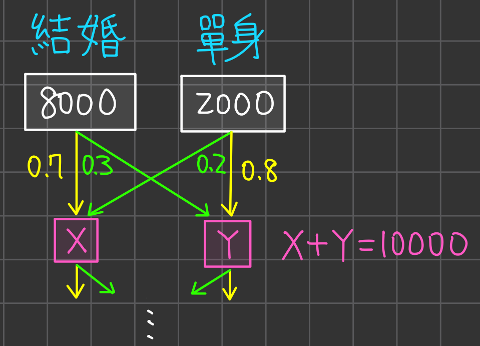
Eigenvalues and eigenvectors

- ex:



Let $\underline{W_0} = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix}$, $\underline{W_1} = \underbrace{\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 8000 \\ 2000 \end{bmatrix}}_{W_0} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}$

$\underline{W_n} = A^n \underline{W_0}$, $\lim\limits_{n\to\infty} \underline{W_n} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix} = \underset{\text{steady state}}{\underline{W}}$, $A\underline{W} = \underline{W}$

Suppose $\underline{x_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $A\underline{x_1} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \underline{x_1}$

看成 basis $\underline{x_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then $A\underline{x_2} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2}\underline{x_2}$

We can have $\underline{W_0} = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = 2000\underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{x_1} - 4000\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{x_2} = 2000\underline{x_1} - 4000\underline{x_2}$

It follows that $\underline{W_1} = A\underline{W_0} = A(2000\underline{x_1} - 4000\underline{x_2}) = 2000A\underline{x_1} - 4000A\underline{x_2}$

$= 2000\underline{x_1} - 4000(\frac{1}{2}\underline{x_2})$

$\underline{W_2} = A\underline{W_1} = A(2000\underline{x_1} - 4000(\frac{1}{2}\underline{x_2})) = 2000\underline{x_1} - 4000(\frac{1}{2})^2\underline{x_2}$

In general $\underline{W_n} = A^n\underline{W_0} = 2000\underline{x_1} - 4000(\frac{1}{2})^n\underline{x_2} \xrightarrow{n\to\infty} 2000\underline{x_1} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$

Def For $A_{n\times n}$, if $A\underline{x} = \lambda\underline{x}$ for a nonzero vector $\underline{x}$, then λ is called an eigenvalue of A and $\underline{x}$ is the associated eigenvector

- $A-\lambda I$: Nullspace $\neq \{0\}$, singular , $\det(A-\lambda I)=0$ characteristic polynomial
- 1. $\det(A-\lambda I) = 0$, solve λ (characteristic polynomial)
  2. $(A-\lambda I)\underline{x} = \underline{0}$, solve $\underline{x}$ ($N(A-\lambda I)$)
- If A is singular, then λ will have 0 ex: projection matrix ($P^2=P^T=P$)

  pf: $(A-\lambda I)\underline{x} = \underline{0}$, λ 代 0 => $A\underline{x} = \underline{0}$, $\underline{x}$ is nonzero => A is singular
- rotation matrix Q 有複數 λ ,因旋轉後沒 $\underline{x}$ 保持原來方向 (except $\underline{x}=\underline{0}$)
- If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2 ,且相應 $\underline{x}$ remains the same

  pf: $A\underline{x} = \lambda\underline{x}$, $A^2\underline{x} = AA\underline{x} = A\lambda\underline{x} = \lambda A\underline{x} = \lambda\lambda\underline{x} = \lambda^2\underline{x}$

Claim $\prod \lambda_i = \det A$

pf: $\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & & \cdots \\ & \ddots & \\ \cdots & & a_{nn}-\lambda \end{vmatrix}$ → a polynomial in λ of degree n

$= (\lambda_1-\lambda)\cdot \cdots \cdot(\lambda_n-\lambda)$ ← 做因數分解可得

Putting $\lambda = 0$, $\det A = \lambda_1\lambda_2\cdots\lambda_n$

Claim $\sum \lambda_i = \operatorname{trace}(A)$ $\boxed{\operatorname{trace} A = \sum a_{ii}}$

pf: 比較 λ^{n-1} 的係數, the only term in $\det(A-\lambda I)$ contains λ^{n-1} terms is

$(a_{11}-\lambda)\cdot\cdots\cdot(a_{nn}-\lambda)$ (若其中一列不取 $a_{ii}-\lambda$ 則至少有另一列 $a_{jj}-\lambda$ 不能取

, 而其最高次為 λ^{n-2}). We can have $(-1)^{n-1}(a_{11}+\cdots+a_{nn}) = (-1)^{n-1}(\lambda_1+\cdots+\lambda_n)$

$\Rightarrow a_{11}+a_{22}+\cdots+a_{nn} = \lambda_1+\lambda_2+\cdots+\lambda_n$

Diagonalizing a matrix

- Suppose $A_{n\times n}$ has n L.I. eigenvectors $\underline{x_1},\cdots,\underline{x_n}$

$$\begin{cases} A\underline{x_1} = \lambda_1 \underline{x_1} \\ \vdots \\ A\underline{x_n} = \lambda_n \underline{x_n} \end{cases} \Rightarrow A\cdot\underbrace{[\underline{x_1}\cdots\underline{x_n}]}_{S} = [\lambda_1\underline{x_1}\cdots\lambda_n\underline{x_n}] \Rightarrow AS = \underbrace{[\underline{x_1}\cdots\underline{x_n}]}_{S}\underbrace{\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}}_{\text{eigenvalue matrix}} = S\cdot\Lambda$$

$\Rightarrow S^{-1}AS = \Lambda$ or $A = S\Lambda S^{-1}$

- Then, $A^2 = A\cdot A = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^2 S^{-1}$

$A^k = S\Lambda^k S^{-1}$