Solving the nth-order linear homogeneous difference equation with constant coefficients (if A is diagonalizable)

- In general, $G_{k+n} = a_1 G_{k+n-1} + a_2 G_{k+n-2} + \cdots + a_n G_k$, for $k \geq 0$

  with initial conditions $G_{n-1}, G_{n-2}, \cdots, G_1, G_0$

  Let $\underline{u_k} = \begin{bmatrix} G_{k+n-1} \\ \vdots \\ G_k \end{bmatrix}$, then we have $\underline{u_{k+1}} = \begin{bmatrix} G_{k+n} \\ \vdots \\ G_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} G_{k+n-1} \\ \vdots \\ G_k \end{bmatrix}$, $\underline{u_{k+1}} = \underset{n\times n}{A}\,\underline{u_k}$

- $\underline{u_k} = A^k \underline{u_0}$

(1) Diagonalize A : $A = S\Lambda S^{-1} = [\underline{x_1} \cdots \underline{x_n}] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} [\underline{x_1} \cdots \underline{x_n}]^{-1}$

(2) Write $\underset{\uparrow}{\underline{u_0}} = \begin{bmatrix} G_{n-1} \\ \vdots \\ G_0 \end{bmatrix} = c_1\underline{x_1} + \underset{\uparrow}{\cdots} + c_n\underline{x_n} \Rightarrow \underline{u_0} = [\underline{x_1} \cdots \underline{x_n}] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S \cdot \underline{c} \Rightarrow \underline{c} = S^{-1}\underline{u_0}$

initial conditions　　表成 eigenvectors 的組合

(3) $\underline{u_k} = A^k \underline{u_0} = (S\Lambda^k S^{-1})\,\underline{u_0} = S\Lambda^k \underline{c} = [\underline{x_1} \cdots \underline{x_n}] \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow \underline{u_k} = S\Lambda^k \underline{c}$

(4) G_k can be found as the nth component of $\underline{u_k}$

Solving Differential Equations (if A is diagonalizable)

- 1st differential equation ※ $\frac{du}{dt} = \lambda u(t) \Rightarrow u(t) = c \cdot e^{\lambda t}$

$$\begin{cases} \frac{dy}{dt} = -2y + z, \; y(0) = 2 \\ \frac{dz}{dt} = y - 2z, \; z(0) = 0 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}}_{\frac{d\underline{u}(t)}{dt}} = \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y \\ z \end{bmatrix}}_{\underline{u}(t)}$$

$\frac{d\underline{u}}{dt} = A\underline{u} \Rightarrow \frac{d}{dt}(e^{\lambda_1 t}\underline{x_1}) = \lambda_1 e^{\lambda_1 t}\underline{x_1} = e^{\lambda_1 t}(\lambda_1 \underline{x_1}) = e^{\lambda_1 t}(A\underline{x_1}) = A(e^{\lambda_1 t}\underline{x_1})$

，同理 $\frac{d}{dt}(e^{\lambda_2 t}\underline{x_2}) = A(e^{\lambda_2 t}\underline{x_2})$

$\Rightarrow$ solution : $\underline{u}(t) = c_1 e^{\lambda_1 t}\underline{x_1} + c_2 e^{\lambda_2 t}\underline{x_2} = [\underline{x_1}\;\underline{x_2}] \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Se^{\Lambda t}\underline{c}$, $\underline{c} = S^{-1}\underline{u}(0)$

Recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, For $A_{n\times n}$

Define $e^{At} \triangleq I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$

then $\frac{de^{At}}{dt} = A + A^2 t + \frac{1}{2!}A^3 t^2 + \frac{1}{3!}A^4 t^3 + \cdots = A\left(I + At + \frac{1}{2!}A^2 t^2 + \frac{1}{3!}A^3 t^3 + \cdots\right)$

$= Ae^{At}$

- $e^{As}e^{At} = e^{A(s+t)}$

- $e^{A0} = I$

- Suppose $\underline{u}(t) = e^{At}\underline{u}_I$, then $\frac{d\underline{u}}{dt} = Ae^{At}\underline{u}_I = A\underline{u}$
  , Hence $\underline{u} = e^{At}\underline{u}_I$ solves $\frac{d\underline{u}}{dt} = A\underline{u}$
- $\underline{u}(0) = \underline{u}_I$
- $e^{At} = I + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}}{2!}t^2 + \frac{S\Lambda^3 S^{-1}}{3!}t^3 + \cdots = S\left[I + \Lambda t + \frac{\Lambda^2}{2!}t^2 + \frac{\Lambda^3}{3!}t^3 + \cdots\right]S^{-1}$
  $= Se^{\Lambda t}S^{-1}$, $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$
- $\underline{u}(t) = e^{At}\underline{u}(0) = Se^{\Lambda t}S^{-1}\underline{u}(0) = Se^{\Lambda t}\underline{c}$
- $e^{\Lambda t} = I + \Lambda t + \frac{\Lambda^2}{2!}t^2 + \frac{\Lambda^3}{3!}t^3 + \cdots = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & 0 \\ & \ddots & \\ 0 & & \lambda_n t \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^2 t^2}{2!} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_n^2 t^2}{2!} \end{bmatrix} + \cdots = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$
- $\underline{u}(t) = Se^{\Lambda t}S^{-1}\underline{u}(0) = \left[\underline{x}_1 \cdots \underline{x}_n\right]\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} S^{-1}\underline{u}(0) = c_1 e^{\lambda_1 t}\underline{x}_1 + \cdots + c_n e^{\lambda_n t}\underline{x}_n$, $S^{-1}\underline{u}(0) = \underline{c}$