \ \forall \underline{v} \in V\}$

Remark $V^\perp$ is also a subspace

Claim $\boxed{C(A^T)^\perp = N(A)}$, $N(A)^\perp = C(A^T)$, $C(A)^\perp = N(A^T)$, $N(A^T)^\perp = C(A)$

pf: Suppose $\underline{v} \in C(A^T) \cap \underline{v} \in N(A)$，將 $\underline{v}^T$ 加入 A 的 row 中 $\rightarrow A' = \begin{bmatrix} A_{\text{rows}} \\ \underline{v}^T \end{bmatrix}$

$\because \underline{v} \in C(A^T)$ $\therefore C(A'^T) = C(A^T)$ $\Rightarrow \dim(C(A'^T)) = \dim(C(A^T)) = r$

$\Rightarrow \dim(N(A')) = n - r$ — ①

$A'\underline{x} = \underline{0} \rightarrow \underline{v}^T\underline{x} = \underline{0} \rightarrow \underline{v} \perp \underline{x} \rightarrow \underline{v} \notin \{\underline{x}\} = N(A')$, but $\underline{v} \in N(A)$

$\Rightarrow \dim(N(A')) < \dim(N(A)) = n - r$ — ②

① $\not\leftrightarrow$ ② $\Rightarrow \underline{v}$ is D.N.E. ※

線代基本定理 Part 1 $\dim(C(A)) = r$, $\dim(C(A^T)) = r$, $\dim(N(A)) = n-r$, $\dim(N(A^T)) = m-r$, $\dim(-(A)) = \dim(-(R))$

Part 2 $N(A)^\perp = C(A^T)$ in $\mathbb{R}^n$, $N(A^T)^\perp = C(A)$ in $\mathbb{R}^m$