Claim For a 正定 symmetric matrix A, the following properties are equivalent:

(1) $\underline{x}^T A \underline{x} > 0 \quad \forall \underline{x} \neq \underline{0}$

(2) All $\lambda_i > 0$

(3) All n 左上行列式 > 0 (順序主子式)

(4) All n pivots (without row exchanges) > 0

(5) $A = R^T R$ for a matrix R with L.I. columns

pf: (1) => (2) Suppose $A\underline{x_i} = \lambda_i \underline{x_i}$ ($\underline{x_i}$ is unit) => $\underline{x_i}^T A \underline{x_i} = \underline{x_i}^T \lambda_i \underline{x_i} = \lambda_i \|\underline{x_i}\|^2 = \lambda_i > 0$

(2) => (1) Assume all $\lambda_i > 0$. Since a symmetric matrix has a full set of n orthonormal eigenvectors $\underline{x_1}, \cdots, \underline{x_n}$, we can write for any $\underline{x}$:

$\underline{x} = c_1 \underline{x_1} + \cdots + c_n \underline{x_n} \Rightarrow A\underline{x} = c_1 A \underline{x_1} + \cdots + c_n A \underline{x_n} = c_1 \lambda_1 \underline{x_1} + \cdots + c_n \lambda_n \underline{x_n}$

$\Rightarrow \underline{x}^T A \underline{x} = (c_1 \underline{x_1}^T + \cdots + c_n \underline{x_n}^T)(c_1 \lambda_1 \underline{x_1} + \cdots + c_n \lambda_n \underline{x_n}) = c_1^2 \lambda_1 + \cdots + c_n^2 \lambda_n > 0$

$\forall \underline{x} \neq \underline{0}$ since all $\lambda_i > 0$

(1) => (3) $\det A = \prod \lambda_i > 0$ since A is 正定 and by (2) all $\lambda_i > 0$. We now show that all 左上 submatrices A_k are 正定 and hence $\det A_k > 0$

Let $\underline{x}^T A \underline{x} = \begin{bmatrix} \underline{x_k}^T & 0 \cdots 0 \end{bmatrix} \begin{bmatrix} A_k & \times \\ \times & \times \end{bmatrix} \begin{bmatrix} \underline{x_k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} > 0 \Rightarrow \underline{x_k}^T A_k \underline{x_k} > 0 \quad \forall \underline{x_k} \neq \underline{0}$

(3) => (4) $d_k = \dfrac{\det A_k}{\det A_{k-1}}$ (without row exchanges). Since all $\det A_k > 0$, all $d_k > 0$

(4) => (1) Without row exchanges, for a symmetric matrix A, we have $A = LDL^T$, then $\underline{x}^T A \underline{x} = \underline{x}^T L D L^T \underline{x} = (L^T \underline{x})^T D L^T \underline{x}$

$= \begin{bmatrix} \underline{l_1}^T \underline{x} & \cdots & \underline{l_n}^T \underline{x} \end{bmatrix} \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \begin{bmatrix} \underline{l_1}^T \underline{x} \\ \vdots \\ \underline{l_n}^T \underline{x} \end{bmatrix} = d_1 (\underline{l_1}^T \underline{x})^2 + \cdots + d_n (\underline{l_n}^T \underline{x})^2 > 0$

$\forall \underline{x} \neq \underline{0}$ since all $d_i > 0$

(5) => (1) If $A = R^T R$, $\underline{x}^T A \underline{x} = \underline{x}^T R^T R \underline{x} = (R\underline{x})^T R \underline{x} = \|R\underline{x}\|^2$. Since R has L.I. columns, $R\underline{x} \neq \underline{0}$ if $\underline{x} \neq \underline{0}$. ($x_1 \underline{r_1} + \cdots + x_n \underline{r_n} \neq \underline{0}$)

Therefore, $\underline{x}^T A \underline{x} = \|R\underline{x}\|^2 > 0 \quad \forall \underline{x} \neq \underline{0}$

(1) => (5) There are many choices for R:

