Determinants

Def: The determinant of real $A_{n\times n}$ denoted by $\det A$ or $|A|$, is $\mathbb{R}$ defined by:

1. $\det I = 1$
2. $\det A$ change sign when two rows are exchanged.
3. $\det A$ is a linear function of each row separately.

- If P is a permutation matrix, then $\det P = \pm 1$ (rule 2)
  (+1：偶數次列交換 −1：奇數次列交換)
- $\begin{vmatrix} a^T + b^T \\ B \end{vmatrix} = \begin{vmatrix} a^T \\ B \end{vmatrix} + \begin{vmatrix} b^T \\ B \end{vmatrix}$ (rule 3)
- $\det(tI) = t^n \det I = t^n$

4. If two rows are equal, then $\det A = 0$

pf: 交換兩列不改變 $A \to \det A = -\det A \to \det A = 0$

5. 從一列減去另一列的倍數 $\det A$ 不變

pf: $\det \begin{bmatrix} a^T - \lambda b^T \\ b^T \\ B \end{bmatrix} = \det \begin{bmatrix} a^T \\ b^T \\ B \end{bmatrix} - \lambda \det \begin{bmatrix} b^T \\ b^T \\ B \end{bmatrix} = \det \begin{bmatrix} a^T \\ b^T \\ B \end{bmatrix} - \lambda \cdot 0 = \det \begin{bmatrix} a^T \\ b^T \\ B \end{bmatrix}$

Remark Suppose $A \to U$, then $\det A = \det U$ (Gauss' Elimination without row change)

6. A matrix with a zero row has $\det A = 0$

pf: $\det \begin{bmatrix} 0^T \\ b^T \\ B \end{bmatrix} = \det \begin{bmatrix} b^T \\ b^T \\ B \end{bmatrix} = 0$

7. If A is triangular, then $\det A = \prod a_{ii}$ (對角元素乘積)

pf: Gauss Jordan Elimation $\to \mathrm{Diag}(a_{ii}) \to \det \mathrm{diag}(a_{ii}) = \prod a_{ii} \det I = \prod a_{ii}$

8. If A is singular, then $\det A = 0$

If A is invertible, then $\det A \neq 0$

pf: $A \to U$ if A is invertible, then U has nonzero pivots along its diagonal.

Then $\det A = \pm \det U = \pm \prod a_{ii} \neq 0$