Similar Matrices

Recall $\Lambda = S^{-1}AS$

Def Let M be any 可逆矩陣. Then $B = M^{-1}AM$ is similar to A

Remark 1 The process from A to B is called similarity transformation

Remark 2 A 自 similar $A = I^{-1}AI$ (reflexivity)

Remark 3 $A.B$ 相互 similar $B = M^{-1}AM \Rightarrow A = (M^{-1})^{-1}B(M^{-1})$ (symmetry)

Remark 4 If $A.B$ similar and $B.C$ similar, then $A.C$ similar $B = M_1^{-1}AM_1$, $C = M_2^{-1}BM_2$

$\Rightarrow C = M_2^{-1}(M_1^{-1}AM_1)M_2 = (M_1M_2)^{-1}A(M_1M_2)$ (transitivity)

Claim Similar matrices have the same eigenvalues

pf: Assume $B = M^{-1}AM \Rightarrow A = MBM^{-1}$

Suppose $A\underline{x} = \lambda\underline{x}$, then $MBM^{-1}\underline{x} = \lambda\underline{x} \Rightarrow B(M^{-1}\underline{x}) = \lambda(M^{-1}\underline{x})$

$B\underline{x}' = \lambda\underline{x}'$, $\underline{x}' = M^{-1}\underline{x} \Rightarrow B$ 的 eigenvalue : λ eigenvector : $M^{-1}\underline{x}$

alternative pf: $\det(B-\lambda I) = \det(M^{-1}AM-\lambda I) = \det(M^{-1}(A-\lambda I)M)$

$= \det(M^{-1})\det(A-\lambda I)\det(M) = \det(A-\lambda I) \Rightarrow$ A.B 有相同特徵多項式

$\Rightarrow$ Have the same eigenvalues

Remark If $\underline{x}$ is an eigenvector of A, then $M^{-1}\underline{x}$ is an eigenvector of $B = M^{-1}AM$

- All 2×2 matrices with eigenvalues $1, 0$ are similar to each other (This is generally true for matrices with all distinct eigenvalues)
- Same for A and $B = M^{-1}AM$

1. λ_i
2. Trace and 行列式
3. Rank
4. # of independent eigenvectors
5. Jordan form

3pf: Suppose $A\underline{x}=\underline{0}$, then $MBM^{-1}\underline{x}=\underline{0} \Rightarrow B(M^{-1}\underline{x})=\underline{0}$. If $\underline{x}\in N(A)$, then $M^{-1}\underline{x}\in N(B)$

Conversely, if $\underline{x}'\in N(B)$, then $M\underline{x}'\in N(A)$. $c_1\underline{x_1}+\cdots+c_\ell\underline{x_\ell}=\underline{0} \Leftrightarrow$

$c_1(M^{-1}\underline{x_1})+\cdots+c_\ell(M^{-1}\underline{x_\ell})=\underline{0}$ i.e. $\underline{x_1},\cdots,\underline{x_\ell}$ are L.I. $\Leftrightarrow M^{-1}\underline{x_1},\cdots,M^{-1}\underline{x_\ell}$ are L.I.

$\dim(N(A))=\dim(N(B)) \Leftrightarrow \text{rank}(A)=\text{rank}(B)$

4pf: $\underline{x}$ is an eigenvector of $A \Leftrightarrow M^{-1}\underline{x}$ is an eigenvector of B

5pf: Jordan form

If A has s independent eigenvectors, it is similar to a matrix J that has s Jordan blocks on its diagonal.

Jordan form

$M^{-1}AM=\begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_s \end{bmatrix}$ each blocks in J has one eigenvalue λ_i, one eigenvector and 1's above the diagonal. Jordan block : $J_i=\begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$

Remark 1 A is similar to B iff they share the same Jordan form

Remark 2 If A is diagonalizable, then $J=\Lambda$, the diagonal matrix with λ_i on the diagonal

Singular Value Decomposition (SVD)

Given $A_{m\times n}$, assume $\text{rank}(A)=r$. Since A^TA is $n\times n$ and symmetric, it has a complete set of n orthonormal eigenvectors, $A^TA\underline{v_i}=\lambda_i\underline{v_i}$ for $i=1,\cdots,n$ and each $\lambda_i\in\mathbb{R}$. We can have $\|A\underline{v_i}\|^2=\underline{v_i}^TA^TA\underline{v_i}=\underline{v_i}^T\lambda_i\underline{v_i}=\lambda_i\|\underline{v_i}\|^2=\lambda_i\geq 0$

Recall that $\text{rank}(A^TA)=\text{rank}(A)=r$ and # of nonzero eigenvalues of A equals # of nonzero pivots. There should be r nonzero eigenvalues of A^TA.

Assume $\lambda_1,\cdots,\lambda_r>0$ and $\lambda_{r+1}=\cdots=\lambda_n=0$ Let $\sigma_i=\sqrt{\lambda_i}$ for $i=1,\cdots,r$

These values are called singular values. Let $\underline{u_i}=\frac{A\underline{v_i}}{\sigma_i}$ for $i=1,\cdots,r$

$\|\underline{u_i}\|^2=\frac{\|A\underline{v_i}\|^2}{\lambda_i}=\frac{\lambda_i}{\lambda_i}=1$ and $\underline{u_i}^T\underline{u_j}=\frac{\underline{v_i}^T(A^TA\underline{v_j})}{\sigma_i\sigma_j}=\frac{\lambda_j\underline{v_i}^T\underline{v_j}}{\sigma_i\sigma_j}=0$ if $i\neq j$

Therefore, $\underline{u_1},\cdots,\underline{u_r}$ are orthonormal. By the Gram-Schmidt process, we can extend them to a complete orthonormal basis for $\mathbb{R}^m$: $\underline{u_1},\cdots,\underline{u_r},\underline{u_{r+1}},\cdots,\underline{u_m}$

Then $\underset{m\times n}{A}\,\overset{n\times 1}{\underline{v_i}} = \sigma_i\,\underset{m\times 1}{\underline{u_i}}$ for $i = 1, \cdots, r$ $\Rightarrow$ $\underset{m\times n}{A}\,\underset{n\times r}{\left[\underline{v_1} \cdots \underline{v_r}\right]} = \underset{m\times r}{\left[\underline{u_1} \cdots \underline{u_r}\right]}\underset{r\times r}{\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix}}$

$$\Rightarrow \underset{m\times n}{A}\,\underset{n\times n}{\left[\underline{v_1} \cdots \underline{v_n}\right]} = \underset{m\times m}{\left[\underline{u_1} \cdots \underline{u_m}\right]}\underset{m\times n}{\left[\begin{array}{ccc|c}\sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ \hline & 0 & & 0\end{array}\right]} \Rightarrow AV = U\Sigma \Rightarrow \underset{m\times n}{A} = \underset{m\times m}{U}\,\overset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$$

$$= \underline{u_1}\,\sigma_1\,\underline{v_1}^T + \cdots + \underline{u_r}\,\sigma_r\,\underline{v_r}^T$$ where $V^T = V^{-1}$ and $U^T = U^{-1}$

This is called "Singular Value Decomposition (SVD)"