(i) Since A is 正定 symmetric and no row exchanges, $A = LDL^T$

All the pivots $d_i > 0$, so let $\sqrt{D} \triangleq \begin{bmatrix}\sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n}\end{bmatrix} \Rightarrow A = (L\sqrt{D})(\sqrt{D}L^T)$

$= R^TR$, where $R = \sqrt{D}L^T$

(ii) $A = Q\Lambda Q^T$, all $\lambda_i > 0$, so let $\sqrt{\Lambda} = \begin{bmatrix}\sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n}\end{bmatrix} \Rightarrow A = Q\sqrt{\Lambda}\sqrt{\Lambda}Q^T$

$= R^TR$, where $R = \sqrt{\Lambda}Q^T$

(iii) Let R be one choice with $A = R^TR$. Then multiply R by any matrix with orthonormal columns (call it Q) $\Rightarrow$ Letting $R' = QR$, we have $(R')^TR' = (QR)^TQR = R^TQ^TQR = R^TIR = R^TR = A$

Therefore, $R' = QR$ is another choice, which may not be a 方陣

Def A real symmetric matrix A is said to be positive (negative) semidefinitive if $\underline{x}^TA\underline{x} \ge 0$ $(\le)$ $\forall \underline{x}$

Claim For a symmetric matrix A, the following properties are equivalent:

(1) $\underline{x}^TA\underline{x} \ge 0 \quad \forall \underline{x}$

(2) All $\lambda_i \ge 0$

(3) All n 左上行列式 ≥ 0 (順序主子式)

(4) All n pivots (without row exchanges) ≥ 0

(5) $A = R^TR$ for a matrix R possibly with L.D. columns

ex: Find the axes of this tilted ellipse $5x^2 + 8xy + 5y^2 = 1$

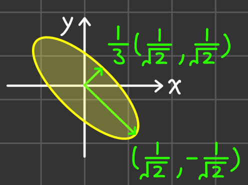
Let $A = \begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}$, then $\underline{x}^TA\underline{x} = 5x^2 + 8xy + 5y^2$

By the spectral theorem, $A = Q\Lambda Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}9 & 0\\ 0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

Then $\underline{x}^TA\underline{x} = \underline{x}^TQ\Lambda Q^T\underline{x} = (Q^T\underline{x})^T\Lambda(Q^T\underline{x}) = 9\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1\left(\frac{x-y}{\sqrt{2}}\right)^2$

Let $X = \frac{x+y}{\sqrt{2}}$ and $Y = \frac{x-y}{\sqrt{2}}$ $\Rightarrow 5x^2 + 8xy + 5y^2 = 9X^2 + Y^2 = 1 \Rightarrow \frac{X^2}{\frac{1}{9}} + \frac{Y^2}{\frac{1}{1}} = 1$

The axes point along eigenvectors $\underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}}$ principal axes



The half-lengths are $\sqrt{\frac{1}{9}} = \frac{1}{3}$ and $\sqrt{\frac{1}{1}} = 1$

$\begin{bmatrix}X\\ Y\end{bmatrix} = Q^T\begin{bmatrix}x\\ y\end{bmatrix}$

In general, suppose $A = Q\Lambda Q^T$ is 正定, so $\lambda_i > 0$. The graph of $\underline{x}^TA\underline{x} = 1$ is an ellipse $[x\ y]Q\Lambda Q^T\begin{bmatrix}x\\ y\end{bmatrix} = [X\ Y]\Lambda\begin{bmatrix}X\\ Y\end{bmatrix} = \lambda_1X^2 + \lambda_2Y^2 = 1$

$$\Rightarrow \frac{X^2}{\left(\frac{1}{\sqrt{\lambda_1}}\right)^2} + \frac{Y^2}{\left(\frac{1}{\sqrt{\lambda_2}}\right)^2} = 1 \quad \text{where } \begin{bmatrix} X \\ Y \end{bmatrix} = Q^T \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = Q \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \underline{q_1} & \underline{q_2} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = X \underline{q_1} + Y \underline{q_2}$$

The axes point along eigenvectors. The half-lengths are $\frac{1}{\sqrt{\lambda_1}}$ and $\frac{1}{\sqrt{\lambda_2}}$