Stability of Differential Equations

- If $\lambda = a+ib$, $e^{\lambda t} = e^{at} \cdot e^{ibt} = e^{at}(\cos bt + i\sin bt)$ and $|e^{\lambda t}| = e^{at}$
- $\frac{du}{dt} = A\underline{u}$ is stable when all $Re(\lambda_i) < 0$ $\left(\lim_{t\to\infty} e^{At} \to 0\right)$

  neutral when all $Re(\lambda_i) \le 0$ and some $Re(\lambda_i) = 0$

  unstable when all $Re(\lambda_i) > 0$ $\left(\lim_{t\to\infty} e^{At} \text{ is unbounded}\right)$

Symmetric Matrices

- $A : A^T = A$

1. A has only real eigenvalues.
2. The eigenvectors can be chosen orthonormal.

- Spectral Theorem
- $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$, where Q is an orthogonal matrix

Claim All the eigenvalues of a real symmetric matrix are real

pf: Suppose $\overset{①}{A\underline{x} = \lambda\underline{x}}$ $\Rightarrow$ $\overline{A\underline{x}} = \overline{\lambda\underline{x}}$ (complex conjugate) $\Rightarrow A\overline{\underline{x}} = \overline{\lambda}\,\overline{\underline{x}} \Rightarrow (A\overline{\underline{x}})^T = (\overline{\lambda}\,\overline{\underline{x}})^T$

$\Rightarrow \overline{\underline{x}}^T A^T = \overline{\lambda}\,\overline{\underline{x}}^T \Rightarrow \overset{②}{\overline{\underline{x}}^T A = \overline{\lambda}\,\overline{\underline{x}}^T}$

We can have $\overset{①}{\overline{\underline{x}}^T(A\underline{x})} = \overline{\underline{x}}^T(\lambda\underline{x}) = \lambda\|\underline{x}\|^2$

$\qquad\qquad\quad \|$

$\qquad\quad \overset{②}{(\overline{\underline{x}}^T A)\underline{x}} = (\overline{\lambda}\,\overline{\underline{x}}^T)\underline{x} = \overline{\lambda}\|\underline{x}\|^2$

Therefore, $\lambda = \overline{\lambda}$ $\Rightarrow \lambda$ is real

Claim Eigenvectors of a real symmetric matrix (when they correspond to different eigenvalues) are always orthogonal

pf: Suppose $A\underline{x}_1 = \lambda_1\underline{x}_1$ and $A\underline{x}_2 = \lambda_2\underline{x}_2$, where $\lambda_1 \neq \lambda_2$

We can obtain $\lambda_1(\underline{x}_1^T\underline{x}_2) = (\lambda_1\underline{x}_1)^T\underline{x}_2 = (A\underline{x}_1)^T\underline{x}_2 = \underline{x}_1^T A^T\underline{x}_2 = \underline{x}_1^T(A\underline{x}_2) = \underline{x}_1^T(\lambda_2\underline{x}_2)$

$= \lambda_2(\underline{x}_1^T\underline{x}_2)$

Since $\lambda_1 \neq \lambda_2$, we must have $\underline{x}_1^T\underline{x}_2 = 0$

- For a 3×3 symmetric matrix A, $A = Q\Lambda Q^T = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \underline{x}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix} \begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \\ \underline{x}_3^T \end{bmatrix}$

$= \lambda_1 \underline{x}_1 \underline{x}_1^T + \lambda_2 \underline{x}_2 \underline{x}_2^T + \lambda_3 \underline{x}_3 \underline{x}_3^T$

If $\lambda_1 = \lambda_2$, $A = Q\Lambda Q^T = \lambda_1 \underbrace{(\underline{x}_1 \underline{x}_1^T + \underline{x}_2 \underline{x}_2^T)}_{P_1} + \lambda_3 \underbrace{\underline{x}_3 \underline{x}_3^T}_{P_3}$

Recall for projection of $\underline{b}$ onto orthonormal $\underline{q}_1, \cdots, \underline{q}_n$

$\underline{p} = \underline{q}_1(\underline{q}_1^T \underline{b}) + \cdots + \underline{q}_n(\underline{q}_n^T \underline{b}) = (\underline{q}_1 \underline{q}_1^T + \cdots + \underline{q}_n \underline{q}_n^T)\underline{b} = P\underline{b}$, hence $A = \lambda_1 P_1 + \lambda_3 P_3$, where

P_1 is the projection matrix onto the subspace spanned by $\underline{x}_1$、$\underline{x}_2$ (eigenspace corresponding to λ_1)

P_3 is the projection matrix onto the eigenspace corresponding to λ_3

In general, for a real symmetric A, if $\lambda_1, \cdots, \lambda_k$ 不同, $A = \lambda_1 P_1 + \cdots + \lambda_k P_k$ Spectral decomposition

where P_i is the projection matrix onto the eigenspace corresponding to λ_i for $i = 1, \cdots, k$

Claim For real matrices, complex eigenvalues and eigenvectors come in "conjugate pairs"

pf: Suppose ① $A\underline{x} = \lambda \underline{x}$, then $\overline{A\underline{x}} = \overline{\lambda \underline{x}} \Rightarrow$ ② $A\overline{\underline{x}} = \overline{\lambda}\overline{\underline{x}}$