Remark Every rule for the rows can apply to the columns ($\because |A^T|=|A|$)

Permutation and cofactors

- The Pivot Formula

  $A_{n\times n}$ can $\rightarrow$ $PA=LU$

  (P: permutation matrix; L: 下三角, diagonal $=1$; U: 上三角, diagonal $=$ pivots $d_1,\cdots,d_n$)

  Then $|P||A|=|L||U| \rightarrow \pm|A|=\prod_{i=1}^{n} d_i \Rightarrow |A|=\pm\prod d_i$

- If $r<n$, then $|A|=0$ and A is singular
- If without row exchange, then A的對角元素 $=$ U的對角元素

  $|A_k|=\prod_{i=1}^{k} d_i$ (A_k: A左上角的 $n\times n$ submatrix)

- The kth pivot $d_k=\dfrac{|A_k|}{|A_{k-1}|}$ (if without row exchanges)
- The Big Formula

  $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \cancel{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \cancel{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}} = ad\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = ad-bc$$

- In general, $|A_{n\times n}|$可分解為 n^n 個 simple determinant (each row has only one entry)

  ,其中有 $n!$ 個有效行列式 (其餘整行元素皆為0者det $=0$)

  pf: each row pick one entry, 且其行皆不重複 $\rightarrow$ 排列組合 $=n!$

- In general, $|A|=\sum_{\sigma}|P_\sigma|\,a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$

  (σ: a permutation of $(1,\cdots,n)$; P_σ為相應排列矩陣; $\sum_{\sigma}$所有可能的σ之和,有$n!$個)

- Cofactor Formula
- $|A|=\sum_{j=1}^{n} a_{ij}C_{ij}$ ($C_{ij}=(-1)^{i+j}|M_{ij}|$; M_{ij}為將A第i列、第j行移除的submatrix)
- $|A|=\sum_{i=1}^{n} a_{ij}C_{ij}$ ($\because |A^T|=|A|$)
- $A_n=\underbrace{\begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}}_{n}$, $D_n=|A_n|$, $D_n=n+1$, $n=1、2、\cdots$