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4月21日 星期三 上午10:19

Claim $\underline{v_1}, \underline{v_2}, \cdots, \underline{v_n}$ are L.I. in $\mathbb{R}^n$ iff $\underline{v_1}, \underline{v_2}, \cdots \underline{v_n}$ span $\mathbb{R}^n$

pf: "$\Rightarrow$" Let $A = [\, \underline{v_1}\, \underline{v_2} \cdots \underline{v_n} \,] \Rightarrow \underline{v_1} \cdots \underline{v_n}$ are L.I. in $\mathbb{R}^n$, then $\mathrm{rank}(A) = n$

$\longrightarrow$ A invertible $\longrightarrow A\underline{x} = \underline{b}$ solvable $\forall \underline{b} \in \mathbb{R}^n \longrightarrow \underline{v_1} \cdots \underline{v_n}$ span $\mathbb{R}^n$

"$\Leftarrow$" If $\underline{v_1}, \cdots, \underline{v_n}$ span $\mathbb{R}^n$, then $A\underline{x} = \underline{b}$ solvable $\forall \underline{b} \in \mathbb{R}^n$

$\longrightarrow \mathrm{rank}(R) = \mathrm{rank}(A) = n \longrightarrow N(A) = \{\underline{0}\} \longrightarrow \underline{v_1}, \cdots, \underline{v_n}$ are L.I.

Remark Any n L.I. vectors form a basis in $\mathbb{R}^n$. Also any n vectors that span $\mathbb{R}^n$ are a basis for $\mathbb{R}^n$.

Claim If $V \perp W$, then $V \cap W = \{\underline{0}\}$

pf: Suppose $\underline{v} \in V \cap W$, then $\underline{v} \in V$ and $\underline{v} \in W$. we have $\underline{v}^T \underline{v} = \|\underline{v}\|^2 = 0$, which implies $\underline{v} = \underline{0}$

Claim $\forall \underline{x} \in \mathbb{R}^n$, we can have $\underline{x} = \underline{x_r} + \underline{x_n}$ where $\underline{x_r} \in C(A^T)$ and $\underline{x_n} \in N(A)$

pf: Let $\underline{v_1}, \cdots, \underline{v_r}$ be a basis for $C(A^T)$, $\underline{w_1}, \cdots, \underline{w_{n-r}}$ be a basis for $N(A)$

Suppose $a_1 \underline{v_1} + \cdots + a_r \underline{v_r} + b_1 \underline{w_1} + \ldots + b_{n-r} \underline{w_{n-r}} = \underline{0}$, let

$\underline{u} = a_1 \underline{v_1} + \cdots + a_r \underline{v_r} = -b_1 \underline{w_1} - \ldots - b_{n-r} \underline{w_{n-r}}$. Then $\underline{u} \in C(A^T)$ and $\underline{u} \in N(A)$

Since $C(A^T) \perp N(A)$, we must have $\underline{u} = \underline{0}$.

Since $\underline{v_1} \cdots \underline{v_n}$ are a basis for $C(A^T)$ and

$\underline{w_1}, \cdots, \underline{w_{n-r}}$ are a basis for $N(A) \longrightarrow a_1 = \ldots = a_r = b_1 = \ldots = b_{n-r} = 0$.

Therefore $\underline{v_1} \cdots \underline{v_r}, \underline{w_1} \cdots \underline{w_{n-r}}$ are L.I. and hence form a basis for $\mathbb{R}^n$

$\forall \underline{x} \in \mathbb{R}^n$, we can have $\underline{x} = c_1 \underline{v_1} + \cdots + c_r \underline{v_r} + c_{r+1} \underline{w_1} + \cdots + c_n \underline{w_{n-r}} = \underset{\in C(A^T)}{\underline{x_r}} + \underset{\in N(A)}{\underline{x_n}}$

Remark The decomposition $\underline{x} = \underline{x_r} + \underline{x_n}$ is unique.

$A\underline{x} = A(\underline{x_r} + \underline{x_n}) = A\underline{x_r} + A\underline{x_n} = A\underline{x_r}$.

$\underline{x_n}$ goes to 0: $A\underline{x_n} = \underline{0}$, $\underline{x_r}$ goes to the column space: $A\underline{x_r} = A\underline{x}$

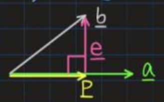
Claim $\forall\, \underline{b}$ in the column space comes from one and only one vector in the row space

pf: Let $\underline{b} = A\underline{x_r} = A\underline{x_r'} \in C(A)$ where $\underline{x_r}, \underline{x_r'} \in C(A^T) \Rightarrow A(\underline{x_r} - \underline{x_r'}) = \underline{0}$

$\Rightarrow \underline{x_r} - \underline{x_r'} \in N(A)$. Since $\underline{x_r} - \underline{x_r'} \in C(A^T)$, we can have $\underline{x_r} - \underline{x_r'} = \underline{0}$

yielding $\underline{x_r} = \underline{x_r'}$ (Note: $C(A^T) \perp N(A)$)

Projection



$\underline{e} = \underline{b} - \underline{p}$

Let $\underline{p} = \hat{x}\underline{a}$, Then $a^T\underline{e} = a^T(\underline{b} - \hat{x}\underline{a}) = 0 \Rightarrow \hat{x} = \dfrac{\underline{a}^T\underline{b}}{\underline{a}^T\underline{a}}$

$\therefore\ \underline{p} = \dfrac{\underline{a}^T\underline{b}}{\underline{a}^T\underline{a}}\underline{a}$

$\underline{p} = \hat{x}\underline{a} = \underline{a}\hat{x} = \underline{a}\dfrac{\underline{a}^T\underline{b}}{\underline{a}^T\underline{a}} = \dfrac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}}\underline{b} = P\underline{b}$, where $P = \dfrac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}}$ is the projection matrix

$P^n = P$ (projecting a second time doesn't change anything) $\longrightarrow P^2 = \dfrac{\underline{a}\,\cancel{\underline{a}^T\underline{a}}\,\underline{a}^T}{\underline{a}^T\underline{a}\,\cancel{\underline{a}^T\underline{a}}} = \dfrac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}} = P$