Singular Value Decomposition (SVD)

$$A_{m\times n} = U\Sigma V^T$$

$(A^TA)\underline{v_i} = \lambda_i \underline{v_i}$, $\lambda_1 \cdots \lambda_r > 0$, $\lambda_{r+1} = \cdots = \lambda_n = 0$, $\underline{u_i} = \dfrac{A\underline{v_i}}{\sigma_i}$, $\sigma_i = \sqrt{\lambda_i} > 0$

Remark 1　半正定 symmetric matrix $A = Q\Lambda Q^T$

Remark 2　$\overset{m\times n}{A}\underset{n\times m}{A^T} = (U\Sigma V^T)(V\Sigma^T U^T) = U\Lambda U^T \Longrightarrow (AA^T)\underline{u_i} = \lambda_i \underline{u_i}$

∘ $\begin{cases} (A^TA)\underline{v_i} = \lambda_i \underline{v_i} & \text{for } i = 1\cdots n \\ (AA^T)\underline{u_i} = \lambda_i \underline{u_i} & \text{for } i = 1\cdots m \end{cases}$

Therefore, $\underline{u_1} \cdots \underline{u_m}$ are orthogonal eigenvectors of AA^T with corresspondіng eigenvalues $\lambda_1 \cdots \lambda_r, 0 \cdots 0$

Recall that $\underline{v_1} \cdots \underline{v_n}$ are orthogonal eigenvectors of A^TA with corresspondіng eigenvalues $\lambda_1 \cdots \lambda_r, 0 \cdots 0$

Remark 3　$A^T(A\underline{v_i}) = \lambda_i \underline{v_i}$　$\lambda_i > 0$ for $i = 1\cdots r$ $\therefore$

$\underline{v_1} \cdots \underline{v_r}$ form an orthonormal basis for $C(A^T)$

$\underline{v_{r+1}} \cdots \underline{v_n}$ form an orthonormal basis for $N(A)$

[$C(A^T)$ 、$N(A)$ is orthogonal complement] $\mathbb{R}^n$

$A\underline{v_i} = \sigma_i \underline{u_i}$ for $i = 1\cdots r$ $\therefore$

$\underline{u_1} \cdots \underline{u_r}$ form an orthonormal basis for $C(A)$

$\underline{u_{r+1}} \cdots \underline{u_m}$ form an orthonormal basis for $N(A^T)$

[$C(A)$ 、$N(A^T)$ is orthogonal complement] $\mathbb{R}^m$

Applications of SVD

$A = U\Sigma V^T = \underline{u_1}\sigma_1\underline{v_1}^T + \cdots + \underline{u_r}\sigma_r\underline{v_r}^T$　Suppose $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_r > 0$

(1) 用於影像壓縮 to approximate A ，只保留足夠大 σ_i 項的和

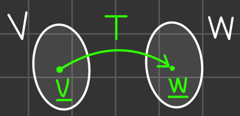
(2) In multiple-input multiple-output (MIMO) transmission, the channel $H = U\Sigma V^T$ where the MIMO channel is dicomposed into r uncoupled parallel subchannels with σ_i as the amplitude channel gain of the ith parallel subchannel.

※ Operator Norm or Norm of a matrix A : $\|A\| = \sigma_{max}$

※ Frobenius Norm of a matrix A : $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2} = \sqrt{\sum |a_{ij}|^2}$, $A = [a_{ij}]$

Linear transformations

Def A transformation T form a vector space V to a vector space W assigns to each input vector $\underline{v} \in V$ an output $T(\underline{v})$ in W. We write $T : V \rightarrow W$



Def A transformation T is linear if all $\underline{v}$ and $\underline{w}$

(i) $T(\underline{v}+\underline{w}) = T(\underline{v}) + T(\underline{w})$

(ii) $T(c\,\underline{v}) = c\,T(\underline{v}) \quad \forall c$

Remark1 $T(\underline{0}) = \underline{0}$ if T is linear $\quad T(\underline{0}) = T(\underline{0}+\underline{0}) = T(\underline{0}) + T(\underline{0}) = 2T(\underline{0})$

Remark2 (i).(ii) $\Rightarrow T(c\underline{v}+d\underline{w}) = cT(\underline{v}) + dT(\underline{w}) \quad \forall c.d.\underline{v}.\underline{w}$

Remark3 $T(c_1\underline{v_1} + \cdots + c_n\underline{v_n}) = \sum_1^n c_i T(\underline{v_i})$ if T is linear

Remark4 If $V = W$, then a linear transformation $T : V \rightarrow V$ is called a linear operator on V

Def Range of $T \triangleq \{T(\underline{v}) : \underline{v} \in V\}$ $\quad$ ex: $C(A)$

Def Kernel of $T \triangleq \{\underline{v} : T(\underline{v}) = \underline{0}\}$ $\quad$ ex: $N(A)$

Remark The range of T is a subspace of W, the kernel of T is a subspace of V

Matrix Representation of a Linear transformation

Suppose $\beta = \{\underline{v_1} \cdots \underline{v_n}\}$ is a basis of V and $\gamma = \{\underline{w_1} \cdots \underline{w_m}\}$ is a basis of W

For every $\underline{v} \in V$, we have $\underline{v} = x_1\underline{v_1} + \cdots + x_n\underline{v_n}$.

Since $T(\underline{v}) \in W$, we can have $T(\underline{v}) = y_1\underline{w_1} + \cdots + y_m\underline{w_m}$.

Let $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$

(coordinates of $\underline{v}$ with respect to β) (coordinates of $T(\underline{v})$ with respect to γ)

Since $T(\underline{v_j}) \in W$ for $j = 1 \cdots n$

, we can have $T(\underline{v_j}) = a_{1j}\underline{w_1} + \cdots + a_{mj}\underline{w_m} = \sum_{i=1}^m a_{ij}\underline{w_i}$

Then $T(\underline{v}) = T(x_1 \underline{v_1} + \cdots + x_n \underline{v_n}) = T(\sum_{j=1}^{n} x_j \underline{v_j}) = \sum_{j=1}^{n} x_j T(\underline{v_j}) = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} a_{ij} \underline{w_i}$

$= \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_j \underline{w_i} = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) \underline{w_i} = \sum_{i=1}^{m} y_i \underline{w_i}$ $\quad \therefore y_i = \sum_{j=1}^{n} a_{ij} x_j$ for $i = 1 \cdots m$

$$\Rightarrow \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \underline{y} = A \underline{x} .$$

We write $[T]_{\beta(n)}^{\gamma(m)} = A_{(m \times n)}$ If $\beta = \gamma$, we write $[T]_{\beta} = A$