- Project on a subspace

將 $\mathbb{R}^m$ 中 $\underline{b}$ 投影在由 $\underline{a_1},\cdots\underline{a_n}$ (which are L.I) 組成的 subspace 上，形成 $\underline{P}$

$\underline{P}=\hat{x}_1\underline{a_1}+\cdots+\hat{x}_n\underline{a_n}$

Let $A=[\underline{a_1}\cdots\underline{a_n}]_{m\times n}$, then $\underline{P}=[\underline{a_1}\cdots\underline{a_n}]\begin{bmatrix}\hat{x}_1\\ \vdots\\ \hat{x}_n\end{bmatrix}=A\underline{\hat{x}}$

The error vector $\underline{e}=\underline{b}-A\underline{\hat{x}}\perp C(A) \Rightarrow \begin{cases}\underline{a_1}^T(\underline{b}-A\underline{\hat{x}})=0\\ \vdots\\ \underline{a_n}^T(\underline{b}-A\underline{\hat{x}})=0\end{cases} \Rightarrow \begin{bmatrix}\underline{a_1}^T\\ \vdots\\ \underline{a_n}^T\end{bmatrix}[\underline{b}-A\underline{\hat{x}}]=\underline{0}$

$\Rightarrow A^T(\underline{b}-A\underline{\hat{x}})=\underline{0} \Rightarrow A^TA\underline{\hat{x}}=A^T\underline{b}$.

A^TA is symmetric, and it is invertible iff $\underline{a_1},\cdots,\underline{a_n}$ are L.I.

$\Rightarrow \underline{\hat{x}}=(A^TA)^{-1}A^T\underline{b} \Rightarrow \underline{P}=A\underline{\hat{x}}=\underbrace{A(A^TA)^{-1}A^T}_{P}\underline{b}$

The projection matrix $P=A(A^TA)^{-1}A^T$

$P^n=P$ pf: $(A(A^TA)^{-1}\cancel{A^T})(\cancel{A}\cancel{(A^TA)^{-1}}A^T)=A(A^TA)^{-1}A^T$

$\underline{P}=P\underline{b}=A\underline{\hat{x}}$, $A^TA\underline{\hat{x}}=A^T\underline{b}$

Claim $\operatorname{rank}(A^TA)=\operatorname{rank}(AA^T)=\operatorname{rank}(A)$ $\langle A: m\times n\rangle$

pf: 先pf $N(\overset{n\times n}{A^TA})=N(\overset{m\times n}{A})$ i.e. $A\underline{x}=\underline{0}\Leftrightarrow A^TA\underline{x}=\underline{0}$

"$\Rightarrow$" $A\underline{x}=\underline{0} \Rightarrow A^TA\underline{x}=A^T\underline{0}=\underline{0} \Rightarrow A^TA\underline{x}=\underline{0}$

"$\Leftarrow$" $A^TA\underline{x}=\underline{0} \Rightarrow \underline{x}^TA^TA\underline{x}=\underline{x}^T\underline{0}=\underline{0} \Rightarrow (A\underline{x})^T(A\underline{x})=\underline{0} \Rightarrow \|A\underline{x}\|^2=0 \Rightarrow A\underline{x}=\underline{0}$

Therefore, $N(A^TA)=N(A)$, which implies $n-\operatorname{rank}(A^TA)=n-\operatorname{rank}(A)$

$\Rightarrow \operatorname{rank}(A^TA)=\operatorname{rank}(A)$, 同理可證 AA^T

Remark 1 A^TA 可逆 iff A has L.I. columns

pf: A has L.I. columns $\Leftrightarrow \operatorname{rank}(A)=n \Leftrightarrow \operatorname{rank}(\overset{n\times n}{A^TA})=n \Leftrightarrow A^TA$ 可逆

Remark 2 When A has L.I. columns, A^TA is square, symmetric, and invertible

- Least squares approximation

$A\underline{x}=\underline{b}$ 不一定有精確解 $\Rightarrow \underline{e}=\underline{b}-A\underline{x}>\underline{0}$

此時我們希望 $\underline{e}$ or $\|\underline{e}\|^2$ 越小越好 $\Rightarrow$ least squares solution