

◦ In general $R^T y = 0 \quad y^T R = 0^T$

$$R = \begin{pmatrix} \left[\begin{array}{cccc} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & & 0 & \dots & 0 \end{array} \right]_r \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \\ 0 & \dots & 0 & \dots & 0 \end{array} \right]_{m-r} \end{pmatrix} \Rightarrow y^T = [0 \dots 0 \quad y_{r+1} \dots y_m]$$

$\therefore (0 \dots 0 \underset{\uparrow r+1}{1} 0 \dots 0), (0 \dots 0 \underset{\uparrow r+2}{1} 0 \dots 0), \dots, (0 \dots 0 \underset{\uparrow m}{1})$ form a basis for $N(R^T)$

In other words, the last $m-r$ rows of $I_{m \times m}$, form a basis for $N(R^T)$

$\therefore \dim(N(R^T)) = m-r$

In general, $EA = R$

$$E_{m \times m} A_{m \times n} = \begin{pmatrix} \left[\begin{array}{cccc} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ 0 & \dots & 0 & \dots & 0 \end{array} \right]_r \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \\ 0 & \dots & 0 & \dots & 0 \end{array} \right]_{m-r} \end{pmatrix} \begin{matrix} \left[\begin{array}{c} \equiv \\ \equiv \\ \equiv \end{array} \right] A = 0 \\ y^T A = 0^T \end{matrix}$$

\uparrow last $m-r$ row of E

Since E 可逆, all rows of E L.I.

$\therefore \dim(N(A^T)) = m-r \Rightarrow$ The last $m-r$ rows of E form a basis for $N(A^T)$

$$\begin{bmatrix} \text{row } r+1 \\ \vdots \\ \text{row } m \end{bmatrix} A = 0 \Rightarrow y = y_1(\text{row } r+1) + \dots + y_r(\text{row } m)$$

◦ Orthogonality 正交性

$\underline{v} \cdot \underline{w} = \underline{v}^T \underline{w}$

$\underline{v} \cdot \underline{v} = \underline{v}^T \underline{v} = v_1^2 + \dots + v_n^2 \triangleq \|\underline{v}\|^2$

length of $\underline{v} \Rightarrow \|\underline{v}\| = (\underline{v}^T \underline{v})^{\frac{1}{2}}$

def: \underline{v} and \underline{w} orthogonal if $\underline{v}^T \underline{w} = 0$ ($\underline{v} \cdot \underline{w} = 0$)

Remark $\underline{v}^T \underline{w} = 0$ ($\underline{w}^T \underline{v} = 0$) $\Leftrightarrow \|\underline{v}\|^2 + \|\underline{w}\|^2 = \|\underline{v} + \underline{w}\|^2$ 勾股推廣

def: Two subspace V and W 正交 if $\underline{v}^T \underline{w} = 0 \quad \forall \underline{v} \in V$ and $\underline{w} \in W$



$$Ax = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Leftrightarrow \begin{matrix} (\text{row } 1) \cdot x = 0 \\ \vdots \\ (\text{row } m) \cdot x = 0 \end{matrix}$$

$\Rightarrow N(A)$ and $C(A^T)$ are orthogonal subspaces of \mathbb{R}^n

$$N(A) \perp C(A^T)$$

Similarly, $A^T y = \begin{bmatrix} (\text{column } 1)^T \\ \vdots \\ (\text{column } n)^T \end{bmatrix} \begin{bmatrix} y \\ \vdots \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\therefore N(A^T)$ and $C(A)$ are orthogonal subspaces of \mathbb{R}^m

$$N(A^T) \perp C(A)$$

def: The orthogonal complement V^\perp "V perp" of a subspace contains every vector that is orthogonal to V

$$V^\perp = \{w : w^T v = 0 \quad \forall v \in V\}$$

Remark V^\perp is also a subspace

Claim $C(A^T)^\perp = N(A)$, $N(A)^\perp = C(A^T)$, $C(A)^\perp = N(A^T)$, $N(A^T)^\perp = C(A)$

Pf: Suppose $v \in C(A^T) \cap v \in N(A)$, 將 v^T 加入 A 的 row 中 $\rightarrow A' = \begin{bmatrix} A_{\text{rows}} \\ v^T \end{bmatrix}$

$$\because v \in C(A^T) \therefore C(A'^T) = C(A^T) \Rightarrow \dim(C(A'^T)) = \dim(C(A^T)) = r$$

$$\Rightarrow \dim(N(A')) = n - r \text{ --- ①}$$

$$A'x = 0 \rightarrow v^T x = 0 \rightarrow v \perp x \rightarrow v \notin \{x\} = N(A'), \text{ but } v \in N(A)$$

$$\Rightarrow \dim(N(A)) < \dim(N(A')) = n - r \text{ --- ②}$$

① \nrightarrow ② $\Rightarrow v$ is D.N.E. ✖

線代基本定理 Part 1 $\dim(C(A)) = r$, $\dim(C(A^T)) = r$, $\dim(N(A)) = n - r$, $\dim(N(A^T)) = m - r$,
 $\dim(-A) = \dim(-R)$

Part 2 $N(A)^\perp = C(A^T)$ in \mathbb{R}^n , $N(A^T)^\perp = C(A)$ in \mathbb{R}^m