Spectral theorem

∘ For a real symmetric matrix A, $A = Q\Lambda Q^T$

Claim Schur's theorem: For a 方陣 A, $A = QTQ^{-1}$ (T is 上三角矩陣, $\overline{Q}^T = Q^{-1}$)

If A has real eigenvalues, then Q and T can be chosen real: $Q^T = Q^{-1}$

Remark Q could be complex. If $\overline{Q}^T = Q^{-1}$, then Q is called a unitary matrix.

$Q\overline{Q}^T = I$. For real matrices, $QQ^T = I$, Q: orthogonal matrix

pf: Schur's theorem

∘ We prove this by induction on n

∘ $n=1$: $A=[a] \Rightarrow a = 1\cdot a \cdot 1^{-1}$

∘ 假設 $k\times k$ matrices 成立 and let A be a $(k+1)\times(k+1)$ matrix

∘ Let λ_1 be a eigenvalue of A and $\underline{q_1}$ be a corresponding unit eigenvector.

Using Gram-Schmidt process, we construct $\underline{q_2}, \cdots, \underline{q_{k+1}}$ s.t. $\underline{q_1}, \cdots, \underline{q_{k+1}}$ constitute an orthonormal basis for $\mathbb{C}^{k+1}$.

Let $Q_1 = [\underline{q_1} \cdots \underline{q_{k+1}}]$, then Q_1 is unitary. ($\overline{Q}^T = Q^{-1}$)

$$\text{And } \overline{Q_1}^T A Q_1 = \begin{bmatrix} \overline{\underline{q_1}}^T \\ \vdots \\ \overline{\underline{q_{k+1}}}^T \end{bmatrix} \begin{bmatrix} A\underline{q_1} \cdots A\underline{q_{k+1}} \end{bmatrix} = \begin{bmatrix} \overline{\underline{q_1}}^T \\ \vdots \\ \overline{\underline{q_{k+1}}}^T \end{bmatrix} \begin{bmatrix} \lambda_1\underline{q_1} & A\underline{q_2} \cdots A\underline{q_{k+1}} \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & x \cdots x \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & A_2 \end{bmatrix}}_{k+1} \Big\} {k+1}$$

$$\left(\lambda_1 \overline{\underline{q_1}}^T \cdot \underline{q_1} = \lambda_1 \|\underline{q_1}\|^2 = \lambda_1, \quad \overline{\underline{q_2}}^T \lambda_1 \underline{q_1} = \lambda_1 \overline{\underline{q_2}}^T \underline{q_1} = 0 \right)$$

∘ By the induction hypothesis, since A_2 is $k\times k$, we have $A_2 = Q_2 T_2 \overline{Q_2}^T$

$\Downarrow$

(Q_2 is unitary, T_2 is 上三角矩陣) $\quad T_2 = \overline{Q_2}^T A_2 Q_2$

Let $Q = Q_1 \begin{bmatrix} 1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & Q_2 \end{bmatrix}_{(k+1)\times(k+1)}$, then Q is unitary, i.e., $Q\overline{Q}^T = I$

$$\left(Q\overline{Q}^T = Q_1 \begin{bmatrix} 1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & Q_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \overline{Q_2}^T \end{bmatrix} \overline{Q_1}^T = Q_1 \begin{bmatrix} 1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_k \end{bmatrix} \overline{Q_1}^T = Q_1 I \overline{Q_1}^T = Q_1 \overline{Q_1}^T = I \right)$$

$$\text{Also, we have } \overline{Q}^T A Q = \underbrace{\begin{bmatrix} 1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \overline{Q_2}^T \end{bmatrix} \overline{Q_1}^T}_{\overline{Q}^T} A \underbrace{Q_1 \begin{bmatrix} 1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & Q_2 \end{bmatrix}}_{Q} = \begin{bmatrix} 1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \overline{Q_2}^T \end{bmatrix} \begin{bmatrix} \lambda_1 & x \cdots x \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & A_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & Q_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \overline{Q_2}^T \end{bmatrix} \begin{bmatrix} \lambda_1 & x \cdots x \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & A_2 Q_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & x \quad \cdots \quad x \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \overline{Q_2}^T A_2 Q_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & x \cdots x \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & T_2 \end{bmatrix}_{(k+1)\times(k+1)} \triangleq T \quad \because T \text{ is 上三角矩陣}$$

∘ If λ_1 is a real eigenvalue, then $\underline{q_1}$ and Q_1 can stay real.

The induction step keeps everything real when A has real eigenvalues. Induction starts with the case of 1×1, and there is no problem.

pf: Spectral theorem

Every real symmetric matrix A has real eigenvalues. By Schur's Theorem, $A = QTQ^T$

(Q is orthogonal: $Q^T = Q^{-1}$, T is 上三角矩陣) Then $T = Q^T A Q$

This is a symmetric matrix since $T^T = Q^T A Q = T$

If T is 三角矩陣 and also symmetric, it must be diagonal $\Rightarrow T = \Lambda$

Therefore, $A = Q\Lambda Q^T$

Claim If A is (real) symmetric, then # (number) of nonzero eigenvalues = rank r = # of nonzero pivots

pf: Since A is diagonalizable, # of zero eigenvalues $=$ AM of $\lambda = 0$ $=$ GM of $\lambda = 0$

$= \dim(\lambda = 0$ 的 eigenspace$) = \dim(N(A)) = n - r = n - (\#$ of nonzero pivots$)$

Positive Definitive Matrices 正定矩陣

ex: $A = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$ $\quad \underline{x}^T A \underline{x} = [x_1\ x_2]\begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 - 4x_1x_2 + 6x_2^2$ (quadratic form)

$B = \begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix}$ $\quad \underline{x}^T B \underline{x} = [x_1\ x_2]\begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 - 4x_1x_2 + 6x_2^2 = \underline{x}^T A \underline{x}$

- In general, if B is not symmetric, then let $A = \frac{1}{2}(B + B^T)$, which is symmetric, and $\underline{x}^T B \underline{x} = \underline{x}^T A \underline{x}$.

  Therefore, we only need to consider symmetric matrices in quadratic form.

Def A real symmetric matrix A is said to be positive (negative) definitive if $\underline{x}^T A \underline{x} > 0$ $(<)$ for every nonzero vector $\underline{x}$

ex: $A = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$ $\quad \underline{x}^T A \underline{x} = x_1^2 - 4x_1x_2 + 6x_2^2 = (x_1 - 2x_2)^2 + 2x_2^2 > 0$ for every nonzero $\underline{x}$

$\therefore$ A is positive definitive

$B = \begin{bmatrix} -1 & 2 \\ 2 & -6 \end{bmatrix} \Rightarrow B$ is negative definitive

Claim If A is (real) symmetric, then # of positive eigenvalues = # of positive pivots