$A\underline{x} = \lambda \underline{x}$

eigenvalue : λ

eigenvector : $\underline{x}$

$\det(A-\lambda I)=0 \Rightarrow \lambda \Rightarrow \underline{x}$

$S = [\underline{x_1} \cdots \underline{x_n}]$

$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$A = S\Lambda S^{-1}$

$A^k = S\Lambda^k S^{-1}$

Claim If $\lambda_1, \cdots, \lambda_n$ are all distinct, then $\underline{x_1}, \cdots, \underline{x_n}$ are L.I.

pf: $A_{2\times 2}$; Suppose $c_1\underline{x_1} + c_2\underline{x_2} = \underline{0}$ (1)

$\Rightarrow A(c_1\underline{x_1} + c_2\underline{x_2}) = A\cdot\underline{0} \Rightarrow c_1(A\underline{x_1}) + c_2(A\underline{x_2}) = \underline{0} \Rightarrow c_1\lambda_1\underline{x_1} + c_2\lambda_2\underline{x_2} = \underline{0}$ (2)

Multiplying (1) by λ_2, we can also have $c_1\lambda_2\underline{x_1} + c_2\lambda_2\underline{x_2} = \underline{0}$ (3)

(2)−(3) $\Rightarrow c_1(\lambda_1-\lambda_2)\underline{x_1} = \underline{0}$, since $\lambda_1 \neq \lambda_2$ and $\underline{x_1} \neq \underline{0}$ we must have $c_1 = 0$

Similarly, $c_2 = 0$. Therefore, $\underline{x_1}$ and $\underline{x_2}$ are L.I.

The proof extends directly to n eigenvectors.

Suppose $c_1\underline{x_1} + \cdots + c_n\underline{x_n} = \underline{0}$ (multiply by A, multiply by λ_{n-1}, and substract)

Removing $\underline{x_n}$ first and then $\underline{x_{n-1}}$, eventually only $\underline{x_1}$ is left.

(multiply by A, multiply by λ_n, and substract)

$\Rightarrow (\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_n)c_1\underline{x_1} = \underline{0} \Rightarrow c_1 = 0$. Similarly, $c_i = 0$ for $i = 2, \cdots, n$

Remark $A_{n\times n}$ that has n 不同 eigenvalues (no repeated eigenvalues) must be diagonalizable.

Question When does $A^k \to O$ (zero matrix)?

Answer All $|\lambda| < 1$. $A^k = S\Lambda^k S^{-1} = S\begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} S^{-1} \to O$

Diagonalizable Matrices

Claim $A_{n\times n}$ is diagonalizable iff A has n L.I. eigenvectors

pf: "$\Leftarrow$" proved previously

"$\Rightarrow$" If A is diagonalizable, then there exist an 可逆 matrix S and a diagonal matrix Λ s.t. $S^{-1}AS = \Lambda$. Let $S = [\underline{x_1} \cdots \underline{x_n}]$ and $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Then $AS = S\Lambda \Rightarrow A[\underline{x_1} \cdots \underline{x_n}] = [\underline{x_1} \cdots \underline{x_n}]\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \Rightarrow A\underline{x_i} = \lambda_i\underline{x_i}$ for $i = 1, \cdots, n$

we can obtain λ_i is an eigenvalue and $\underline{x_i}$ is the associated eigenvector for $i = 1, \cdots, n$

Since S 可逆, $\underline{x_1}, \cdots, \underline{x_n}$ are L.I.

- Suppose λ is an eigenvalue
  1. Algebraic Multiplicity (AM)

     Count the repetitions of λ among the eigenvalues

     (Look at all the roots of $\det(A-\lambda I)=0$)
  2. Geometric Multiplicity (GM)

     Count the independent eigenvectors for λ

     This is the dimension of $N(A-\lambda I)$

     ($N(A-\lambda I)$ is called the eigenspace corresponding to λ)

Claim For every distinct eigenvalue, GM ≤ AM

distinct eigenvalues : $\begin{matrix} \lambda_1, \cdots, \lambda_n \\ \downarrow \quad\quad \downarrow \\ \underline{x_1} \quad\quad \underline{x_n} \end{matrix}$ $\quad \begin{matrix} AM=1 \\ GM=1 \end{matrix}$ $\Rightarrow$ GM = AM

repeated eigenvalues : $\begin{matrix} \lambda_1, \lambda_2, \cdots, \lambda_n \\ \lambda_1=\lambda_2 \end{matrix}$ $\quad \begin{matrix} AM=2 \\ GM=1 \text{ or } 2 \end{matrix}$ $\Rightarrow$ GM ≤ AM

Claim Suppose $A_{n\times n}$ has k distinct eigenvalues : $\lambda_1, \cdots, \lambda_k$. A is diagonalizable iff

(1) $\sum_{i=1}^{k} AM_i = n$

(2) For every distinct eigenvalue λ_i, $GM_i = AM_i$

Solving Difference Equations 差分方程

- $F_{k+2} = F_{k+1} + F_k$, $k \geq 0$, $F_0 = 0$, $F_1 = 1 \longrightarrow 0,1,1,2,3,5,8$ Fibonacci numbers

Let $\underline{u_k} = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$, $\underline{u_{k+1}} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} F_{k+1}+F_k \\ F_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A\underline{u_k} \Rightarrow \underline{u_k} = A^k \underline{u_0}$, $\underline{u_0} = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

將A對角化 : $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$, $\underline{x_1} = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$, $\underline{x_2} = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \Rightarrow A = S\Lambda S^{-1} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1}$

1. $\underline{u_0} = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = C_1\underline{x_1} + C_2\underline{x_2} = C_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}}_{S} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = S \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = S^{-1}\underline{u_0} = \frac{\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1-\lambda_2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \underline{u_0} = \frac{1}{\lambda_1-\lambda_2}(\underline{x_1} - \underline{x_2})$$

2. $\underline{u_k} = A^k \underline{u_0} = A^k \frac{1}{\lambda_1-\lambda_2}(\underline{x_1}-\underline{x_2}) = \frac{1}{\lambda_1-\lambda_2}(\lambda_1^k \underline{x_1} - \lambda_2^k \underline{x_2}) = \frac{\lambda_1^k}{\lambda_1-\lambda_2}\underline{x_1} - \frac{\lambda_2^k}{\lambda_1-\lambda_2}\underline{x_2} = \frac{\lambda_1^k}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{\lambda_2^k}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$\because F_k = \frac{1}{\lambda_1-\lambda_2}(\lambda_1^k - \lambda_2^k) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$, $k \geq 2$

For large k, $F_k \to \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k$. When k is large, $\frac{F_{k+1}}{F_k} \to \frac{1+\sqrt{5}}{2} \doteqdot 1.618$ golden ratio