- ## Cramer's Rule

Try to solve $A\underline{x}=\underline{b}$

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{\underline{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\underline{b}}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} \Rightarrow \underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix}} = \underset{B_1}{\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}}$$

$$\Rightarrow |A| \cdot x_1 = |B_1| \Rightarrow x_1 = \frac{\det B_1}{\det A}$$

Similarly, $[\underline{a_1}\ \underline{a_2}\ \underline{a_3}]\begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = [\underline{a_1}\ \underline{b}\ \underline{a_3}] \Rightarrow x_2 = \frac{\det B_2}{\det A}$

- If $\det A \neq 0$, $A\underline{x}=\underline{b}$ is solved by $x_i = \frac{\det B_i}{\det A}$

, where B_i has the ith column of A replaced by $\underline{b}$.

- Cramer's Rule : $O((n+1)!) \gg O(\frac{n^3}{3})$: Guass Elimination

- ## Inverse

$$AA^{-1} = I \Rightarrow \underset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow x_{11} = \frac{\begin{vmatrix} 1 & a_{12} \\ 0 & a_{22} \end{vmatrix}}{|A|} = \frac{C_{11}}{|A|} \Rightarrow (A^{-1})_{ij} = \frac{C_{ji}}{|A|}$$

- $A^{-1} = \frac{C^T}{|A|}$ where $C = [C_{ij}]$

∘ Area

Claim Area $= \det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \det A$

Remark 正值 if (x_1, y_1) to (x_2, y_2) is 逆時針

pf: 1. when $A = I$, Area $= 1$ and $\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

2. when 列交換, $\det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = -\det\begin{bmatrix} x_2 & y_2 \\ x_1 & y_1 \end{bmatrix}$

$\Rightarrow$ value (area) 不變, sign 變

3.(a) If row 1 $\times t$, $\det\begin{bmatrix} tx_1 & ty_1 \\ x_2 & y_2 \end{bmatrix} = t\det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$

$\Rightarrow$ area $\times t$

(b) Suppose a new row (x_1', y_1') is added to (x_1, y_1)

$\det\begin{bmatrix} x_1+x_1' & y_1+y_1' \\ x_2 & y_2 \end{bmatrix} = \det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} + \det\begin{bmatrix} x_1' & y_1' \\ x_2 & y_2 \end{bmatrix}$

$\Rightarrow$ area 相加

∘ Similarly, Volumn $= \det\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$ (符合右手定則 value >0, 反之)

volumn 也遵守上述三規則

Claim 由 (x_1, y_1)、(x_2, y_2)、(x_3, y_3) 組成之 triangle has

area: $\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$